Cumulative and Averaging Unfusion of Beliefs*

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Abstract

Belief fusion is the principle of combining separate beliefs or bodies of evidence originating from different sources. Depending on the situation to be modelled, different belief fusion methods can be applied. Cumulative and averaging belief fusion is defined for fusing opinions in subjective logic, and for fusing belief functions in general. The principle of unfusion is the opposite of fusion, namely to eliminate the contribution of a specific belief from an already fused belief, with the purpose of deriving the remaining belief. This paper describes unfusion of cumulative belief as well as unfusion of averaging belief in subjective logic. These operators can for example be applied to belief revision in Bayesian belief networks, where the belief contribution of a given evidence source can be determined as a function of a given fused belief and its other contributing beliefs.

Keywords: Fusion, unfusion, subjective logic, belief, uncertainty

1 Introduction

Subjective logic is a type of probabilistic logic that explicitly takes uncertainty and belief ownership into account. In general, subjective logic is suitable for modeling and analysing situations involving uncertainty and incomplete knowledge [1, 2]. For example, it can be used for modeling trust networks [6] and for analysing Bayesian networks [5].

Arguments in subjective logic are subjective opinions about propositions. The opinion space is a subset of the belief function space used in Dempster-Shafer belief theory. The term belief will be used interchangeably with opinions throughout this paper. A binomial opinion applies to a single proposition, and can be represented as a Beta distribution. A multinomial opinion applies to a collection of propositions, and can be represented as a Dirichlet distribution. Through the correspondence between opinions and Beta/Dirichlet distributions, subjective logic provides an algebra for these functions.

The two types of fusion defined for subjective logic are *cumulative fusion* and *averaging fusion* [4]. Situations that can be modelled with the cumulative operator are for example when fusing beliefs of two observers who have assessed separate and independent evidence, such as when they have observed the outcomes of a given process over two separate non-overlapping time periods. Situations that can be modelled with the averaging operator are for example when fusing beliefs of two observers who have assessed the same evidence and possibly interpreted it differently.

Dempster's rule also represents a method commonly applied for fusing beliefs. However, it is not used in subjective logic and will not be discussed here.

There are situations where it is useful to separate a

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fused belief in its contributing belief components, and this process is called belief unfusion. This requires the already fused belief and one of its contributing belief components as input, and will produce the remaining contributing belief component as output. Unfusion is basically the opposite of fusion, and the formal expressions for unfusion can be derived by rearranging the expressions for fusion. This will be described in the following sections.

Fission of beliefs is related to unfusion of beliefs but is different and will not be discussed here. Fission simply means that a belief is split into several parts without specifying any of its contributing factors. A belief can for example be split into two equal contributing beliefs. Belief fission will be discussed in future work.

2 Fundamentals of Subjective Logic

Subjective opinions express subjective beliefs about the truth of propositions with degrees of uncertainty, and can indicate subjective belief ownership whenever required. An opinion is usually denoted as ω_x^A where A is the subject, also called the belief owner, and x is the proposition to which the opinion applies. An alternative notation is $\omega(A:x)$. The proposition x is assumed to belong to a frame of discernment (also called state space) e.g. denoted as X, but the frame is usually not included in the opinion notation. The propositions of a frame are normally assumed to be exhaustive and mutually disjoint, and subjects are assumed to have a common semantic interpretation of propositions. The subject, the proposition and its frame are attributes of an opinion. Indication of subjective belief ownership is normally omitted whenever irrelevant.

2.1 Binomial Opinions

Let x be a proposition. Entity A's binomial opinion about the truth of a x is the ordered quadruple $\omega_x^A = (b, d, u, a)$ with the components:

- b: belief that the proposition is true
- d: disbelief that the proposition is true (i.e. the belief that the proposition is false)
- u: uncertainty about the probability of x (i.e. the amount of uncommitted belief)
- a: base rate of x (i.e. probability of x in the absence of belief)

These components satisfy:

$$b,\ d,\ u,\ a \in [0,1] \quad (1)$$

$$\text{and} \qquad b + d + u = 1 \quad (2)$$

The characteristics of various opinion classes are listed below. An opinion where:
$b = 1$: is equivalent to binary logic TRUE,
$d = 1$: is equivalent to binary logic FALSE,
$b + d = 1$: is equivalent to a probability,
$0 < (b + d) < 1$: expresses uncertainty, and
$b + d = 0$: is vacuous (i.e. totally uncertain).

