

# Base Rates for Belief Functions <sup>0</sup>

Audun Jøsang  
University of Oslo  
Norway  
Email: josang@mn.uio.no

Stephen O'Hara  
21st Century Systems  
USA  
Email: sohara@21csi.com

Kristi O'Grady  
Cobham  
USA  
Email: kristi.ogradey@cobham.com

**Abstract**—The concept of base rates is central in the theory of probability. Base rates are for example useful for default and for conditional reasoning. Traditional belief theory does not specify base rates. The strength of belief theory is the principle of sub-additivity, meaning that the sum of belief masses on disjoint elements in a frame can be less than one. Without base rates however, there are many situations where belief theory does not provide an adequate model for expressing beliefs. This paper specifies base rates for belief functions and shows how it can be used for probability and Dirichlet projections.

**Keywords:** Base Rates, Probability, Dirichlet.

## I. INTRODUCTION

Probability representations and belief representations both apply to state spaces. The term used for a state space in belief theory is normally *frame of discernment* which we will simply call *frame* here. From this common basis it is interesting to notice the main difference between the probability and belief representations.

Belief theory abandons the additivity principle of classical probability theory which requires that the sum of probability values of mutually disjoint subsets in a frame always equals 1. This gives belief representations certain advantages over traditional probability representations with regard to representing ignorance and uncertainty. For example, let “*I don't know*” be a subject's answer to the question “*Which of  $x$ ,  $y$  or  $z$  is true?*”. The closest representation of that answer in probabilistic terms would be  $p(x) = p(y) = p(z) = \frac{1}{3}$  which could be interpreted as “*I believe that either  $x$ ,  $y$  or  $z$  is true, and that their likelihoods of being true are equal*”. However, the answer “*I don't know*” can be much more elegantly and concisely expressed with the belief function  $m(x \cup y \cup z) = 1$ , meaning that the total belief mass is assigned to the union of  $x$ ,  $y$  and  $z$ , which could be interpreted as: “*I believe that either  $x$ ,  $y$  or  $z$  is true, but I don't know which of them*”. In general, belief functions allow a rich set of belief structures to be expressed, e.g. with belief mass assigned to partly overlapping subsets, which has no equivalent in traditional probability representation. This expressiveness is the main advantage of belief theory.

However, base rates have traditionally not been defined for belief functions whereas base rates are often defined for probabilistic models. The concept of base rates, which normally is interpreted as the relative frequency of occurrence

of a phenomenon in a population, is important in order to model many practical situations. For example, the base rates of various diseases are commonly known for specific populations, which is an important when medical practitioners are making diagnoses.

This paper defines base rates for belief functions and describes how base rates can be used for projecting general belief functions onto probabilities and onto Dirichlet belief distributions. This is useful when general beliefs are used as input arguments in traditional statistical analysis or in subjective logic.

## II. BELIEF REPRESENTATIONS

Belief theory has its origin in a model for upper and lower probabilities proposed by Dempster in 1960. Shafer later proposed a model for expressing beliefs [1]. The main idea behind belief theory is to abandon the additivity principle of probability theory, i.e. the principle that the sum of probabilities on all pairwise disjoint states must add up to one. The main advantage of this approach is that ignorance, i.e. the lack of information, can be explicitly expressed e.g. by assigning belief mass to (partially) overlapping subsets or to the whole frame.

Classical belief representation is very general, and allows complex belief structures to be expressed on arbitrarily large frames. Shafer's book [1] describes many aspects of belief theory, but the two main elements are: 1) a flexible way of expressing beliefs, and 2) a method for conjunctive fusion of beliefs, commonly known as Dempster's Rule. We will not be concerned with Dempster's rule here.

### A. Fundamental Belief Representation Concepts

In order to introduce the notation and to make this presentation self contained, central concepts from the Dempster-Shafer theory of evidence [1] are recalled. Let  $X = \{x_i, i = 1, \dots, k\}$  where  $k \geq 2$  denote a finite set of exhaustive and disjoint possible values for a state variable of interest. This is usually called a *frame of discernment*, and means the same as a *state space* in traditional set theory terminology. The frame can for example be the set of six possible outcomes of throwing a dice, where the (unknown) outcome of a particular instance of throwing the dice becomes the state variable. The powerset of  $X$  is defined as  $2^X = \{x'_i \subseteq X | i = 1, \dots, (2^k - 1)\}$ , meaning that the elements of  $2^X$  are the subsets of  $X$ . We let the first  $k$  elements of  $2^X$  be equal to the corresponding  $k$  singleton

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subsets from  $X$ , expressed as  $x'_i = \{x_i\}$  for  $1 \leq i \leq k$ , which with a slight abuse of notation also can be written as  $x'_i = x_i$ .

