Multiplication of Multinomial Subjective Opinions*

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Abstract. Multinomial subjective opinions are a special type of belief functions, where belief mass can be assigned to singletons of the frame as well as to the whole frame, but not to overlapping subsets of the frame. The multiplicative product of two multinomial opinions applies to the Cartesian product of the two corresponding frames. The challenge when multiplying multinomial opinions is that the raw product initially produces belief mass terms on overlapping subsets which does not fit into the opinion requirement of only having belief mass on singletons and on the whole frame. It is therefore necessary to reassign belief mass from overlapping subsets to singletons and to the frame in a way that preserves consistency for multinomial opinions. This paper describes a method for computing multinomial products of opinions according to this principle.

1 Introduction

Arguments in subjective logic are called *"subjective opinions"* or just *"opinions"* for short[1, 2], and are traditionally denoted as ω. A binomial opinion applies to a single proposition/state in a frame. A multinomial opinion applies to the whole frame, i.e. to all the propositions/states in the frame. A binomial opinion is represented by the quadruple consisting of belief mass, disbelief mass, uncertainty mass and base rate, denoted as $\omega = (b, d, u, a)$. A multinomial opinion is represented by the composite function consisting of a belief vector, uncertainty mass and a base rate vector, denoted as $\omega = (\vec{b}, u, \vec{a})$. The uncertainty mass is interpreted as *"uncertainty about probabilities"*, i.e. as the second order complement probability of the first order probability expectation values.

It is relatively straightforward to define operators for subjective opinions that generalize classical binary logic and probabilistic operators. The literature describes a variety of practical operators that provide a basis for modeling and analyzing situations where

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input arguments are incomplete or affected by uncertainty. Binomial and multinomial opinions are equivalent to Beta and Dirichlet probability density functions respectively. Through this equivalence subjective logic provides a calculus for reasoning with probability density functions. In addition to generalizing the set of basic operators traditionally used in binary logic and classical probability calculus, subjective logic also contains some non-traditional operators which are specific to subjective logic.

In this manuscript we describe multinomial opinion multiplication which previously has not been described in the literature. Multinomial products are useful e.g. when combining opinions about different aspects of the same phenomenon or object.

It is straightforward to compute multinomial products in traditional probability calculus, which simply consists of multiplication of the argument probability vectors. In binary logic, the product of two binary frames with their binary truth values produces a quaternary frame with corresponding truth values, which in turn can be multiplied with other frames. The related analytical framework of the Dempster-Shafer belief theory [5] traditionally does not define multiplication of bbas that apply to separate frames, but it would be straightforward to do. The approach would simply be to multiply the belief mass terms of the argument belief functions and assign the product belief masses to the corresponding subsets of the product frame. Multiplication of binomial opinions has also been described in the literature [3].

The challenge with multinomial multiplication in subjective logic is that it initially produces belief mass terms that do not fit into the multinomial representation of $\omega = (\vec{b}, u, \vec{a})$. What is needed therefore is a transformation of the initial product terms into a product opinion that conforms with the required representation. This approach to computing the multinomial opinion product is described below.

2 The Multinomial Opinion Representation

Uncertainty comes in many flavours, and a good taxonomy is described in [6]. In subjective logic, the uncertainty relates to probability values. For example, let the probability estimate of a future event x be expressed as $P(x) = 0.5$, e.g. for obtaining heads when flipping a coin. In subjective logic, the probability P expressed without uncertainty is interpreted as dogmatic and expresses a crisp value, even though the outcome of the event itself might be totally unpredictable. The probability of an event is thus separated from the certainty/uncertainty of its probability. With this separation subjective logic can be applied in case of an event with very likely outcome but where the probability of the outcome still can be totally uncertain. This is possible by including the base rate of an event in the belief representation. For example the *a priori* likelihood that a given person selected at random is immune against tetanus[3] is close to 1, simply due to the base rate of tetanus immunity in the population. However, before actually testing the person, the immunity is still uncertain. The extreme case of an absolutely likely event

[3] Assuming a random person from the population of the developed world

that still has an uncertain probability is theoretically possible but is at the same time a singularity in subjective logic.

