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reading:
- Weinberg, chapters 5-8

4 Evolution of density perturbations

4.1 Statistical description

The cosmological principle is obviously only valid on very large scales, not very much smaller than the size of the horizon. The generally accepted picture, however, is that the universe started off in an extremely homogeneous and isotropic state, with initial conditions provided by an era of accelerated expansion called *inflation*. The tiny primordial density fluctuations, generated during inflation from quantum fluctuations of the vacuum, would later grow under the influence of gravity and eventually collapse to form the structures that we observe today, like galaxies, clusters and super-clusters.

The energy density in the early universe can thus be written as

$$\rho(\mathbf{x},t) = \bar{\rho}(t)[1+\delta(\mathbf{x},t)]\,, \tag{43}$$

where $\delta \ll 1$ and $\bar{\rho}$ is the homogeneous background density that corresponds to the FRW space-time. Usually, it is much more convenient to work in Fourier space so we will in the following use $\delta\rho(\mathbf{x},t) \to \delta_{\mathbf{k}}(t)$. In analogy, for a multi-component fluid, one can define $\delta^i \equiv \rho^i/\bar{\rho}^i - 1$. Assuming Gaussian statistics for the primordial density fluctuations, as is the case in most models of inflation, all modes are uncorrelated:

$$\langle \delta_{\mathbf{k}} \delta^*_{\mathbf{k}'} \rangle \equiv \frac{2\pi^2}{k^3} \mathcal{P}_\delta(k)\, \delta(\mathbf{k}-\mathbf{k}')\,. \tag{44}$$

This equation also defines the *power spectrum* which often is assumed to follow a simply power-law[8]

$$\mathcal{P}_\delta(k) \propto k^{3+n}\,. \tag{46}$$

The probability (density) to find an average density contrast δ in a spherical region of size R is then given by

$$p_R(\delta) = \frac{1}{\sqrt{2\pi}\sigma(R)} \exp\left[-\frac{\delta^2}{2\sigma^2(R)}\right]\,, \tag{47}$$

[8]The way the spectral index n is introduced here has historical reasons. For $n=1$, in particular, one recovers the *scale-invariant* spectrum proposed independently by Harrison and Zel'dovich: In this case, the spectrum *evaluated at the time* t_k *of horizon crossing of the scale* k (defined by $aH = k$) does not depend on k:

$$\mathcal{P}_\delta(k)\big|_{t_k} \propto k^{n-1}\,. \tag{45}$$

The reason for this scaling will become apparent later when taking into account how δ evolves with time – which is not shown explicitly in Eqs. (44) and (46 - 48).

where the mass variance $\sigma(R)$ is computed by convolving the power spectrum with a top-hat window function:

$$\sigma^2(R) = \int_0^\infty W_{\rm TH}^2(kR)\mathcal{P}_\delta(k)\frac{dk}{k}\,. \tag{48}$$

Here, $W_{\rm TH}(x) = 3j_1(x)/x = 3\,x^{-3}\,(\sin x - x\cos x)$ denotes the Fourier transform of the top-hat window function. Note that the introduction of a window function is necessary in order to regulate the divergence of the integral $\int W_{\rm TH}^2(kR)\mathcal{P}_\delta(k)dk/k$ at both large and small k.

4.2 Newtonian treatment

In order to get a first, intuitive understanding of how density fluctuations evolve, let us start with a Newtonian analysis, keeping in mind that this will necessarily be restricted to $p \ll \rho$ and scales $\lambda \lesssim (aH)^{-1}$. The starting point are then the continuity, Euler[9] and Poisson equations:

$$\begin{aligned} \dot\rho + \nabla(\rho\mathbf{v}) &= 0 & (49)\\ \dot{\mathbf{v}} + (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\nabla\Phi_{\rm gr} - \frac{1}{\rho}\nabla p & (50)\\ \Delta\Phi_{\rm gr} &= 4\pi G\rho, & (51)\end{aligned}$$

where $\Phi_{\rm gr}$ is the Newtonian gravitational potential. The velocity of the fluid elements can be written as $\mathbf{v} = \frac{d}{dt}(a(t)\mathbf{x}) = \dot a(t)\mathbf{x} + a(t)\mathbf{u}$, where $\mathbf{u}$ is the peculiar velocity (which is treated as a small quantity); similarly, in a co-moving frame we have $\nabla_i = a^{-1}\partial_i$. Expanding all quantities that appear above as in Eq. (43), and using $\overline\rho \propto a^3$, one can combine these equations at first order in δ to