A bba (basic belief assignment<sup>1</sup>) on  $X$  denoted by  $m_X$  is defined as a belief mass distribution function  $m_X : 2^X \rightarrow [0, 1]$  satisfying:

$$m_X(\emptyset) = 0 \quad \text{and} \quad \sum_{x'_i \subseteq X} m_X(x'_i) = 1. \quad (1)$$

Values of a bba are called *belief masses*. Each subset  $x'_i \subseteq X$  such that  $m_X(x'_i) > 0$  is called a focal element of  $m_X$ .

From a bba  $m_X$  can be derived a set of non-additive belief functions  $\text{Bel} : 2^X \rightarrow [0, 1]$ , defined as

$$\text{Bel}(x'_i) \triangleq \sum_{\emptyset \neq x'_j \subseteq x'_i} m_X(x'_j) \quad \forall x'_i \subseteq X. \quad (2)$$

The quantity  $\text{Bel}(x'_i)$  can be interpreted as a measure of the total of one's belief committed to the hypothesis that  $x'_i$  is true. Note that the functions  $m_X$  and  $\text{Bel}$  are in one-to-one correspondence [1] and can be seen as two facets of the same piece of information.

A general bba  $m_X$  can be projected to a scalar probability value through the probability projection [3], also known as the pignistic transformation [4], [5], denoted by  $\wp(x'_i)$ , defined as:

$$\wp(x'_i) = \sum_{x'_j \subseteq X} m_X(x'_j) \frac{|x'_i \cap x'_j|}{|x'_j|}, \quad \forall x'_i \in 2^X. \quad (3)$$

By using Eq.(3), it is possible to derive a probability expectation value for every subset of the frame as a function of the bba. Of course, this process is affected by information loss, so it would not be possible to recover the original bba from the set of probability expectation values produced through the projection.

A few special classes of bba can be mentioned. A vacuous bba has  $m_X(X) = 1$ , i.e. no belief mass committed to any proper subset of  $X$ . A *Bayesian* bba is when all the focal elements are singletons, i.e. one-element subsets of  $X$ . If all the focal elements are nestable (i.e. linearly ordered by inclusion) then we speak about *consonant* bba. A *dogmatic* bba is defined as a bba for which  $m_X(X) = 0$  [6]. Let us note, that trivially, every Bayesian bba is dogmatic.

### B. Base Rate Augmented Belief Functions

The traditional probability projection of belief functions defined by Eq.(3) assumes a default subset base rate that is equal to the subset's relative atomicity. In other words, the default base rate of a subset is equal to the relative number of singletons in the subset with respect to the total number of singletons in the whole frame. Subsets also have default *relative base rates* with respect to every other fully or partly overlapping subset of the frame. Thus, when projecting a bba to scalar probability values, Eq.(3) dictates that belief

masses on subsets contribute to the projected probabilities as a function of the default base rates on those subsets.

However, in practical situations it would be possible and useful to apply base rates that are different from the default base rates. For example, when considering the base rate of a particular infectious disease in a specific population, the frame can be defined as  $\{\text{"infected"}, \text{"not infected"}\}$ . Assuming that an unknown person enters a medical clinic, the physician would *a priori* be ignorant about whether that person is infected or not before having assessed any evidence. This ignorance should intuitively be expressed as a vacuous belief function, i.e. with the total belief mass assigned to ( $\text{"infected"} \cup \text{"not infected"}$ ). The probability projection of a vacuous belief function using Eq.(3) would dictate that the *a priori* probability of having the disease is 0.5. Of course, the base rate of diseases is normally much lower, and can be determined by relevant statistics from a given population. Traditional belief functions are not well suited for representing this situation. Using only traditional belief functions, the base rate of a disease would have to be expressed through a bba that assigns some belief mass to either *"infected"* or *"not infected"* or both. Then after assessing the results of e.g. a medical test, the bba would have to be conditionally updated to reflect the test evidence in order to derive the *a posteriori* bba. Unfortunately, no computational method for conditional updating of traditional bbas according to this principle exists. The methods that have been proposed, e.g. [7], have been shown to be flawed [8] because they are subject to the base rate fallacy [9]. Base rates for belief functions represents a necessary prerequisite for conditional belief reasoning, in particular for abductive belief reasoning. This paper only focuses on describing base rates, and leaves the investigation of conditional belief reasoning for future research.