A general multinomial opinion is a composite function consisting of a belief vector $\vec{b}$, an uncertainty mass u and a base rate vector $\vec{a}$. These components are defined next.

Definition 1. Belief Mass Vector
Let $X = \{x_i | i = 1, \dots k\}$ *be a frame of cardinality* k *and let* $\vec{b}$ *be a vector function from the singletons of* X *to* $[0,1]^k$ *satisfying:*

$$\vec{b}(\emptyset) = 0 \quad \textit{and} \quad \sum_{x \in X} \vec{b}(x) \leq 1 \; . \tag{1}$$

Then $\vec{b}$ *is called a belief mass vector, or belief vector for short.*

The parameter $\vec{b}(x_i)$ is interpreted as belief mass on x_i, i.e. the amount of positive belief that x_i is true. The belief vector can be interpreted as a sub-additive probability function because the sum can be less than one. Additivity is achieved by including the uncertainty mass defined below.

Definition 2. Uncertainty Mass
Let $X = \{x_i | i = 1, \dots k\}$ *be a frame with a belief vector* $\vec{b}$*. Let* u *be a function from* X *to* $[0,1]$ *representing uncertainty over* X *satisfying:*

$$u + \sum_{x \in X} \vec{b}(x) = 1 \; . \tag{2}$$

The parameter u *is then called an uncertainty mass.*

The uncertainty mass can be interpreted as the lack of committed belief about the truth of any of the propositions of X. In other words, uncertainty mass reflects that the belief owner does not know which of the propositions of X in particular is true, only that one of them must be true.

In case the belief vector is subadditive, i.e. $\sum_{x \in X} \vec{b}(x) < 1$, the base rate vector together with base rates will determine the probability expectation values over X. The base rate vector is defined below.

Definition 3. Base Rate Vector
Let $X = \{x_i | i = 1, \dots k\}$ *be a frame and let* $\vec{a}$ *be a vector function from the singletons of* X *to* $[0,1]^k$ *representing non-informative a priori probability over* X *satisfying:*

$$\vec{a}(\emptyset) = 0 \quad \textit{and} \quad \sum_{x \in X} \vec{a}(x) = 1 \; . \tag{3}$$

Then $\vec{a}$ *is called a base rate vector.*

Having defined the belief vector, the uncertainty mass and the base rate vector, the general opinion can be defined.

Definition 4. Subjective Opinion
Let $X = \{x_i | i = 1, \ldots k\}$ *be a frame, i.e. a set of* k *exhaustive and mutually disjoint propositions* x_i*. Let* $\vec{b}$ *be a belief vector, let* u *be the corresponding uncertainty mass, and let* $\vec{a}$ *be the base rate vector over* X*, all seen from the viewpoint of a subject entity* A*. The composite function* $\omega_X^A = (\vec{b}, u, \vec{a})$ *expresses* A*'s subjective beliefs over* X*. This represents the traditional belief notation of opinions.*

We use the convention that the subscript on the multinomial opinion symbol indicates the frame to which the opinion applies, and that the superscript indicates the subject owner of the opinion so that ω_X^A denotes A's opinion about X. Subscripts can be omitted when it is clear and implicitly assumed to which frame an opinion applies, and superscripts can be omitted when it is irrelevant who the owner is.

Assuming that the frame X has cardinality k, then the belief vector $\vec{b}$ and the base rate vector $\vec{a}$ will have k parameters each. The uncertainty parameter u is a simple scalar. A multinomial opinion over a frame of cardinality k will thus contain $2k + 1$ parameters. However, given the constraints of Eq.(2) and Eq.(3), the multinomial opinion will actually only have $2k - 1$ degrees of freedom. A binomial opinion will for example be 3-dimensional.

The introduction of the base rate vector allows the probabilistic transformation to be independent from the internal structure of the frame. The probability expectation of multinomial opinions is a vector expressed as a function of the belief vector, the uncertainty mass and the base rate vector.