$$\ddot\delta_{\mathbf{k}} + 2H\dot\delta_{\mathbf{k}} + \left(\frac{c_s^2k^2}{a^2} - 4\pi G\overline\rho\right)\delta_{\mathbf{k}} = 0. \tag{52}$$

where $c_s^2 := \frac{\partial p}{\partial\rho} = w$ is the sound velocity. An important length scale of this equation is the *Jeans length*:

$$\lambda_J = 2\pi\frac{a}{k_J} \qquad \text{where} \qquad \left(\frac{k_J}{a}\right)^2 \equiv \frac{4\pi G\overline\rho}{c_s^2}. \tag{53}$$

For density fluctuations on scales $\lambda_{phys} \lesssim \lambda_J$, Eq. (52) takes the form of a damped harmonic oscillator: the perturbations oscillate and slowly decay; this is referred to as *acoustic oscillations*. For $\lambda_{phys} \gtrsim \lambda_J$, on the other hand,

[9] This is essentially just the Jeans equation, which we already encountered in Eq. (36).

there is both a growing and a decaying solution to the above ODE. Using $4\pi G\bar{\rho} = \frac{3}{2}H^2 = \frac{2}{3}t^{-2}$, one easily finds that the former is given by

$$\delta_{\mathbf{k}}(t) \propto t^{\frac{2}{3}} \propto a(t) \tag{54}$$

and soon starts to dominate over the latter ($\propto t^{-1}$).

4.3 Relativistic analysis

For the full relativistic treatment, one has to consider arbitrary perturbations of the metric and stress-energy tensor,

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, & (55)\\ T_{\mu\nu} &= \bar{T}_{\mu\nu} + \delta T_{\mu\nu}\,, & (56) \end{aligned}$$

where $\bar{g}_{\mu\nu}$ is the FRW background given in Eq. (1) and $\bar{T}_{\mu\nu}$ is the unperturbed stress-energy tensor that takes the form of Eq. (2). One of the complications compared to the non-relativistic case is that among the degrees of freedom introduced above there are unphysical modes that correspond to a mere gauge transformation. In particular the density contrast δ itself is gauge-dependent and thus not a physical observable. For typically adopted gauge choices, however, it reduces to the Newtonian quantity – which was used in Eqs. (49-51) – on scales much smaller than the horizon. In what follows, this is addressed in some more detail.

The perturbed metric can always be brought into the form

$$\begin{aligned} g_{\mu\nu}dx^{\mu\nu} &= -[1+E]dt^2 + a(t)\left[\partial_i F + G_i\right] dt dx^i \\ &\quad + a^2(t)\left[1 + A\delta_{ij} + \partial_i\partial_j B + \partial_i C_j + \partial_j C_i + D_{ij}\right] dx^i dx^j\,, \end{aligned} \tag{57}$$

where the perturbations $A...G$ are in principle arbitrary functions of $\mathbf{x}$ and t that satisfy the following conditions:

$$\partial_i C_i = \partial_i G_i = \partial_i D_{ij} = 0, \quad D_{ii} = 0\,. \tag{58}$$

Here, we are not interested in tensor modes (gravitational waves) described by D_{ij} nor in vector modes (which can be shown to have only decaying solutions and are thus of minor cosmological relevance) described by G_i and C_i, and will thus neglect them in the following. In a similar fashion, $\delta T_{\mu\nu}$ can be de-composed as

$$\begin{aligned} \delta T_{ij} &= \bar{p}\,h_{ij} + a^2\left(\delta_{ij}\delta p + \partial_i\partial_j\pi^S + \partial_i\pi_j^V + \partial_j\pi_i^V + \pi_{ij}^T\right) & (59)\\ \delta T_{i0} &= \bar{p}\,h_{i0} - (\bar{\rho}+\bar{p})\left(\partial_i\delta u + \delta u_i^V\right) & (60)\\ \delta T_{00} &= -\bar{\rho}\,h_{00} + \delta\rho\,, & (61) \end{aligned}$$

where the terms containing a "π" represent dissipative corrections to the inertia tensor. In the above equations, the vector and tensor perturbations are divergence-free and traceless as in Eq. (58); for the same reason as before, they will be neglected in the following discussion.

