Abductive Reasoning with Uncertainty*

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Abstract

Abductive reasoning in general describes the process of discovering hypotheses and rules that would entail a given conclusion. Abductive reasoning consists of assessing the likelihood that a specific hypothesis entails a given conclusion. Abductive reasoning based on probabilities is used in many disciplines, such as medical diagnostics, where medical test results combined with conditional probabilities are used to determine the likelihood of possible diseases. In this paper we focus on abductive reasoning in subjective logic. The advantage of our approach over a purely probabilistic approach is that degrees of ignorance can be explicitly included as input and during the analysis.

Keywords: Abduction, deduction, subjective logic, belief, uncertainty

1 Introduction

Abductive reasoning is a general approach to finding the hypotheses that would best explain the given evidence. Discovering or generating relevant hypothesis in general is a hard task which can require considerable computational effort when searching over a large space of information [1]. We focus on simple abductive reasoning in the sense that one or several proposed hypotheses are given, and these are simply analysed for their likelihood given the evidence. For example, simple probabilistic abduction consists of determining the probability of the hypothesis given the evidence as well as a set of conditionals between the hypothesis and the evidence.

Deductive reasoning, which is related to abductive reasoning, consists of deriving conclusions from the given evidence. In that sense, abduction can be described as the inverse of deduction.

Both abductive and deductive reasoning require conditionals. Conditional probabilities relate to conditional propositions which typically are of the form *"If we reduce the* CO_2 *emission, global warming will be halted"*, which are of the form "IF x THEN y", where x denotes the antecedent and y the consequent. An equivalent way of expressing conditionals is through the concept of implication, so that the above proposition can be expressed as *"Reducing the* CO_2 *emission implies that global warming is halted"*.

When making assertions of conditionals with antecedent and consequent, which can be evaluated as TRUE or FALSE propositions, we are in fact evaluating a proposition which can itself be considered TRUE or FALSE.

The idea of having a conditional connection between an antecedent and a consequent can be traced back to Ramsey [10] who articulated what has become known as Ramsey's Test: *To decide whether you believe a conditional, provisionally or hypothetically add the antecedent to your stock of beliefs, and consider whether to believe the consequent*. This idea was translated into a formal language by Stalnaker [11] in the form of the so-

*Appears in the Proceedings of the 12th International Conference on Information Processing and Management of Uncertainty (IPMU2008), Malaga, June 2008.

called Stalnaker's Hypothesis, formally expressed as: $p(\text{IF } x \text{ THEN } y) = p(y|x)$. The interpretation of Stalnaker's Hypothesis is that the probability of the conditional proposition "IF x THEN y" is equal to the probability of the proposition y given that the proposition x is TRUE.

However, Lewis [8] argued that conditional propositions do not have truth-values and that they do not express propositions. This means that given any propositions x and y, there is no proposition z for which $p(z) = p(y|x)$, so the conditional probability can not be the same as the probability of conditionals. Without going into detail we support Stalnaker's Hypothesis, and would argue against Lewis by simply saying that it is meaningful to assign a probability to a conditional proposition like "$y|x$", which is defined in case x is true, and undefined in case x is false.

A meaningful conditional abduction requires that the antecedent is relevant to the consequent, or in other words that the consequent depends on the antecedent, as explicitly expressed in relevance logics [3]. Conditionals that are based on the dependence between consequent and antecedent are considered to be universally valid, and are called *logical conditionals* [2]. Deduction with logical conditionals reflect human intuitive conditional reasoning.

Both binary logic and probability calculus have mechanisms for conditional reasoning. In binary logic, Modus Ponens (MP) and Modus Tollens (MT) are the classical operators which are used in any field of logic that requires conditional deduction. In probability calculus, binomial conditional deduction is expressed as:

$$p(y\|x) = p(x)p(y|x) + p(\overline{x})p(y|\overline{x}) \qquad (1)$$

where the terms are interpreted as follows:

$p(y\|x)$:	probability of y given x is TRUE
$p(y\|\overline{x})$:	probability of y given x is FALSE
$p(x)$:	probability of the antecedent x
$p(\overline{x})$:	complement probability $= 1 - p(x)$
$p(y\|x)$:	derived probability of consequent y

We follow the convention whereby conditional relationship are denoted on the form "consequent | antecedent", i.e. with the consequent first and the antecedent second.