The probability expectation value of a binomial opinion is:

$$p(\omega_x) = b_x + a_x u_x \ . \quad (3)$$

The expression of Eq.(3) is equivalent to the pignistic probability in traditional belief function theory [10], and is based on the principle that the belief mass assigned to the whole frame is split equally among the singletons of the frame. In Eq.(3) the base rate a_x must be interpreted in the sense that the relative proportion of singletons contained in x is equal to a_x.

Binomial opinions can be represented on an equilateral triangle as shown in Fig.1 below.

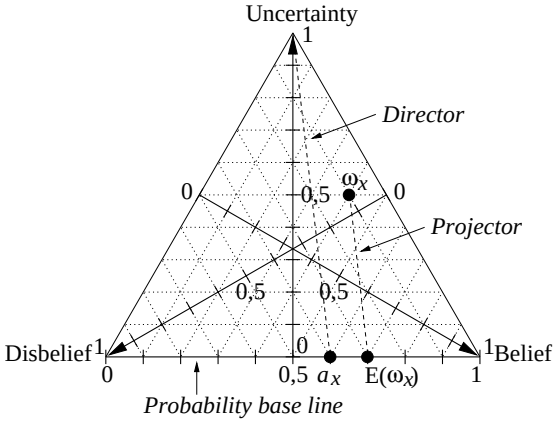

Figure 1: Opinion triangle with example opinion

A point inside the triangle represents a (b, d, u) triple. The b,d,u-axes run from one edge to the opposite vertex indicated by the Belief, Disbelief or Uncertainty label. For example, a strong positive opinion is represented by a point towards the bottom right Belief vertex. The base rate, also called relative atomicity, is shown as a red pointer along the probability base line, and the probability expectation, E, is formed by projecting the opinion onto the base, parallel to the base rate projector line. As an example, the opinion $\omega_x = (0.4,\ 0.1,\ 0.5,\ 0.6)$ is shown on the figure.

Uncertainty about probability values can be interpreted as ignorance, or second order uncertainty about the first order probabilities. In this paper, the term "uncertainty" will be used in the sense of *"uncertainty about the probability values"*. A probabilistic logic based on belief theory therefore represents a generalisation of traditional probabilistic logic.

2.2 Multinomial Opinions

Let X be a frame, i.e. a set of exhaustive and mutually disjoint propositions x_i. Entity A's multinomial opinion over X is the composite function $\omega_X^A = (\vec{b}, u, \vec{a})$, where $\vec{b}$ is a vector of belief masses over the propositions of X, u is the uncertainty mass, and $\vec{a}$ is a vector of base rate values over the propositions of X. These components satisfy:

$$\vec{b}(x_i), u, \vec{a}(x_i) \in [0, 1], \quad \forall x_i \in X \tag{4}$$

$$u + \sum_{x_i \in X} \vec{b}(x_i) = 1 \tag{5}$$

$$\sum_{x_i \in X} \vec{a}(x_i) = 1 \tag{6}$$

Visualising multinomial opinions is not trivial. Trinomial opinions can be visualised as points inside a triangular pyramid as shown in Fig.2, but the 2D aspect of printed paper and computer monitors make this impractical in general.

Opinions with dimensions larger than trinomial do not lend themselves to traditional visualisation.

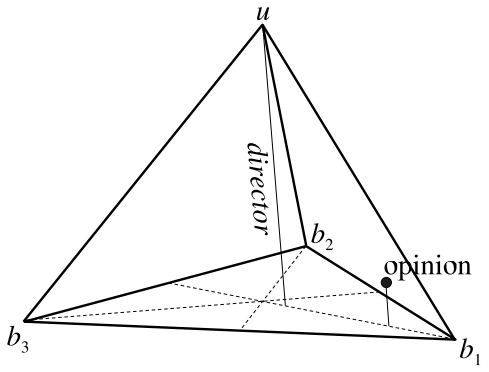

Figure 2: Opinion pyramid with example trinomial opinion

3 Fusion of Multinomial Opinions

In many situations there will be multiple sources of evidence, and fusion can be used to combine evidence from different sources.