In order to have a better and more intuitive probability projection from belief functions in general and from vacuous belief functions in particular we propose to augment traditional bbas with a base rate function. When taking a bba combined with a base rate function and then projecting it to probabilities, the share of contributing belief mass from subsets of the frame will be a function of this base rate function. Let  $a$  denote the base rate function so that  $a(x_i)$  represents the base rate of the elements  $x_i \in X$ . The base rate function is formally defined below.

*Definition 1 (Base Rate Function):* Let  $X = \{x_i | i = 1, \dots, k\}$  be a frame, and let  $a_X$  be the function from  $X$  to  $[0, 1]^k$  satisfying:

$$a_X(\emptyset) = 0, \quad a_X(x_i) \in [0, 1] \quad \text{and} \quad \sum_{i=1}^k a_X(x_i) = 1. \quad (4)$$

Then  $a_X$  is called a *base rate function*.

Events that can be repeated many times are typically frequentist in nature, meaning that the base rates for these often can be derived from statistical observations. For events that can only happen once, the analyst must often extract base rates from subjective intuition or from analyzing the

<sup>1</sup>Called *basic probability assignment* in [1], and *Belief Mass Assignment* (BMA) in [2].

nature of the phenomenon at hand and any other relevant evidence. However, in many cases this can lead to considerable uncertainty about the base rate, and when nothing else is known, the default base rate of the singletons in a frame must be defined to be equally partitioned between them. More specifically, when there are  $k$  singletons in the frame, the default base rate of each element is  $1/k$ .

The difference between the concepts of subjective and frequentist probabilities is that the former can be defined as subjective betting odds – and the latter as the relative frequency of empirically observed data, where the subjective probability normally converges toward the frequentist probability in the case where empirical data is available [10]. The concepts of *subjective* and *empirical* base rates can be interpreted in a similar manner where they also converge and merge into a single base rate when empirical data about the population in question is available.

It is possible to define base rate functions that assign specific base rates to the elements of the frame, and that in general are different from the default base rate of  $1/k$  for for each singleton.

The intended meaningful interpretation of the base rate function emerges from its application as the basis for probability projection of bbas. Because a bba can be specified to assign belief mass to any subset of the frame, i.e. also to subsets larger than singletons, it is necessary to also represent the base rates of such subsets. This is defined below.

*Definition 2 (Subset Base Rates):* Let  $X = \{x_i | i = 1, \dots, k\}$  be a frame, and let  $2^X = \{x'_i \subseteq X | i = 1, \dots, (2^k - 1)\}$  be the powerset of  $X$ , with  $x'_i = x_i$  for  $1 \leq i \leq k$ . Assume that a base rate function  $a_X$  is defined over  $X$  according to Def.1. Then the base rates of the elements of the powerset  $2^X$  are expressed according to the base rate function  $a'$  defined below.

$$a_{(2^X)}(\emptyset) = 0 \quad \text{and} \quad a_{(2^X)}(x'_i) = \sum_{x_j \subseteq x'_i} a_X(x_j). \quad (5)$$

Trivially, it can be seen that  $a_{(2^X)}(x'_i) = a_X(x_i)$  for  $1 \leq i \leq k$ , i.e. that elements in  $2^X$  that correspond to singletons in  $X$  have a base rate equal to that of their corresponding singleton subsets. Because of this strong correspondence between  $a_X$  and  $a_{(2^X)}$  we will simply denote both base rate functions as  $a_X$ .

Equipped with the subset base rate function of Def.2 the expression for probability projection of bbas as a function of base rates can be defined.

*Definition 3 (Probability Projection with Base Rates):* Let  $X = \{x_i | i = 1, \dots, k\}$  be a frame. Assume that  $m_X$  is a bba on  $X$  and that  $a_X$  is a base rate function on  $X$ , then the probability projection of  $m_X$  as a function of  $a_X$  is expressed according to Eq.(6) below.

$$\wp'(x'_i) = \sum_{x'_j \subseteq X} m_X(x'_j) \frac{a_X(x'_i \cap x'_j)}{a_X(x'_j)}, \quad \forall x'_i \in 2^X. \quad (6)$$

Trivially, when  $a_X(x_i) = \frac{1}{k}, \forall x_i \in X$ , i.e. when the base rate function  $a_X$  simply expresses the default base rates, it can be seen that  $\wp'(x'_i) = \wp(x'_i)$  from Eq.(3).