The notation $y\|x$, introduced in [7], denotes that the truth or probability of proposition y is derived as a function of the probability of the antecedent x together with the conditionals. The expression $p(y\|x)$ thus represents a derived value, whereas the expressions $p(y|x)$ and $p(y|\overline{x})$ represent input values together with $p(x)$. Below, this notational convention will also be used for opinions in subjective logic.

This paper describes how the same principles for conditional inference outlined above can be formulated in the framework of subjective logic when applied to binomial opinions. The advantage of this approach is to allow conditional inference to take place in the presence of uncertainty and partial ignorance. This will also allow the analyst to appreciate the relative proportions of firm evidence and uncertainty as contributing factors to the derived probabilistic likelihoods.

A more general description of both abduction and deduction for multinomial opinions is provided in [5]. Binomial opinions represent a special case of general multinomial opinions.

2 Probabilistic Conditional Reasoning

In this section, classical results from probabilistic abduction are briefly reviewed in order to provide a benchmark for abduction with subjective logic, described in Sec.3.

2.1 Binomial Conditional Reasoning

Abduction is used extensively in areas where conclusions need to be derived from probabilistic input evidence, such as for making diagnoses from medical tests. For example, a pharmaceutical company that develops a test for a particular infection disease will typically determine the reliability of the test by letting a group of infected and a group of non-infected people undergo the test. The result of these trials will then determine the reliability of the test in terms of its *sensitivity* and *false positive rate*. This can be expressed in terms of the binomial conditionals $p(x|y)$ and $p(x|\overline{y})$, where x: *"Positive Test"*, y: *"Infected"* and $\overline{y}$: *"Not infected"*. Their interpretations can be expressed as follows:

- $p(x|y)$: *"The probability of positive test given infection"*
- $p(y|\overline{x})$: *"The probability of positive test in the absence of infection"*.

In other words $p(x|y)$ expresses the rate of true positives, and $p(x|\overline{y})$ expresses the rate of false positives of the test. The problem with applying this in a practical setting is that the conditionals are expressed in the opposite direction to what the practitioner needs in order to apply the expression of Eq.(1). The conditionals needed for making the diagnosis are:

- $p(y|x)$: *"The probability of infection given positive test"*
- $p(y|\overline{x})$: *"The probability of infection given negative test"*

but these are usually not directly available to the medical practitioner.

The base rate fallacy in medicine consists of making the erroneous conclusion that if the patient tests positive (i.e. $p(x) = 1$), the probability of having the disease is $p(x|y)$, which is equivalent to making the false assumption that $p(y|x) = p(x|y)$. While this reasoning error often can give a relatively good approximation of the correct probability value, it can lead to a completely wrong result and wrong diagnosis in case the base rate of the disease in the population is very low and the reliability of the test is not perfect.

The required conditionals can be derived by inverting the available conditionals using Bayes rule. The inverted conditionals are obtained as follows:

$$\begin{cases} p(x|y) = \frac{p(x \wedge y)}{p(y)} \\ \\ p(y|x) = \frac{p(x \wedge y)}{p(x)} \end{cases} \quad \Rightarrow \quad p(y|x) = \frac{p(y)p(x|y)}{p(x)} \,. \tag{2}$$

Here $p(y)$ represents the base rate of the disease in the population, and $p(x)$ represents the the expected rate of positive tests as a function of the base rate of the disease in the population, which can be computed with Eq.(1) where x and y are swapped in every term. The notation $a(x)$ and $a(y)$ will be used to denote the base rate of x and y respectively. The full expression for the required positive conditional is then:

$$p(y|x) = \frac{a(y)p(x|y)}{a(y)p(x|y) + a(\overline{y})p(x|\overline{y})} \tag{3}$$

A medical test result is typically positive or negative, so it can be assumed that either $p(x)$ = 1 (positive) or $p(\overline{x})$ = 1 (negative). In case the patient tests positive, Eq.(1) can be simplified to $p(y\|x) = p(y|x)$ so that Eq.(3) will give the correct likelihood that her or she actually has contracted the disease.