In order to provide an interpretation of fusion in subjective logic it is useful to consider a process that is observed by two sensors. A distinction can be made between two cases.

1. The two sensors observe the process during disjoint time periods. In this case the observations are independent, and it is natural to simply add the observations from the two sensors, and the resulting fusion is called *cumulative fusion*.

2. The two sensors observe the process during the same time period. In this case the observations are dependent, and it is natural to take the average of the observations by the two sensors, and the resulting fusion is called *averaging fusion*.

3.1 Cumulative Fusion

Assume a frame X containing k elements. Assume two observers A and B who have independent opinions over the frame X. This cane for example result from having observed the outcomes of a process over two separate time periods.

Let the two observers' respective opinions be expressed as $\omega_X^A = (\vec{b}_X^A, u_X^A, \vec{a}_X^A)$ and $\omega_X^B = (\vec{b}_X^B, u_X^B, \vec{a}_X^B)$.

The cumulative fusion of these two bodies of evidence is denoted as $\omega A \diamond B_X = \omega_X^A \oplus \omega_X^B$. The symbol "$\diamond$" denotes the fusion of two observers A and B into a single imaginary observer denoted as $A \diamond B$. The mathematical expressions for cumulative fusion is described below.

Definition 1 The Cumulative Fusion Operator

Let ω_X^A and ω_X^B be opinions respectively held by agents A and B over the same frame $X = \{x_i \mid i = 1, \cdots, k\}$. Let $\omega_X^{A \diamond B}$ be the opinion such that:

Case I: For $u_X^A \neq 0 \ \vee \ u_X^B \neq 0$:

$$\begin{cases} b_{x_i}^{A \diamond B} & = \frac{b_{x_i}^A u_X^B + b_{x_i}^B u_X^A}{u_X^A + u_X^B - u_X^A u_X^B} \\ u_X^{A \diamond B} & = \frac{u_X^A u_X^B}{u_X^A + u_X^B - u_X^A u_X^B} \end{cases} \tag{7}$$

Case II: For $u_X^A = 0 \ \wedge \ u_X^B = 0$:

$$\begin{cases} b_{x_i}^{A \diamond B} & = \gamma \, b_{x_i}^A + (1 - \gamma) b_{x_i}^B \\ u_X^{A \diamond B} & = 0 \end{cases} \tag{8}$$

$$\text{where} \quad \gamma = \lim_{\substack{u_X^A \to 0 \\ u_X^B \to 0}} \frac{u_X^B}{u_X^A + u_X^B}$$

Then $\omega_X^{A \diamond B}$ is called the cumulatively fused opinion of ω_X^A and ω_X^B, representing the combination of independent opinions of A and B. By using the symbol '$\oplus$' to designate this belief operator, we define $\omega_X^{A \diamond B} \equiv \omega_X^A \oplus \omega_X^B$.

The cumulative fusion operator is equivalent to *a posteriori* updating of Dirichlet distributions. Its proof and derivation is based on the bijective mapping between multinomial opinions and and an augmented representation of the Dirichlet distribution [4].

It can be verified that the cumulative fusion operator is commutative, associative and non-idempotent. In Case II of Def.1, the associativity depends on the preservation of relative weights of intermediate results, which requires the additional weight variable γ. In this case, the cumulative operator is equivalent to the weighted average of probabilities.

The cumulative fusion operator represents a generalisation of the consensus operator [3, 2] which emerges directly from Def.1 by assuming a binary frame.

3.2 Averaging Fusion

Assume a frame X containing k elements. Assume two observers A and B who have dependent opinions over the frame X. This can for example result from observing the outcomes of the process over the same time periods.

Let the two observers' respective opinions be expressed as $\omega_X^A = (\vec{b}_X^A, u_X^A, \vec{a}_X^A)$ and $\omega_X^B = (\vec{b}_X^B, u_X^B, \vec{a}_X^B)$.

