Fission of Opinions in Subjective Logic*

Audun Jøsang
University of Oslo - UNIK Graduate Center
josang @ matnat.uio.no

Abstract – *Opinion fusion in subjective logic consists of combining separate observers' opinions about the same frame of discernment. The principle of fission is the opposite of fusion, namely to split an opinion into two separate opinions which when fused would produce the original opinion. Different fusion methods can be applied, depending on the situation to be modelled. The cumulative fusion operator and the averaging fusion operator are defined for subjective opinions, and for belief functions in general. This paper describes fission of opinions which is the opposite of cumulative fusion. The fission operator can for example be applied to transforming trust networks with dependent trust paths into trust networks in canonical form where all trust paths are independent.*

Keywords: Fusion, fission, unfusion, subjective logic, belief, opinion, uncertainty, trust.

1 Introduction

Belief fusion is a term used to denote various methods of combining beliefs on the same frame of discernment, or frame for short[1], where it is assumed that the separate beliefs originate from different sources. Belief fission can be seen as the opposite of fusion. While fusion is used to merge beliefs, fission is used to split beliefs.

The two types of fusion defined for subjective logic are *cumulative fusion* and *averaging fusion* [5]. Situations that can be modelled with the cumulative fusion operator are for example when fusing beliefs of two observers who have assessed separate and independent evidence about the same frame, such as when they have observed the outcomes of a given process over two separate non-overlapping time periods. Situations that can be modelled with the averaging fusion operator are for example when fusing beliefs of two observers who have assessed the same evidence and possibly interpreted it differently.

Subjective logic is a form of probabilistic logic where belief ownership and uncertainty about probability values are explicitly expressed. Subjective logic is therefore suitable for modeling and analysing situations involving uncertainty and incomplete knowledge [1, 2]. For example, it can be used for computational trust [9] and for modelling and analysing Bayesian networks [6].

Arguments in subjective logic are subjective opinions about propositions. The opinion space is related to the classical belief function space used in Dempster-Shafer belief theory. The difference is that belief functions allow belief mass to be assigned to arbitrary subsets of a frame, whereas subjective opinions only allow belief mass to be assigned to singletons as well as to the whole frame. In addition, subjective opinions include base rates of singletons, whereas classical belief functions do not.

The operator most commonly used for belief fusion in Dempster-Shafer belief theory is the so-called Dempster's rule, also known as the normalised conjunctive rule of combination [12]. The equivalent of Dempster's rule in subjective logic would be a normalised form of the multiplication operator [10]. It is normally assumed that the arguments of multiplication apply to separate frames and originate from the same observer. However, when using multiplication as a form of fusion as with Dempster's rule the arguments must apply to the same frame and originate from separate observers. We will not be concerned with multiplication here. A large number of other belief fusion operators have been defined for belief functions, see e.g. [13].

A binomial opinion applies to a single proposition, i.e. to a binary frame consisting of a proposition and its complement, and can be represented as a Beta distribution over a binary frame. A multinomial opinion applies to a frame, i.e. a set of propositions, and can be represented as a Dirichlet distribution over the frame. Multinomial opinions represent a generalisation of binomial opinions in the same way that the Dirichlet distribution represents a generalisation of the Beta distribution. Through the correspondence between opinions and Beta/Dirichlet distributions, subjective logic provides an algebra for the latter functions.

Assuming that an opinion can be considered as the actual or virtual result of fusion, there are situations where it

*In the proceedings of the 12th International Conference on Information Fusion (FUSION 2009), Seattle, July, 2009

[1] A frame of discernment is equivalent to a state space.

is useful to split it into two separate opinions, and this process is called opinion fission. This operator, which requires an opinion and a fission parameter as input arguments, will produce two separate opinions as output. Fission is basically the opposite operation to fusion. The mathematical formulation of fission will be described in the following sections.