2.2 Example 1: Probabilistic Medical Reasoning

Let the sensitivity of a medical test be expressed as $p(x|y) = 0.9999$ (i.e. an infected person will test positive in 99.99% of the cases) and the false positive rate be $p(x|\overline{y}) = 0.001$ (i.e. a non-infected person will test positive in 0.1% of the cases). Let the base rate of infection in population A be 1% (expressed as $a(y_A)$=0.01) and let the base rate of infection in population B be 0.01% (expressed as $a(y_B)$=0.0001). Assume that a person from population A tests positive, then Eq.(1) and Eq.(3) lead to the conclusion that $p(y_A\|x) = p(y_A|x) = 0.9099$ which indicates a 91% likelihood that the person is infected. Assume that a person from population B tests positive, then Eq.(1) and Eq.(3) produces $p(y_B\|x) = p(y_B|x) = 0.0909$ which indicates only a 9% likelihood that the person is infected.

By using the correct method in this example, the base rate fallacy is avoided.

2.3 Binomial Probabilistic Abduction

In the general case where the the truth of the antecedent is expressed as a probability, and not just binary TRUE and FALSE, the negative conditional is also needed as specified in Eq.(1). In case the negative conditional is not directly available, it can be derived according to Eq.(3) where x is replaced with $\overline{x}$ in every term. This produces:

$$\begin{aligned} p(y|\overline{x}) &= \frac{a(y)p(\overline{x}|y)}{a(y)p(\overline{x}|y)+a(\overline{y})p(\overline{x}|\overline{y})} \\ &= \frac{a(y)(1-p(x|y))}{a(y)(1-p(x|y))+a(\overline{y})(1-p(x|\overline{y}))} \end{aligned} \tag{4}$$

Eq.(3) and Eq.(4) make it possible to perform conditional reasoning when the required conditionals are expressed in the reverse direction to what is needed by the analyst.

We will use the term *"parent state"* and *"child state"* to denote the reasoning direction, meaning that the parent is what the analyst has evidence about, and the child is what the analyst wants to derive an opinion about. Defining parent and child is thus equivalent with defining the reasoning direction.

Forward conditional inference, called *deduction*, is when the parent and child states of the reasoning are the antecedent and consequent states respectively of the available conditionals.

Reverse conditional inference, called *abduction*, is when the parent state of the reasoning is the consequent of the conditionals, and the child state of the reasoning is the antecedent state of the conditionals.

The deductive reasoning principle is illustrated in Fig.1 where x denotes the parent state and y denotes the child state of the reasoning. Conditionals are expressed as $p(\text{consequent}\,|\text{antecedent})$, i.e. with the consequent first, and the antecedent last.

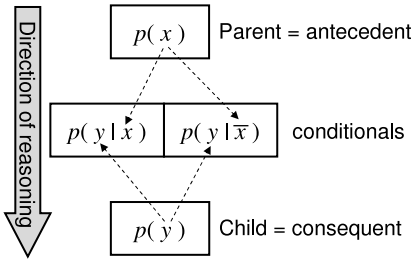

Figure 1: Deduction principle

The abductive reasoning principle is illustrated in Fig.2. It can be seen that the order of the propositions in the conditionals is inverted in comparison to deduction.