Definition 5. Probability Expectation Vector
Let $X = \{x_i | i = 1, \ldots k\}$ *be a frame and let* ω_X *be an opinion on* X *with belief vector* $\vec{b}$*, uncertainty mass* u*, and base rate vector* $\vec{a}$*. The function* $\vec{\mathrm{E}}_X$ *from the singletons of* X *to* $[0, 1]^k$ *expressed as:*

$$\vec{\mathrm{E}}_X(x_i) = \vec{b}(x_i) + \vec{a}(x_i)u \ . \tag{4}$$

is then called the probability expectation vector over X*.*

It can be shown that $\vec{\mathrm{E}}_X$ satisfies the additivity principle:

$$\vec{\mathrm{E}}_X(\emptyset) = 0 \quad \text{and} \quad \sum_{x \in X} \vec{\mathrm{E}}_X(x) = 1 \ . \tag{5}$$

The base rate vector of Def.3 expresses non-informative *a priori* probability, whereas the probability expectation function of Eq.(4) expresses informed probability estimates, i.e. that are based on evidence which comes in addition to the base rates.

Given a frame of cardinality k, the default base rate of each element in the frame is $1/k$, but it is possible to define arbitrary base rates for all elements of the frame, as long as the additivity constraint of Eq.(3) is satisfied.

Two different multinomial opinions on the same frame will normally share the same base rate vectors. However, it is obvious that two different observers can assign different base rates to the same frame, in addition to assigning different beliefs to the frame. This naturally reflects different views, analysis and interpretations of the same context and situation seen by different observers.

The largest multinomial opinions that can be easily visualized are trinomial, in which case it can be represented as a point inside an equal-sided tetrahedron (pyramid with triangular base), as shown in Fig.1 below.

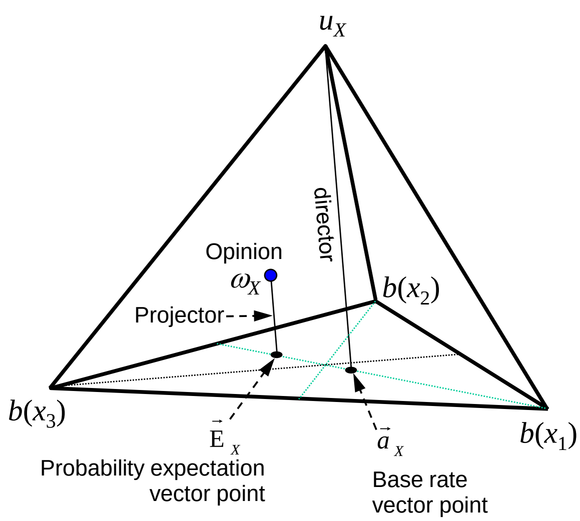

Fig. 1. Opinion tetrahedron with example opinion

In Fig.1 the vertical elevation of the opinion point inside the tetrahedron represents the uncertainty mass. When considering a triangular side plane and the opposite vertex corresponding to a given state x_i, the orthogonal distances from the plane to the opinion point represents the belief mass value on the state x_i. This geometric structure is commonly called a *barycentric* coordinate system, so named by August Ferdinand Möbius (1827). It can be shown that the opinion point is the center of mass when it is assumed that the uncertainty mass and the belief masses are placed on their respective vertices.

The base rate vector $\vec{a}_X$ is indicated as a point on the base plane. The line that joins the tetrahedron apex and the base rate vector point represents the director. The probability expectation vector point is geometrically determined by drawing a projection from the opinion point parallel to the director onto the base plane.

In general, the triangle and tetrahedron belong to the *simplex* family of geometrical shapes. Multinomial opinions on frames of cardinality k can in general be represented as a point in a simplex of dimension $(k + 1)$. For example, binomial opinions can be represented inside a triangle which is a 3D simplex, and trinomial opinions can be represented inside a tetrahedron which is a 4D simplex.

The 2D aspect of paper and computer display units makes it impractical to visualize larger than 4D simplexes, meaning that opinions larger than trinomial do not lend themselves to traditional visualization.