Under a "small" coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)\,, \tag{62}$$

the metric will be transformed to $g'_{\mu\nu}(x') = g_{\lambda\kappa}(x)\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\kappa}{\partial x'^\nu}$. Correspodingly, this induces the following change in the definition of the metric perturbation:

$$\Delta\delta g_{\mu\nu}(x) = -\bar{g}_{\lambda\mu}(x)\frac{\partial\epsilon^\lambda(x)}{\partial x^\nu} - \bar{g}_{\lambda\nu}(x)\frac{\partial\epsilon^\lambda(x)}{\partial x^\mu} - \epsilon^\lambda(x)\frac{\partial\bar{g}_{\mu\nu}(x)}{\partial x^\lambda}\,. \tag{63}$$

Similarly, such a coordinate transformation induces a change in the definition of $\delta T_{\mu\nu}$, which (after some algebra) can be expressed as

$$\Delta\delta p = \dot{\bar{p}}\epsilon_0, \quad \Delta\delta\rho = \dot{\bar{\rho}}\epsilon_0, \quad \Delta\delta u = -\epsilon_0\,. \tag{64}$$

(The other degrees of freedom of $T_{\mu\nu}$ are gauge-invariant). This means, as mentioned before, that the density contrast δ is gauge-dependent and thus not a physical observable! Popular gauge choices that recover the classical interpretation on sub-horizon scales include

- the *Newtonian gauge*,

  $$B = F = 0, \quad E \equiv 2\Phi, \quad A \equiv -2\Psi\,, \tag{65}$$

  where the potentials Φ and Ψ are not actually physically independent: from the field equations, it follows that for vanishing anisotropic stress ($\pi^S = 0$, like for a perfect fluid) one has $\Psi = \Phi$. The reason for the name of this gauge is that, on scales much smaller than the horizon, Ψ takes the role of the classical Newtonian potential for non-relativistic matter; in particular, the field equations in this limit reproduce the Poisson equation ($\Delta\Psi = 4\pi G\rho$).

- the *synchronous gauge*,

  $$E = F = 0\,, \tag{66}$$

  where the effect of gravitation on the total fluid (but not its individual components) is entirely governed by the quantity $\psi \equiv \partial_t\left[h_{ii}/(2a^2)\right]$. A potential problem of this gauge is that there is a residual gauge invariance left. It can, however, be removed in a natural way if there is a non-relativistic species (like dark matter) for which we can impose $\delta u_\chi = 0$; this completely removes any gauge ambiguities.

- the co-moving or *total matter gauge*,

  $$F = 0 \quad \text{and} \quad \delta u = 0\,, \tag{67}$$

  where one chooses a frame that moves with the cosmological fluid.

A gauge-invariant definition of the *curvature perturbation* is given by $\mathcal{R} \equiv A/2 + H\,\delta u$. In comoving gauge, it can also be introduced as

$$H^2 \equiv \frac{8\pi G}{3}\rho + \frac{2}{3}\nabla^2\mathcal{R}\,, \tag{68}$$

so its physical interpretation is the spatial curvature seen by a comoving observer. Using the field equations, one can show that this quantity stays constant on scales much larger than the horizon, $k \ll aH$, for *adiabatic perturbations* (for which $\delta_i/(\bar{\rho}_i + \bar{p}_i)$ is the same for each fluid component on these scales; these are the ones produced by inflation) and is related to the density contrast by

$$\mathcal{R}_k = \frac{5+3w}{3+3w}\frac{2}{3}\left(\frac{aH}{k}\right)^2 \delta_k \,. \tag{69}$$

Since the numerical value of this quantity at horizon crossing is very close (for standard gauge choices) to the value on much larger scales it follows, in particular, that $\mathcal{P}_\mathcal{R}(k)|_{t=0} \simeq \mathcal{P}_\delta(k)|_{t=t_k} \propto k^{n-1}$ as claimed in Eq. (45). It also follows that before horizon crossing, all density contrasts evolve as[10]

$$\delta_k^i \propto t^2/a^2 \,, \tag{70}$$

as long as the equation of state stays constant. This implies, e.g., $\delta_k^i \propto a$ during matter domination and $\delta_k^i \propto a^2$ during radiation domination.