The concepts of *"causal"* and *"derivative"* reasoning are related to deductive and abductive reasoning. By assuming that the conditionals express a causal relationship between the antecedent and the consequent (i.e. that the antecedent actually causes the consequent) then causal reasoning is equivalent to deductive reasoning, and derivative reasoning is equivalent to abductive reasoning.

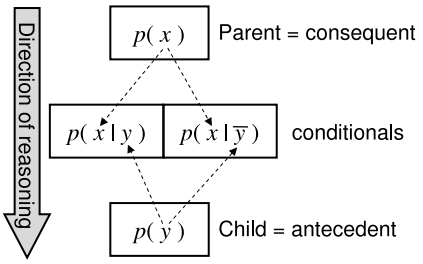

Figure 2: Abduction principle

In medical reasoning for example, the infection causes the test to be positive, not the other way. The reliability of medical tests is expressed as causal conditionals, whereas the practitioner needs to apply the inverted derivative conditionals. Starting from a positive test to conclude that the patient is infected therefore represents derivative reasoning. People usually find causal reasoning more natural, and therefore have a tendency to reason in a causal manner even in situations where derivative reasoning is required. In other words, derivative situations are often confused with causal situations, which provides an explanation for the base rate fallacy in medical diagnostics.

3 Abduction in Subjective Logic

Subjective logic, which will be described here, takes both the uncertainty and individuality of beliefs into account while still being compatible with standard logic and probability calculus. This is achieved by adding an uncertainty dimension to the single valued probability measure, and by taking the individuality of beliefs into account.

3.1 Subjective Logic Fundamentals

Subjective logic [4] is a probabilistic logic that takes *opinions* as input. An opinion denoted by $\omega_x^A = (b, d, u, a)$ expresses the relying party A's belief in the truth of statement x. Here b, d, and u represent belief, disbelief and uncertainty respectively, where $b, d, u \in [0, 1]$ and $b + d + u = 1$. The parameter $a \in [0, 1]$ is called the base rate, and is used for computing an opinion's probability expectation value that can be determined as $\mathrm{E}(\omega_x^A) = b + au$. In the absence of any specific evidence about a given party, the base rate determines the *a priori* trust that would be put in any

member of the community.

The opinion space can be mapped into the interior of an equal-sided triangle, where, for an opinion $\omega_x = (b_x, d_x, u_x, a_x)$, the three parameters b_x, d_x and u_x determine the position of the point in the triangle representing the opinion. Fig.3 illustrates an example where the opinion about a proposition x from a binary state space has the value $\omega_x = (0.7,\ 0.1,\ 0.2,\ 0.5)$.

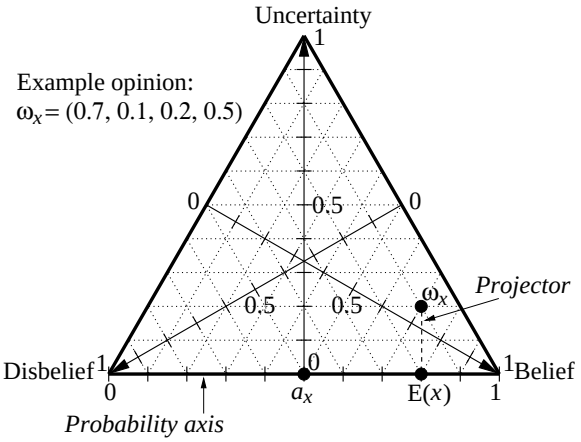

Figure 3: Opinion triangle with example opinion

The top vertex of the triangle represents uncertainty, the bottom left vertex represents disbelief, and the bottom right vertex represents belief. The parameter b_x takes value 0 on the left side edge and takes value 1 at the right side belief vertex. The parameter d_x takes value 0 on the right side edge and takes value 1 at the left side disbelief vertex. The parameter u_x takes value 0 on the base edge and takes value 1 at the top uncertainty vertex. The base of the triangle is called the probability axis. The base rate is indicated by a point on the probability axis, and the projector starting from the opinion point is parallel to the line that joins the uncertainty vertex and the base rate point on the probability axis. The point at which the projector meets the probability axis determines the expectation value of the opinion, i.e. it coincides with the point corresponding to expectation value $\mathrm{E}(\omega_x^A)$.