The base rate function has the same syntactic constraints as a traditional probability function, such as additivity, i.e. the sum of base rates over all mutually exclusive subsets equals one.

### C. Example: Base Rates of Diseases

The base rate of diseases within a community can be estimated. Typically, data is collected from hospitals, clinics and other sources where people diagnosed with the disease are treated. The amount of data that is required to calculate the base rate of the disease will be determined by some departmental guidelines, statistical analysis, and expert opinion about the data that it is truly reflective of the actual number of infections – which is itself a subjective assessment. After the guidelines, analysis and opinion are all satisfied, the base rate will be determined from the data, and can then be used with medical tests to provide a better indication of the likelihood of specific patients having contracted the disease [11].

### D. The Dirichlet bba

We define a special type of bbas called *Dirichlet bba* which are equivalent to opinions used in subjective logic. Dirichlet bbas are characterized by having only mutually disjoint focal elements, except the whole frame  $X$  itself. This is defined as follows.

*Definition 4 (Dirichlet bba):* Let  $X$  be a frame. A bba  $m_X^D$  where the only focal elements are  $X$  and/or mutually exclusive subsets of  $X$  (singletons or sets of singletons), is called a *Dirichlet belief mass distribution function*, or *Dirichlet bba* for short.

The probability projection of Dirichlet bbas is particularly simple and can be expressed as a function of the bba and the base rate function.

$$\wp(x_i) = m_X^D(x_i) + a_X(x_i)m_X^D(X) \quad (7)$$

where  $m_X^D$  denotes a Dirichlet bba on  $X$ .

The number of singletons in a frame  $X$  is the same as its cardinality  $|X|$ . The number of focal elements of a Dirichlet bba can be at most  $|X| + 1$ , which happens when every singleton as well as the whole frame is a focal element.

The name "Dirichlet" bba is used because bbas of this type can be interpreted as equivalent to Dirichlet probability density functions under a specific mapping. A bijective mapping between Dirichlet bbas and Dirichlet probability density functions is defined in [12].

### E. Comparison to the Imprecise Dirichlet Model

The Imprecise Dirichlet Model (IDM) for multinomial data is described by Walley [13] as a method for determining upper and lower probabilities. The model is based on varying the base rate over all possible outcomes. The probability expectation value of an outcome resulting from assigning the total base rate (i.e. equal to one) to that outcome produces

the upper probability, and the probability expectation value of an outcome resulting from assigning a zero base rate to that outcome produces the lower probability. The upper and lower probabilities are interpreted as the upper and lower bounds for the relative frequency of the outcome. While this is an interesting interpretation of the Dirichlet distribution, it can not be taken literally, as shown below.

According to the Imprecise Dirichlet Model (IDM) [13] the upper and lower probability values for an outcome  $x_i$  are defined as:

$$\text{IDM Upper probability: } \bar{P}(x_i) = \frac{r(x_i) + W}{W + \sum_{i=1}^k r(x_i)} \quad (8)$$

$$\text{IDM Lower probability: } \underline{P}(x_i) = \frac{r(x_i)}{W + \sum_{i=1}^k r(x_i)} \quad (9)$$

where  $r(x_i)$  denotes the amount of evidence for outcome  $x_i$  and  $W$  denotes the prior non-informative weight of the Dirichlet distribution. It is normal to set  $W = 2$  although other values for the non-informative prior weight are possible.

It can easily be shown that the IDM Upper and IDM Lower values can not be literally interpreted as upper and lower bounds for the probability. For example, assume a bag contains 9 red marbles and 1 black marble, meaning that the relative frequencies of red and black marbles are  $p(\text{red}) = 0.9$  and  $p(\text{black}) = 0.1$ . The *a priori* weight is set to  $W = 2$ . Assume further that an observer picks one marble which turns out to be black. According to Eq.(9) the lower probability is then  $\underline{P}(\text{black}) = \frac{1}{3}$ . It would be incorrect to literally interpret this value as the lower bound for the probability because it obviously is greater than the actual relative frequency of black balls. In other words, if  $\underline{P}(\text{black}) > p(\text{black})$  then  $\underline{P}(\text{black})$  can impossibly be the lower bound. This case shows that the upper and lower probabilities defined by the IDM should be interpreted as a rough probability interval, because that would allow actual relative frequencies can be outside the range.