The averaging fusion of these two bodies of evidence is denoted as $\omega A \underline{\diamond} B_X = \omega_X^A \underline{\oplus} \omega_X^B$. The symbol "$\underline{\diamond}$" denotes the averaging fusion of two observers A and B into a single imaginary observer denoted as $A \underline{\diamond} B$. The mathematical expressions for averaging fusion is described below.

Definition 2 The Averaging Fusion Operator

Let ω_X^A and ω_X^B be opinions respectively held by agents A and B over the same frame $X = \{x_i \mid i = 1, \cdots, k\}$. Let $\omega_X^{A \underline{\diamond} B}$ be the opinion such that:

Case I: For $u_X^A \neq 0 \ \vee \ u_X^B \neq 0$:

$$\begin{cases} b_{x_i}^{A \underline{\diamond} B} & = \frac{b_{x_i}^A u_X^B + b_{x_i}^B u_X^A}{u_X^A + u_X^B} \\ u_X^{A \underline{\diamond} B} & = \frac{2 u_X^A u_X^B}{u_X^A + u_X^B} \end{cases} \tag{9}$$

Case II: For $u_X^A = 0 \ \wedge \ u_X^B = 0$:

$$\begin{cases} b_{x_i}^{A \diamond B} & = \gamma \, b_{x_i}^A + (1 - \gamma) b_{x_i}^B \\ u_X^{A \diamond B} & = 0 \end{cases} \tag{10}$$

$$\text{where} \quad \gamma = \lim_{\substack{u_X^A \to 0 \\ u_X^B \to 0}} \frac{u_X^B}{u_X^A + u_X^B}$$

Then $\omega_X^{A \underline{\diamond} B}$ is called the averaged opinion of ω_X^A and ω_X^B, representing the combination of the de-

pendent opinions of A and B. By using the symbol '$\underline{\oplus}$' to designate this belief operator, we define $\omega_X^{A\underline{\diamond}B} \equiv \omega_X^A \underline{\oplus} \omega_X^B$.

The averaging operator is equivalent to averaging the evidence of Dirichlet distributions. Its proof derivation is based on the bijective mapping between multinomial opinions and an augmented representation of Dirichlet distributions [4].

It can be verified that the averaging fusion operator is commutative and idempotent, but not associative.

The averaging fusion operator represents a generalisation of the consensus operator for dependent opinions defined in [7].

4 Unfusion of Multinomial Opinions

The principle of belief unfusion is the opposite to belief fusion. This section describes the unfusion operators corresponding to the cumulative and averaging fusion operators described in the previous section.

4.1 Cumulative Unfusion

Assume a frame X containing k elements. Assume two observers A and B who have observed the outcomes of a process over two separate time periods. Assume that the observers' beliefs have been cumulatively fused into $\omega_X^{A\diamond B} = \omega_X^C = (\vec{b}_X^C, u_X^C, \vec{a}_X^C)$, and assume that entity B's contributing opinion $\omega_X^B = (\vec{b}_X^B, u_X^B, \vec{a}_X^B)$ is known.

The cumulative unfusion of these two bodies of evidence is denoted as $\omega_X^{C\overline{\diamond}B} = \omega_X^A = \omega_X^C \ominus \omega_X^B$, which represents entity A's contributing opinion. The mathematical expressions for cumulative unfusion is described below.

Definition 3 The Cumulative Unfusion Operator

Let $\omega_X^C = \omega_X^{A\diamond B}$ be the cumulatively fused opinion of ω_X^B and the unknown opinion ω_X^A over the frame $X = \{x_i \mid i = 1, \cdots, k\}$. Let $\omega_X^A = \omega_X^{C\overline{\diamond}B}$ be the opinion such that:

Case I: For $u_X^C \neq 0 \ \vee \ u_X^B \neq 0$:

$$\begin{cases} b_{x_i}^A = b_{x_i}^{C\overline{\diamond}B} & = \frac{b_{x_i}^C u_X^B - b_{x_i}^B u_X^C}{u_X^B - u_X^C + u_X^B u_X^C} \\ u_X^A = u_X^{C\overline{\diamond}B} & = \frac{u_X^B u_X^C}{u_X^B - u_X^C + u_X^B u_X^C} \end{cases} \tag{11}$$