3.2 Abduction in Subjective Logic

Abduction is related to deduction. The algebraic expression for conditional deducting in subjective logic is relatively long and is therefore omitted here. However, it is relatively simple and can be computed extremely efficiently. A full presentation of the expressions for conditional deduction in subjective logic is given in [7]. Only the notation is provided here.

Let ω_x, $\omega_{y|x}$ and $\omega_{y|\overline{x}}$ be an agent's respective opinions about x being true, about y being true given that x is true, and about y being true given that x is false. Then the opinion $\omega_{y\|x}$ is the conditionally derived opinion, expressing the belief in y being true as a function of the beliefs in x and the two sub-conditionals $y|x$ and $y|\overline{x}$. The conditional deduction operator is a ternary operator, and by using the function symbol '$\circledcirc$' to designate this operator, we write:

$$\omega_{y\|x} = \omega_x \circledcirc (\omega_{y|x}, \omega_{y|\overline{x}}) \ . \tag{5}$$

Abduction requires the conditionals to be inverted. Let x be the parent node, and let y be the child node. In this situation, the input conditional opinions are $\omega_{x|y}$ and $\omega_{x|\overline{y}}$. That means that the original conditionals are expressed in the opposite direction to what is needed.

The inverted conditional opinions, can be derived from knowledge of the supplied conditionals, $\omega_{x|y}$ and $\omega_{x|\overline{y}}$, and knowledge of the base rate of the child, a_y.

Definition 1 (Abduction) *Given knowledge of the base rate a_y of the child state where ω_y^{vac} is a vacuous subjective opinion about the base rate of the hypothesis, defined as*

$$\omega_y^{\mathrm{vac}} = (b_y, d_y, u_y, a_y) \begin{cases} b_y & = 0 \\ d_y & = 0 \\ u_y & = 1 \\ a_y & = \textit{base rate of } y \end{cases} \tag{6}$$

and given the logical conditionals $\omega_{x|y}, \omega_{x|\overline{y}}$, then the inverted conditionals $\omega_{y|x}, \omega_{y|\overline{x}}$ can be derived using the following formula

$$\begin{aligned} \omega_{y|x} &= \frac{\omega_y^{\mathrm{vac}} \cdot \omega_{x|y}}{\omega_y^{\mathrm{vac}} \circledcirc \left(\omega_{x|y}, \omega_{x|\overline{y}}\right)} \\ \omega_{y|\overline{x}} &= \frac{\omega_y^{\mathrm{vac}} \cdot \neg\, \omega_{x|y}}{\omega_y^{\mathrm{vac}} \circledcirc \left(\neg\, \omega_{x|y}, \neg\, \omega_{x|\overline{y}}\right)} \end{aligned} \tag{7}$$