Utkin (2005) [14] defines a method for deriving beliefs and plausibilities based on the IDM, where the lower probability is interpreted as the belief and the upper probability is interpreted as the plausibility. This method can produce unreasonable results in practical applications, and Utkin provides extensions to the Imprecise Dirichlet Model to overcome some of these problems. In our view the belief and plausibility functions can not be based on the base rate uncertainty of the Dirichlet distributions. The base rates are determined by the structure of the state space when it is known, and must be estimated on a subjective basis when not known [6]. In belief theory, the state space structure is used when e.g. computing the probability projection, but it is independent of the bba.

### III. PROJECTING GENERAL BBAS TO DIRICHLET BBAS

Let  $X$  be a frame over which a general bba  $m_X$  and a base rate function  $a_X$  are specified. We will define a projection from  $m_X$  to a Dirichlet bba  $m_X^D$  which preserves the probability expectation value of every subset of  $X$

In case the original bba  $m_X$  already is a Dirichlet bba, no projection is needed. Def.(5) provides a test for determining whether the original bba is a Dirichlet bba or not.

*Definition 5 (Dirichlet bba Test):* Let  $m_X$  be a bba defined over the frame  $X$ , and let  $\widehat{X} \subset 2^X$  denote the set of focal elements of  $m_X$ , not including the frame  $X$  itself. Then  $m_X$  is a Dirichlet bba if the following is true.

$$x'_i \neq x'_j \rightarrow x'_i \cap x'_j = \emptyset, \forall x'_i, x'_j \in \widehat{X}$$

where  $\rightarrow$  denotes material implication.

Note that the case  $(x'_i = x'_j) \wedge (x'_i \cap x'_j = \emptyset)$  does not apply.

Assuming that a bba is **not** a Dirichlet bba, then it can be projected onto a Dirichlet bba. We require that the projection preserve the probability expectation values of each subset in the frame, and also that it preserve the bba's degree of uncertainty as will be explained below. The degree of uncertainty is here interpreted as inversely proportional to dogmatism, i.e. the larger the focal elements and the more belief mass assigned to them, the greater the degree of uncertainty.

The powerset  $2^X$  of a frame  $X$  of  $n$  singletons has  $2^n - 1$  elements, i.e. each subset in  $X$  is an element in  $2^X$ . Subsets of  $X$  contains an integer number of singletons in the range  $[1, n]$ . Trivially there are exactly  $n$  subsets that are themselves singletons, and exactly 1 element which contains  $n$  singletons and which therefore is the whole frame itself.

We are interested in the total belief mass assigned to subsets of specific sizes. A bba with belief mass assigned mainly to large subsets contains more uncertainty than a bba with belief mass assigned mainly to small subsets. This observation forms the basis for defining *relative uncertainty weights* to specific subset sizes, where the weight 1 is assigned to the whole frame and the weight 0 is assigned to singletons. The relative uncertainty weight of intermediate subset sizes is evenly distributed according to the definition below.

*Definition 6 (Relative Uncertainty Weights):* Let  $X$  be a frame of size  $k$ . The relative uncertainty weights  $U(x'_i)$  of subsets  $x'_i \subseteq X$  are defined according to:

$$\begin{cases} U(x'_i) = 0 & \text{for } 1 \leq i \leq k \\ U(x'_i) = a_X(x'_i) & \text{for } k < i \leq (2^k - 1) \\ U(X) = 1 \end{cases} \quad (10)$$

Def.6 is ad hoc and is based on the following rationale. The uncertainty weight is a function of the prior base rate. More precisely, belief mass assigned to subsets with relatively high base rate signifies relatively high uncertainty, whereas belief mass assigned to subsets with relatively low base rate signifies relatively low uncertainty. The actual number of states in a frame is irrelevant.

The relative uncertainty of a bba can be defined as a function of the bba and the distribution of uncertainty weights over the elements of the powerset.