2 Fundamentals of Subjective Logic

Subjective opinions express subjective beliefs about the truth of propositions with degrees of uncertainty, and can indicate subjective belief ownership whenever required. A multinomial opinion is usually denoted as ω_X^A where A is the subject, also called the belief owner, and X is the set of proposition to which the opinion applies. An alternative notation is $\omega(A:X)$. Binomial opinions are denoted as ω_x^A where the singleton proposition x is assumed to belong to a frame e.g. denoted as X, but the frame is usually not included in the notation for binomial opinions. The propositions of a frame are normally assumed to be exhaustive and mutually disjoint, and subjects are assumed to have a common semantic interpretation of propositions. The subject owner, the proposition and its frame are attributes of an opinion. Indication of subjective opinion ownership can be omitted whenever irrelevant.

2.1 Binomial Opinions

Let x be a proposition. Entity A's binomial opinion about the truth of a x is the ordered quadruple $\omega_x^A = (b, d, u, a)$ with the components:

b: belief in the proposition being true
d: disbelief in the proposition being true
(i.e. belief in the proposition being false)
u: uncertainty about the probability of x
(i.e. the amount of uncommitted belief)
a: base rate of x, (i.e. *a priori* probability of x)

These components satisfy:

$$b,\ d,\ u,\ a \in [0,1] \tag{1}$$

$$\text{and} \qquad b + d + u = 1 \tag{2}$$

The characteristics of various binomial opinion classes are listed below. An opinion where:
$b = 1$: is equivalent to binary logic TRUE,
$d = 1$: is equivalent to binary logic FALSE,
$b + d = 1$: is equivalent to the probability $p(x) = b$,
$0 < (b + d) < 1$: expresses levels of uncertainty, and
$b + d = 0$: is vacuous (i.e. totally uncertain).

The probability expectation value of a binomial opinion is:

$$\mathrm{E}(\omega_x) = b + au\ . \tag{3}$$

The expression of Eq.(3) is equivalent to the pignistic probability defined in classical belief theory [14], and is based on the principle that the belief mass assigned to the whole frame is split equally among the singletons of the frame. This interpretation requires that the base rate a_x is equal to the proportion of singletons contained in x relative to the frame X.

Binomial opinions can be represented on an equilateral triangle as shown in Fig.1 below. A point inside the triangle represents a (b, d, u) triple. The b,d,u-axes run from one edge to the opposite vertex indicated by the Belief, Disbelief or Uncertainty labels. For example, a strong positive opinion is represented by a point toward the bottom right Belief vertex. The base rate a_x, also called relative atomicity, is shown as a point on the probability base line. The probability expectation value $\mathrm{E}(\omega_x)$ is formed by projecting the opinion onto the base, parallel to the base rate director line. As an example, the opinion $\omega_x = (0.4,\ 0.1,\ 0.5,\ 0.6)$ is shown on the figure.

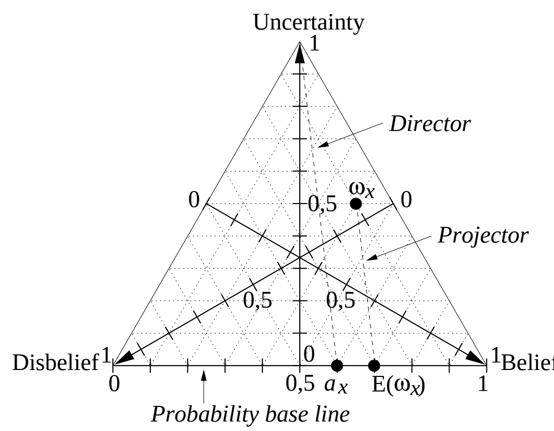

Figure 1: Opinion triangle with example opinion

Uncertainty about probability values can be interpreted as ignorance, or second order uncertainty about the first order probabilities. In this paper, the term "uncertainty" will be used in the sense of "*uncertainty about probability values*". Subjective logic therefore represents a generalisation of traditional probabilistic logic.