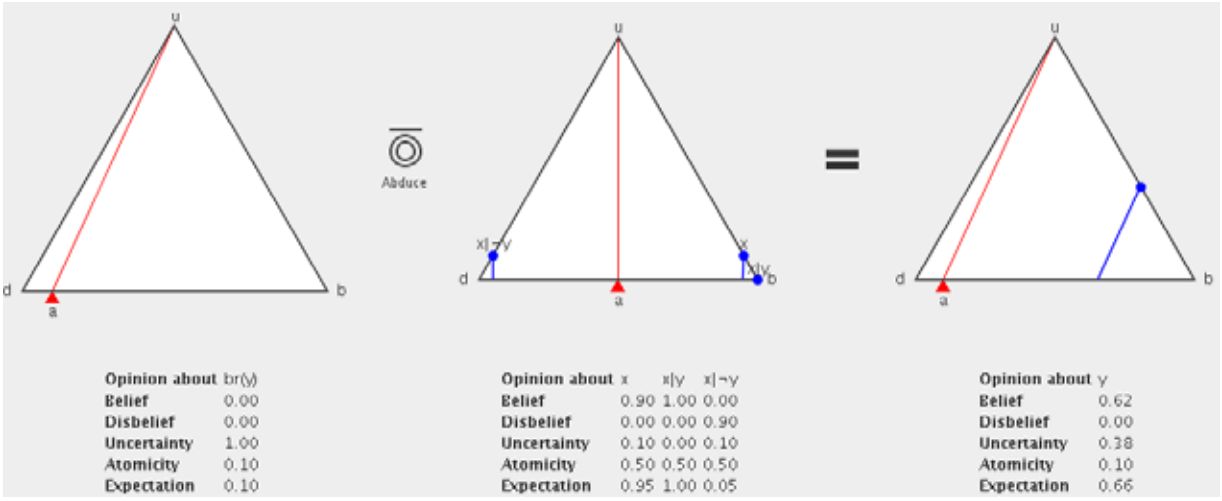

Figure 4: Likelihood of man made CO_2 emission with IPCC's conditionals

The abduction operator, $\overline{\circledcirc}$, is written as $\omega_{y\|x} = \omega_x \overline{\circledcirc} \left(\omega_{x|y},\ \omega_{x|\overline{y}},\ a_y\right)$. Details on the multiplication and division operators can be found in [6].

The advantage of subjective logic over probability calculus and binary logic is its ability to explicitly express and take advantage of ignorance and belief ownership. Subjective logic can be applied to all situations where probability calculus can be applied, and to many situations where probability calculus fails precisely because it can not capture degrees of ignorance. Subjective opinions can be interpreted as probability density functions, making subjective logic a simple and efficient calculus for probability density functions. An online demonstration of subjective logic can be accessed at: http://folk.uio.no/josang/sl/.

4 Example

Let us assume that the conditional relevance between CO_2 emission and global warming is known. Let x:*"Global warming"* and y:*"Man made CO_2 emission"*. The hypothetical question we will ask is whether it could be concluded that man made CO_2 emission is occurring simply based on observing global warming. This is easier to imagine by considering an alien civilisation that observe the temperature of the earth from millions of kilometers distance without actually observing the industrialised CO_2 emission taking place.

4.1 IPCC's View

There have been approximately equally many periods of global warming as global cooling over the history of the earth, so the base rate of global warming is set to 0.5. According to the IPCC (International Panel on Climate Change) [9] the relevance between CO_2 emission and global warming is expressed as:

$$\omega_{x|y}^{\text{IPCC}} = (1.0,\ 0.0,\ 0.0,\ 0.5) \tag{8}$$

$$\omega_{x|\overline{y}}^{\text{IPCC}} = (0.8,\ 0.0,\ 0.2,\ 0.5) \tag{9}$$

$$\tag{10}$$

Similarly, over the history of the earth, man made CO_2 emission has occurred very rarely, meaning that $a_y = 0.1$ for example.

Let us further assume the evidence of global warming, i.e. that an increase in temperature can be observed, expressed as:

$$\omega_x = (0.9,\ 0.0,\ 0.1,\ 0.5) \tag{11}$$

Having received the IPCC's view, the alien civilisation will conclude that there is man made CO_2 emission with the likelihood $\omega_{y\|x}^{\text{IPCC}} = (0.62,\ 0.00,\ 0.38,\ 0.10)$, as illustrated in Fig.4.