3 Products of Multinomial Opinions

Evaluating the products of two separate multinomial opinions involves the Cartesian product of the respective frames to which the opinions apply. Let ω_X and ω_Y be two independent multinomial opinions that apply to the separate frames

$$X = \{x_1, x_2, \dots x_k\} \;\text{ with cardinality } k \tag{6}$$

$$Y = \{y_1, y_2, \dots y_l\} \;\text{ with cardinality } l\ . \tag{7}$$

The Cartesian product $X \times Y$ with cardinality kl is expressed as the matrix:

$$X \times Y = \begin{pmatrix} (x_1, y_1), & (x_2, y_1), & \cdots & (x_k, y_1) \\ (x_1, y_2), & (x_2, y_2), & \cdots & (x_k, y_2) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (x_1, y_l), & (x_2, y_l), & \cdots & (x_k, y_l) \end{pmatrix} \tag{8}$$

We now turn to the product of the multinomial opinions. The raw terms produced by $\omega_X \cdot \omega_Y$ can be separated into different groups.

1. The first group of terms consists of belief masses on singletons of $X \times Y$:

$$b^{\mathrm{I}}_{X\times Y} = \begin{cases} b_X(x_1)b_Y(y_1), & b_X(x_2)b_Y(y_1), & \dots & b_X(x_k)b_Y(y_1) \\ b_X(x_1)b_Y(y_2), & b_X(x_2)b_Y(y_2), & \dots & b_X(x_k)b_Y(y_2) \\ . & . & \dots & . \\ b_X(x_1)b_Y(y_l), & b_X(x_2)b_Y(y_l), & \dots & b_X(x_k)b_Y(y_l) \end{cases} \tag{9}$$

2. The second group of terms consists of belief masses on rows of $X \times Y$:

$$b^{\mathrm{Rows}}_{X\times Y} = \big(u_X b_Y(y_1), \quad u_X b_Y(y_2), \quad \dots \quad u_X b_Y(y_l) \big) \tag{10}$$

3. The third group of terms consists of belief masses on columns of $X \times Y$:

$$b^{\mathrm{Columns}}_{X\times Y} = \big(b_X(x_1) u_Y, \quad b_X(x_2) u_Y, \quad \dots \quad b_X(x_k) u_Y \big) \tag{11}$$

4. The last term is simply the belief mass on the whole product frame:

$$u^{\mathrm{Frame}}_{X\times Y} = u_X u_Y \tag{12}$$

The singleton terms of Eq.(9) and the term on the whole frame are unproblematic because they conform with the opinion representation of having belief mass only on singletons and on the whole frame. In contrast, the terms on rows and columns apply to overlapping subsets which is not compatible with the required opinion format, and therefore need to be reassigned. Some of it can be reassigned to singletons, and some to the whole frame. There are several possible strategies for determining the amount of uncertainty mass to be assigned to singletons and to the frame. Two methods are described below.

3.1 Determining Uncertainty Mass

1. **The Method of Assumed Belief Mass:** The simplest method is to assign the belief mass from the terms of Eq.(10) and Eq.(11) to singletons. Only the uncertainty mass from Eq.(12) is then considered as uncertainty in the product opinion, expressed as:

$$u_{X \times Y} = u_X u_Y \; . \tag{13}$$

   A problem with this approach is that it in general produces less uncertainty than intuition would dictate.
2. **The Method of Assumed Uncertainty Mass:** A method that preserves more uncertainty is to consider the belief mass from Eq.(10) and Eq.(11) as potential uncertainty mass that together with the uncertainty mass from Eq.(12) can be called intermediate uncertainty mass. The intermediate uncertainty mass is thus:

$$u^{\mathrm{I}}_{X \times Y} = u^{\mathrm{Rows}}_{X \times Y} + u^{\mathrm{Columns}}_{X \times Y} + u^{\mathrm{Frame}}_{X \times Y} \tag{14}$$