Case II: For $u_X^C = 0 \ \wedge \ u_X^B = 0$:

$$\begin{cases} b_{x_i}^A = b_{x_i}^{C\overline{\diamond}B} & = \gamma^B b_{x_i}^C - \gamma^C b_{x_i}^B \\ u_X^A = u_X^{C\overline{\diamond}B} & = 0 \end{cases} \tag{12}$$

$$\text{where} \begin{cases} \gamma^B = \lim\limits_{\substack{u_X^C \to 0 \\ u_X^B \to 0}} \frac{u_X^B}{u_X^B - u_X^C + u_X^B u_X^C} \\ \gamma^C = \lim\limits_{\substack{u_X^C \to 0 \\ u_X^B \to 0}} \frac{u_X^C}{u_X^B - u_X^C + u_X^B u_X^C} \end{cases}$$

Then $\omega_X^{C\overline{\diamond}B}$ is called the cumulatively unfused opinion of ω_X^C and ω_X^B, representing the result of eliminating the opinions of B from that of C. By using the symbol '$\ominus$' to designate this belief operator, we define $\omega_X^{C\overline{\diamond}B} \equiv \omega_X^C \ominus \omega_X^B$.

Cumulative unfusion is the inverse of cumulative fusion. Its proof and derivation is based on rearranging the mathematical expressions of Def.1

It can be verified that the cumulative rule is non-commutative, non-associative and non-idempotent. In Case II of Def.3, the unfusion rule is equivalent to the weighted subtraction of probabilities.

4.2 Averaging Unfusion

Assume a frame X containing k elements. Assume two observers A and B who have observed the same outcomes of a process over the same time period. Assume that the observers' beliefs have been averagely fused into $\omega_X^C = \omega_X^{A\underline{\diamond}B} = (\vec{b}_X^C, u_X^C, \vec{a}_X^C)$, and assume that entity B's contributing opinion $\omega_X^B = (\vec{b}_X^B, u_X^B, \vec{a}_X^B)$ is known.

The averaging unfusion of these two bodies of evidence is denoted as $\omega_X^A = \omega_X^{C\underline{\overline{\diamond}}B} = \omega_X^C \underline{\ominus} \omega_X^B$, which represents entity A's contributing opinion. The mathematical expressions for averaging unfusion is described below.

Definition 4 The Averaging Unfusion Operator

Let $\omega_X^C = \omega_X^{A \diamond B}$ be the fused average opinion of ω_X^B and the unknown opinion ω_X^A over the frame $X = \{x_i \mid i = 1, \cdots, k\}$. Let $\omega_X^A = \omega_X^{C \overline{\oslash} B}$ be the opinion such that:

Case I: For $u_X^C \neq 0 \ \vee \ u_X^B \neq 0$:

$$\begin{cases} b_{x_i}^A = b_{x_i}^{C \overline{\oslash} B} = \frac{2 b_{x_i}^C u_X^B - b_{x_i}^B u_X^C}{2u_X^B - u_X^C} \\ u_X^A = u_X^{C \overline{\oslash} B} = \frac{u_X^B u_X^C}{2u_X^B - u_X^C} \end{cases} \tag{13}$$

Case II: For $u_X^C = 0 \ \wedge \ u_X^B = 0$:

$$\begin{cases} b_{x_i}^A = b_{x_i}^{C \overline{\oslash} B} = \gamma^B b_{x_i}^C - \gamma^C b_{x_i}^B \\ u_X^A = u_X^{C \overline{\oslash} B} = 0 \end{cases} \tag{14}$$

$$\text{where} \begin{cases} \gamma^B = \lim\limits_{\substack{u_X^C \to 0 \\ u_X^B \to 0}} \frac{2u_X^B}{2u_X^B - u_X^C} \\ \gamma^C = \lim\limits_{\substack{u_X^C \to 0 \\ u_X^B \to 0}} \frac{u_X^C}{2u_X^B - u_X^C} \end{cases}$$

Then $\omega_X^{C \overline{\oslash} B}$ is called the average unfused opinion of ω_X^C and ω_X^B, representing the result of eliminating the opinions of B from that of C. By using the symbol '$\ominus$' to designate this belief operator, we define $\omega_X^{C \overline{\oslash} B} \equiv \omega_X^C \underline{\ominus} \omega_X^B$.