*Definition 7 (Degree of Uncertainty):* Let  $X$  be a frame of size  $k$ , and let  $m_X$  be a bba on  $X$ . The degree of uncertainty  $u(m_X)$  of  $m_X$  is defined according to:

$$u(m_X) = \sum_{x'_i \subseteq X} m_X(x'_i) U(x'_i) \quad (11)$$

Equipped with the above definitions it is possible to define a projection from general bbas to Dirichlet bbas. The Dirichlet projection is done in three steps. First produce a trivial Bayesian bba  $m_X^P$ . Secondly, the Bayesian bba is uncertainty-maximized as  $\hat{m}_X$  to identify the maximum uncertainty that the projected bba can contain. Finally, the uncertainty  $m_X^D(X)$  is set to best match the degree uncertainty of Eq.(11).

- 1) The probability projection according to Eq.(6) can be used as a basis for a trivial dogmatic Dirichlet bba, i.e. with zero belief mass assigned to the frame itself. This is done by assigning the projected probability of each singleton as belief mass to that singleton. The resulting bba, denoted by  $m_X^P$  is a dogmatic Dirichlet bba, which is also a Bayesian bba.
- 2) Uncertainty maximization of  $m_X^P$  consists of converting as much belief mass as possible into uncertainty mass, i.e. belief mass on  $X$ , while preserving consistent probability expectation values according to Eq.(6). The result is the uncertainty maximized bba denoted as  $\hat{m}_X$ . The equation

$$\wp(x_i) = m_X(x_i) + a_X(x_i)m_X(X), \quad \forall x_i \in X \quad (12)$$

is by definition satisfied for  $m_X^P(x_i)$  where  $m_X^P(X) = 0$ , and we require that  $\hat{m}_X(x_i)$  also satisfies Eq.(12). In general Eq.(12) defines the bbas  $m_X$  that project to the same Bayesian bba as that of Eq.(6). The particular bba  $\hat{m}_X(x_i)$  is uncertainty-maximized when Eq.(12) is satisfied and at least one belief mass of  $\hat{m}_X$  is zero. In general, not all belief masses can be zero simultaneously; that is only possible for vacuous bbas. In order to find the state(s) that can have zero belief mass, the belief mass will be set to zero in Eq.(12) successively for each dimension  $x_i \in X$ , resulting in  $k$  different uncertainty values defined as:

$$m_X^i(X) = \frac{\wp(x_i)}{a_X(x_i)}, \quad \text{where } i = 1 \dots k. \quad (13)$$

The smallest uncertainty for which one of the belief masses is zero, expressed as

$$m_X^t(X) = \min\{m_X^i(X), \text{ for } i = 1 \dots k\} \quad (14)$$

determines the state for which the belief mass is zero, and all other belief masses are either zero or positive. The reason why the state that results in the smallest uncertainty must be chosen is that setting the belief mass to zero for any other state could result in negative belief mass for other states. Assume that  $x_t$  is the state for which the resulting uncertainty is smallest. The uncertainty maximized bba can then be determined as:

$$\hat{m}_X : \begin{cases} \hat{m}_X(x_i) &= \wp(x_i) - a_X(x_i)m_X^t(X), \\ \hat{m}_X(X) &= m_X^t(X) \end{cases} \quad (15)$$

- 3) If the degree of uncertainty from Def.7 is less or equal to that of the uncertainty-maximized bba, then it is possible to adjust the projected bba to exactly match the degree

of uncertainty. If that is not the case, the uncertainty-maximized bba from Eq.(15) is the best match. The Dirichlet projected bba is then computed as:

$$\begin{aligned} &\text{IF } \hat{m}_X(X) \geq u(m_X) \\ &\text{THEN} \\ &\quad m_X^D : \begin{cases} m_X^D(x_i) &= \wp(x_i) - a_X(x_i)u(m_X), \\ m_X^D(X) &= u(m_X) \end{cases} \\ &\text{ELSE} \\ &\quad m_X^D = \hat{m}_X \\ &\text{ENDIF} \end{aligned} \quad (16)$$

This projection of general bbas into Dirichlet bbas is a useful tool for subjective logic practitioners. When a bba has been converted into a Dirichlet bba, then it is equivalent to a multinomial subjective opinion [12]. One can thus use this method to convert any Dempster-Shafer basic belief assignments into subjective opinions.

#### IV. EXAMPLE

Below is a simple example to illustrate how base rates can be used for probabilistic and Dirichlet projection.