In order to calculate the evolution of density perturbations after horizon entry, one now proceeds in a way analogous to Eqs. (49-51) by specifying

- a gravitational field equation that follows directly from the Einstein equations (aka a linear combination of the two "Friedmann" equations for the perturbed metric),
- equations for energy conservation from $T^{0\mu}_{;\mu} = 0$ (one for each of the fluid components) and
- equations for momentum conservation from $T^{i\mu}_{;\mu} = 0$ (also one for each fluid component j – recall that we neglected the $\delta\vec{u}_j^V$ and kept only the δu_j).

These equations present a system of coupled differential equations that takes a different form depending on the gauge choice. Presenting the full set of solutions would be way beyond this lecture, so we will just have a look at the most important aspects for our context. We will only discuss growing adiabatic modes. In principle, there is also a decaying adiabatic mode that arises as a solution to the evolution equations which, however, is not important in practice.[11] Finally, there are *isocurvature* modes that describe fluctuations with $\mathcal{R} = 0$; they are not produced in standard models of inflation and observationally disfavored to be the dominant source of perturbations.

During radiation domination – and in fact essentially until recombination – density perturbations in photons, baryons (tightly coupled to the photons) and neutrinos stop to grow well inside the horizon and oscillate as

$$\delta_\gamma \simeq \delta_b \propto \cos kr_s \,, \tag{71}$$

where $r_s \equiv \int_0^t dt\, c_s/a \approx c_s d_H(t)$ is the comoving size of the *sound horizon* and $c_s = \sqrt{dp/d\rho} \approx 1/\sqrt{3}$ the sound speed. Note that the *phase* of the

[10] In synchronous gauge, a similar relation as Eq. (69) holds, albeit with a different normalization. In Newtonian gauge, on the other hand, $\delta \propto \mathcal{R}$ and all δ_i are constant outside the horizon.

[11] The idea that the very early universe was very close to homogeneous would also be in conflict with such a mode that grows when looking back in time.

oscillations is uniquely determined by the requirement of only keeping the growing adiabatic mode. Dark matter perturbations well inside the horizon, on the other hand, grow logarithmically during radiation domination, $\delta_\chi \propto \log t$.

After matter-radiation equality, the cold dark matter component starts to grow as $\delta_\chi \propto a(t) \propto t^{2/3}$ as expected from Eq. (54). At that time, baryons and photons are still tightly coupled and oscillate like before – though with a slightly increasing amplitude as they start to feel the increasing gravitational potential created by dark matter. At recombination, the baryons then decouple and fall into the gravitational wells created by dark matter. This leads to a rapid increase of δ_b and, after a while

$$\delta_b \simeq \delta_\chi \propto a(t)\,. \tag{72}$$

At recombination, the density contrast in baryons (measured by the CMB, see below) is given by $\delta_b \sim 10^{-5}$. Without a dark matter component, it would have grown until today to only $\delta_b(t_0) \sim 10^{-5} z_r \sim 10^{-2}$ – which is still in the linear regime: without DM, no gravitational structures would yet have had time to collapse! As soon as the universe enters into the vacuum energy-dominated regime, finally, the further growth of structures is strongly suppressed and δ_b and δ_χ stay essentially constant.

One more effect to be discussed is the *free streaming* of non-interacting, relativistic particles that can very effectively reduce the density contrast in over-dense regions. To describe this effect, one needs to go beyond the perfect fluid description and employ the full Boltzmann equation. Roughly, however, one can say that the free streaming scale is given by

$$\lambda_{\rm FS} \simeq \int \frac{v}{a}{\rm d}t = \int \frac{v}{aH}{\rm d}a\,, \tag{73}$$

where v is the velocity of the relativistic particles. If these particles contribute significantly to the total density, there will be no clustering for scales $\lambda < \lambda_{\rm FS}$ – which is a very efficient way to constrain any warm or hot DM candidate as well as standard neutrinos.

One may now go ahead and implement the density evolution in the various components in detailed numerical simulations. These turn out to result in perfect agreement with large-scale structure observations – if and only if the main building block of matter is taken to be cold and non-interacting.