Averaging unfusion is the inverse of averaging fusion. Its proof and derivation is based on rearranging the mathematical expressions of Def.2

It can be verified that the averaging unfusion operator is idempotent, non-commutative and non-associative.

5 Examples

5.1 Simple Belief Unfusion

Assume that A has an unknown opinion about x. Let B's opinion and the cumulatively fused opinion between A's and B's opinions be know as:

$$\omega_x^{A \diamond B} = (0.90,\ 0.05,\ 0.05,\ \tfrac{1}{2}) \text{ and}$$

$$\omega_x^B = (0.70,\ 0.10,\ 0.20,\ \tfrac{1}{2})$$

respectively. Using the cumulative unfusion operator it is possible to derive A's opinion. This situation is illustrated in Fig.3.

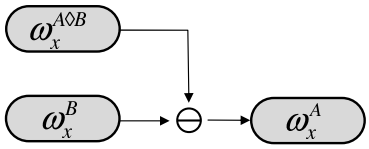

Figure 3: Principle of belief unfusion

By inserting the opinions values into Eq.(7) the contributing opinion from A can be derived as

$$\omega_x^A = (0.93,\ 0.03,\ 0.06,\ \frac{1}{2})$$

5.2 Inverse Reasoning in Bayesian Networks

Bayesian belief networks represent models of conditional relationships between propositions of interest. Subjective logic provides operators for conditional deduction [8] and conditional abduction [9] which allows reasoning to take place in either direction along a conditional edge. Fig.4 shows a simple Bayesian belief network where x and y are parent evidence nodes and z is the child node.

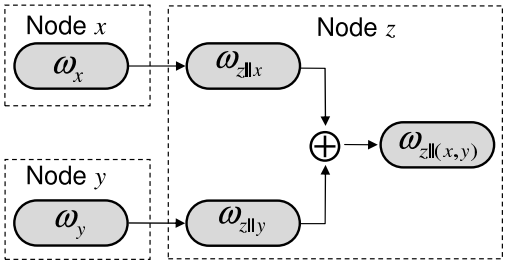

Figure 4: Bayesian network with belief fusion

In order to derive the deduced opinions $\omega_{z\|x}$ and $\omega_{z\|y}$ using the deduction operator, the opinions ω_x and ω_y, as well as the conditional opinions $\omega_{z|x}$, $\omega_{z|\overline{x}}$, $\omega_{z|y}$ and $\omega_{z|\overline{y}}$ are needed. Assuming that the contributions of $\omega_{z\|x}$ and $\omega_{z\|y}$ are independent, they can be fused with the cumulative fusion operator to produce the derived opinion $\omega_{z\|(x,y)}$.

Belief revision based on the unfusion operator can be useful in case a very certain opinion about z has been determined from other sources, and it is in conflict with the opinion derived through the

Bayesian network. In that case, the reasoning can be applied in the inverse direction using the unfusion operator to revise the opinions about x and y or about the conditional relationships $z|x$ and $z|y$.

Opinion ownership in the form of a superscript to the opinions is not expressed in this example. It can be assumed that the analyst derives input opinion values as a function of evidence collected from different sources. The origin of the opinions are therefore implicitly represented as the evidence sources in this model.

6 Conclusion

The principle of belief fusion is used in numerous applications. The opposite principle of belief unfusion is less commonly used. However, there are situations where unfusion can be useful. In this paper we have described the unfusion operators corresponding to cumulative and averaging fusion in subjective logic. The derivation of the unfusion operators are based on rearranging the expressions for the corresponding fusion operators.

In future work we will also define belief fission which consists of splitting a belief function into several parts without explicitly specifying any of its contributing beliefs.

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