   The probability expectation values of each singleton in the product frame can easily be computed as the product of the expectation values of each pair of states from X and Y, as expressed in Eq.(15).

$$\begin{aligned} \mathrm{E}((x_i, y_j)) &= \mathrm{E}(x_i)\mathrm{E}(y_j) \\ &= (b_X(x_i) + a_X(x_i)u_X)(b_Y(y_j) + a_Y(y_j)u_Y) \end{aligned} \tag{15}$$

   We also require that the probability expectation values of the states in the product frame can be computed as a function of the product opinion according to Eq.(16).

$$\mathrm{E}((x_i, y_j)) = b_{X \times Y}((x_i, y_j)) + a_X(x_i) a_Y(y_j) u_{X \times Y} \tag{16}$$

   In order to find the correct uncertainty mass for the product opinion, each state $(x_i, y_j) \in X \times Y$ will be investigated in turn to find the smallest uncertainty mass that satisfies both Eq.(16) and Eq.(17).

$$\frac{b^{\mathrm{I}}_{X \times Y}((x_i, y_j))}{u^{\mathrm{I}}_{X \times Y}} = \frac{b_{X \times Y}((x_i, y_j))}{u_{X \times Y}} \tag{17}$$

The uncertainty mass that satisfies both Eq.(16) and Eq.(17) for state (x_i, y_j) can be expressed as:

$$u^{(i,j)}_{X\times Y} = \frac{u^{\mathrm{I}}_{X\times Y}\,\mathrm{E}((x_i, y_j))}{b^{\mathrm{I}}_{X\times Y}((x_i, y_j)) + a_X(x_i)a_Y(y_j)u^{\mathrm{I}}_{X\times Y}} \tag{18}$$

The product uncertainty can now be determined as the smallest $u^{(i,j)}_{X\times Y}$ among all the states, expressed as:

$$u_{X\times Y} = \min\left\{u^{(i,j)}_{X\times Y} \ \text{ where } \ (x_i, y_j) \in X\times Y\right\} \tag{19}$$

3.2 Determining Belief Mass

Having determined the uncertainty mass, either according to Eq.(13) or according to Eq.(19), the expression for the product expectation of Eq.(15) can be used to compute the belief mass on each element in the product frame, as expressed by Eq.(20).

$$b_{X\times Y}((x_i, y_j)) = \mathrm{E}((x_i, y_j)) - a_X(x_i)a_Y(y_j)u_{X\times Y} \tag{20}$$

It can be shown that the additivity property of Eq.(21) is preserved.

$$u_{X\times Y} + \sum_{(x_i, y_j)\in X\times Y} b_{X\times Y}((x_i, y_j)) = 1 \tag{21}$$

From Eq.(20) it follows directly that the product operator is commutative. It can also be shown that the product operator is associative.

4 Example

We consider the scenario where a GE (Genetic Engineering) process can produce Male (M) or Female (F) eggs, and that in addition, each egg can have genetical mutation S or T independently of its gender. This constitutes two binary frames $X = \{\mathrm{M}, \mathrm{F}\}$ and $Y = \{\mathrm{S}, \mathrm{T}\}$, or alternatively the quaternary product frame $X\times Y = \{\mathrm{MS}, \mathrm{MT}, \mathrm{FS}, \mathrm{FT}\}$. Sensor A observes whether each egg is M or F, and Sensor B observes whether the egg has mutation S or T.

Assume that an opinion regarding the gender of a specific egg is derived from Sensor A data, and that an opinion regarding its mutation is derived from Sensor B data. Sensors A and Sensor B have thus observed different and orthogonal aspects, so their respective opinions can be combined with multiplication. This is illustrated in Fig.2.