According to IPCC's view, it can be concluded that man made CO_2 emission is very likely during periods of global warming on earth. This is obviously a questionable conclusion since all but one period of global warming during the history

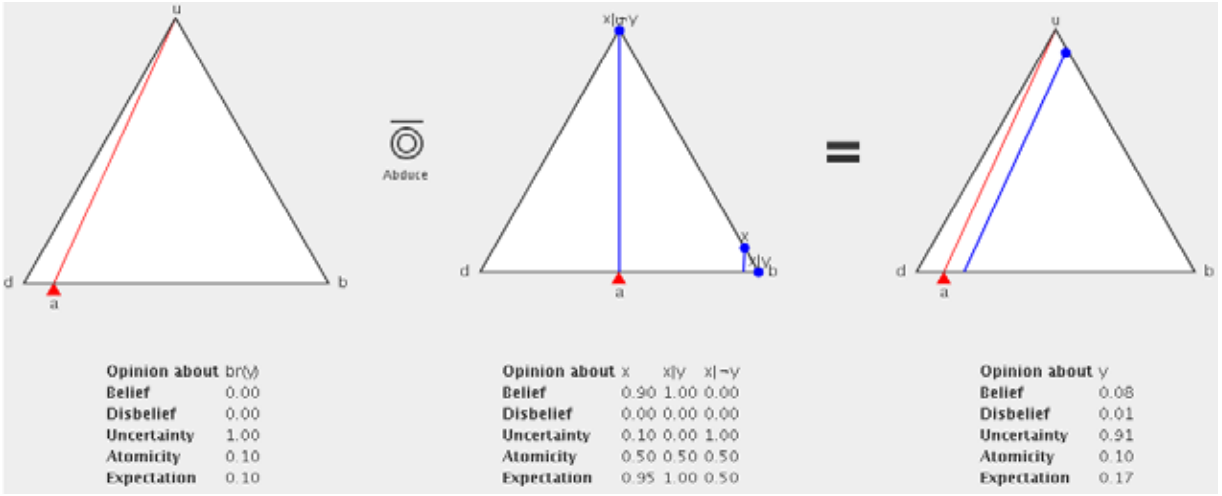

Figure 5: Likelihood of man made CO_2 emission with the sceptic's conditionals

of the earth has taken place without man made CO_2 emission.

4.2 The Sceptic's View

Martin Duke is a journalist who produced the BBC documentary *"The Great Global Warming Swindle"* and who is highly sceptical about IPCC. Let us take sceptic Martin Dukin's view that we don't know anything about whether a reduction in man made CO_2 emission would have had an effect on global warming expressed as:

$$\omega_{x|y}^{\text{Sceptic}} = (1.0,\ 0.0,\ 0.0,\ 0.5) \quad (12)$$

$$\omega_{x|\overline{y}}^{\text{Sceptic}} = (0.0,\ 0.0,\ 1.0,\ 0.5) \quad (13)$$

$$(14)$$

Having received the sceptic's view, the alien civilisation will conclude that there is man made CO_2 emission with the likelihood $\omega_{y\|\overline{x}}^{\text{Sceptic}} = (0.08,\ 0.01,\ 0.91,\ 0.10)$, as illustrated in Fig.5.

According to the sceptic's view, the likelihood of man made CO_2 emission is both low and very uncertain during periods of global warming on earth. This conclusion seems more reasonable in light of the history of the earth.

5 Conclusion

Subjective logic is a belief calculus which takes into account the fact that perceptions about the world always are subjective. This translates into using a belief model that can express degrees of uncertainty about probability estimates, and we use the term *opinion* to denote such subjective beliefs. In addition, ownership of opinions is assigned to particular agents in order to reflect the fact that opinions always are individual. The operators of subjective logic use opinions about the truth of propositions as input parameters, and produce an opinion about the truth of a proposition as output parameter.

We have shown that the principle of abduction from probability calculus can be extended to subjective logic. This allows advanced types of conditional reasoning to be performed in presence of uncertainty and incomplete information.

This paper focuses on the abduction operator for binomial opinions. It should be noted that abduction can be extended to multinomial opinions. Visualisation is particularly simple with binomial opinions and is almost impossible with multinomial opinions. This paper therefore serves to illustrate the principle of abduction in subjective logic. Multinomial abduction is very general and can be applied to parent and child state spaces of any cardinality.

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