A submarine has been detected entering a strait, and a naval analyst wants to express belief in what type submarine might be. Assuming that there are mainly 4 possible types, and many other relatively unlikely types, the frame can be defined as  $X = \{x_1, x_2, x_3, x_4, x_5\}$ , where  $x_5$  denotes other ‘‘types’’. The base rate  $a_X$  specified below defines the average occurrence rates of each submarine type passing through that strait.

$$a_X : \begin{cases} a_X(x_1) &= 0.25 \\ a_X(x_2) &= 0.20 \\ a_X(x_3) &= 0.15 \\ a_X(x_4) &= 0.35 \\ a_X(x_5) &= 0.05 \end{cases} \quad (17)$$

Based on intelligence reports the analyst defines the bba as expressed in Eq.(18) below. The belief assignment is also depicted in Figure 1, where the mass has been assigned to subsets  $x'_1 = \{x_1\}$ ,  $x'_6 = \{x_2 \cup x_4\}$ ,  $x'_7 = \{x_3 \cup x_4\}$ , and  $X$ .

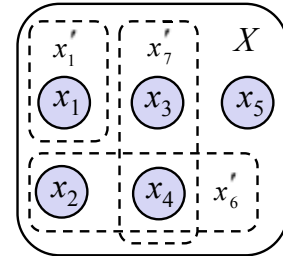


Figure 1. Example Belief Frame X.

$$m_X : \begin{cases} m_X(x'_1) &= 0.15 \\ m_X(x'_6) &= 0.50 \\ m_X(x'_7) &= 0.20 \\ m_X(X) &= 0.15 \end{cases} \quad (18)$$

Note that  $m_X$  is not a Dirichlet bba because  $x'_6 \cap x'_7 \neq \emptyset$ . The probability projection of the bba as a function of the base rates according to Eq.(6) is then expressed as the Bayesian bba below (rounded to 3 significant digits):

$$m_X^P : \begin{cases} m_X^P(x_1) = 0.188 \\ m_X^P(x_2) = 0.212 \\ m_X^P(x_3) = 0.083 \\ m_X^P(x_4) = 0.511 \\ m_X^P(x_5) = 0.008 \end{cases} \quad (19)$$

The degree of uncertainty according to Eq.(11) can be computed as  $u(m_X) = 0.525$ .

The smallest uncertainty according to Eq.(14) is  $m_X^t(X) = m_X^5(X) = 0.16$ . The Dirichlet projection of the original bba can then be computed to be as follows.

$$m_X^D : \begin{cases} m_X^D(x_1) = 0.148 \\ m_X^D(x_2) = 0.180 \\ m_X^D(x_3) = 0.059 \\ m_X^D(x_4) = 0.455 \\ m_X^D(x_5) = 0.000 \\ m_X^D(X) = 0.16 \end{cases} \quad (20)$$

The Dirichlet projection thus has a smaller degree of uncertainty than the original bba. This reduction in uncertainty is caused by the projection. Unlike the Bayesian bba projection  $m_X^P$ , the Dirichlet projection assigns no belief mass to  $x_5$ , which is in better agreement with the original bba. Dirichlet bbas like  $m_X^D$  can easily be converted to subjective logic opinions (see [12]), which in turn provides a rich set of operators for belief reasoning.

## V. DISCUSSION AND CONCLUSION

Base rates are fundamental to probabilistic analysis, and there is no reason why they should be less important in belief theory. For example, in the case of determining the likelihood of hypothesis based on observations through sensors or tests, base rates are necessary. This is because the test results' influence on the conclusion is determined by conditionals. Unless base rates are taken into account the analyst runs the danger of falling victim to a particular reasoning error which commonly is known as the base rate fallacy [9] in medicine or the prosecutor's fallacy [15] in legal reasoning. An extreme example of the base rate fallacy is to conclude that a male person is pregnant just because he tests positive in a pregnancy test. Obviously, the base rate of male pregnancy is zero, and assuming that the test is not perfect, it would be correct to conclude that the male person is not pregnant. The correct reasoning that takes base rates into account can be formalized mathematically in probability calculus.

Proposals for defining conditional reasoning in belief theory have mainly been put forward by Smets 1993 [16] but it has been shown that Smets' method is flawed [8]. Subjective logic which is related to belief theory uses opinions where base rates are explicitly defined. This allows conditional reasoning with subjective logic without the risk of falling victim to the base rate fallacy [8].

In order to advance belief theory to include conditional reasoning it is necessary to include base rates. This paper has demonstrated that belief functions can easily be combined with base rates. An interesting area of future research is to see whether the expression of base rates can contribute to a sound definition of belief-based conditional reasoning.

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