The result of the opinion multiplication can be considered as an opinion based on a single observation where both aspects are observed at the same time. Let the observation opinions be:

$$\text{Gender } \ \omega^A_X : \begin{cases} \vec{b}^A_X = (0.8, 0.1) \\ u^A_X = 0.1 \\ \vec{a}^A_X = (0.5, 0.5) \end{cases} \qquad \text{Mutation } \ \omega^B_Y : \begin{cases} \vec{b}^B_Y = (0.7, 0.1) \\ u^B_Y = 0.2 \\ \vec{a}^B_Y = (0.2, 0.8) \end{cases} \tag{22}$$

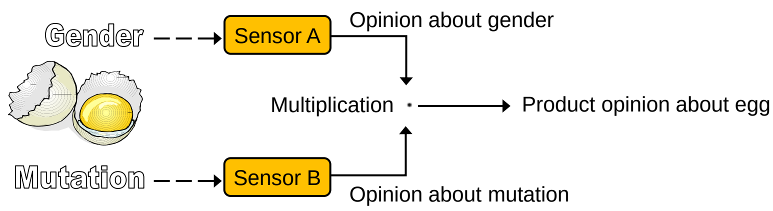

Fig. 2. Multiplication of opinions on orthogonal aspects of GE eggs

The Cartesian product frame can be expressed as:

$$X \times Y = \begin{pmatrix} \mathrm{MS}, & \mathrm{FS} \\ \mathrm{MT}, & \mathrm{FT} \end{pmatrix} \tag{23}$$

According to Eq.(15) the product expectation values are:

$$\mathrm{E}(X \times Y) = \begin{pmatrix} 0.629, & 0.111 \\ 0.221, & 0.039 \end{pmatrix} \tag{24}$$

Below are described the results of both methods proposed in Sec.3.1.

1. When applying the method of *Assumed Belief Mass* where the uncertainty mass is determined according to Eq.(13), the product opinion is computed as:

$$b_{X \times Y} = \begin{pmatrix} 0.627, & 0.109 \\ 0.213, & 0.031 \end{pmatrix}, \quad u_{X \times Y} = 0.02, \quad a_{X \times Y} = \begin{pmatrix} 0.1, & 0.4 \\ 0.1, & 0.4 \end{pmatrix} \tag{25}$$

2. When applying the method of *Assumed Uncertainty* where the uncertainty mass is determined according to Eq.(18) and Eq.(19), the product opinion is computed as:

$$b_{X \times Y} = \begin{pmatrix} 0.620, & 0.102 \\ 0.185, & 0.003 \end{pmatrix}, \quad u_{X \times Y} = 0.09, \quad a_{X \times Y} = \begin{pmatrix} 0.1, & 0.4 \\ 0.1, & 0.4 \end{pmatrix} \tag{26}$$

The results indicate that there can be a significant difference between the two methods, and that the safest approach is to use the *assumed uncertainty* method because it preserves the most uncertainty in the product opinion.

5 Discussion and Conclusion

Multiplication of multinomial opinions is useful in many situations, such as when combining input from sensors that observe different aspects of a target. Two methods for computing the product of multinomial opinions are presented in this paper, where the method of assumed uncertainty is recommended because it preserves the most uncertainty and thereby better reflects the uncertainty of the input arguments.

Subjective opinions are related to general bbas. One of the differences is that a bba can assign belief mass to any subset of a frame, whereas an opinion can only assign belief mass to singletons and to the whole frame. The other difference is that bbas do not include base rates, whereas opinions do. Consequently opinions represent both a subset of, and an extension of general bbas.

Opinions can be derived from bbas if it can be assumed that base rates can be defined separately [4]. It is thus possible to use general bbas as input to subjective logic models in general and to multiplication of opinions in particular.

The advantage of subjective logic over traditional probability calculus and probabilistic logic is that real world situations can be modeled and analyzed more realistically. The analyst's partial ignorance and lack of information can be taken explicitly into account during the analysis, and explicitly expressed in the conclusion. When used for decision support, subjective logic allows decision makers to be better informed about uncertainties affecting the assessment of specific situations. At the same time subjective logic is compatible with traditional statistical analysis.

While the belief representation of opinions is less flexible than that of general bbas, it has the advantage that traditional statistical analysis can be directly applied and that the set of operators such as conditional deduction and abduction can be used for modeling Bayesian networks and the transitivity operator can be used to model trust networks.

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