2.2 Multinomial Opinions

Let X be a frame, i.e. a set of exhaustive and mutually disjoint propositions x_i. Entity A's multinomial opinion over X is the composite function $\omega_X^A = (\vec{b}, u, \vec{a})$, where $\vec{b}$ is a vector of belief masses over the propositions of X, u is the uncertainty mass, and $\vec{a}$ is a vector of base rate values over the propositions of X. These components satisfy:

$$\vec{b}(x_i), u, \vec{a}(x_i) \in [0,1],\ \ \forall x_i \in X \tag{4}$$

$$u + \sum_{x_i \in X} \vec{b}(x_i) = 1 \tag{5}$$

$$\sum_{x_i \in X} \vec{a}(x_i) = 1 \tag{6}$$

Visualising multinomial opinions is not trivial. Trinomial opinions can be visualised as points inside a triangular pyramid as shown in Fig.2, but the 2D aspect of printed paper and computer monitors makes visualisation of multinomial opinions impractical in general.

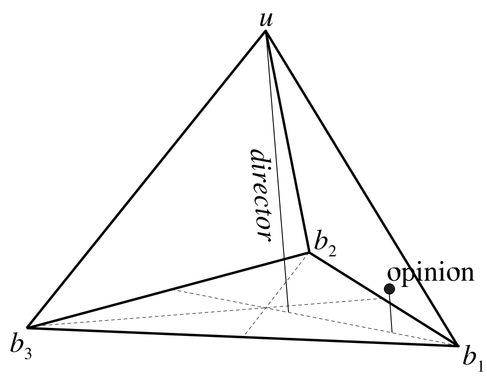

Figure 2: Opinion pyramid with example trinomial opinion

Opinions with dimensions larger than trinomial do not lend themselves to traditional visualisation.

2.3 Dirichlet Representation

A multinomial opinion over a frame $X = \{x_1, \ldots x_k\}$ of cardinality k can be represented as a Dirichlet distribution over the k-component random probability variable $\vec{p}(x_i),\ i = 1 \ldots k$ with sample space $[0,1]^k$, subject to the simple additivity requirement $\sum_{i=1}^{k} \vec{p}(x_i) = 1$.

The Dirichlet distribution with prior captures evidence about the k possible states with k positive real evidence parameters $\vec{r}(x_i),\ i = 1 \ldots k$, each corresponding to one of the possible states. In order to have a compact notation we denote the vector $\vec{p} = \{\vec{p}(x_i) \mid 1 \leq i \leq k\}$ as the k-component probability variable, and the vector $\vec{r} = \{r_i \mid 1 \leq i \leq k\}$ as the k-component evidence variable.

In order to distinguish between the *a priori* base rate, and the *a posteriori* evidence, the notation for the Dirichlet distribution must also include the prior information represented as the base rate vector $\vec{a}$ over the frame. the Dirichlet distribution, denoted as $\mathrm{Dir}(\vec{p} \mid \vec{r}, \vec{a})$ is then expressed as:

$$\mathrm{Dir}(\vec{p} \mid \vec{r}, \vec{a}) =$$

$$\frac{\Gamma\left(\sum_{i=1}^{k}(\vec{r}(x_i)+W\vec{a}(x_i))\right)}{\prod_{i=1}^{k}\Gamma(\vec{r}(x_i)+W\vec{a}(x_i))} \prod_{i=1}^{k} \vec{p}(x_i)^{(\vec{r}(x_i)+W\vec{a}(x_i)-1)}\ ,$$

$$\text{where} \begin{cases} \sum_{i=1}^{k} \vec{p}(x_i) = 1 \\ \\ \vec{p}(x_i) \geq 0, \forall i \end{cases} \quad \text{and} \quad \begin{cases} \sum_{i=1}^{k} \vec{a}(x_i) = 1 \\ \\ \vec{a}(x_i) > 0, \forall i\ . \end{cases} \tag{7}$$

and where W denotes the so-called non-informative prior weight.

It can be noted that Eq.(7) simply is a generalisation of the Beta distribution. The multinomial probability expectation values of the k random probability variables are expressed as:

$$\vec{\mathrm{E}}(x_i) = \mathrm{E}(\vec{p}(x_i) \mid \vec{r}, \vec{a}) = \frac{\vec{r}(x_i) + W\vec{a}(x_i)}{W + \sum_{i=1}^{k} \vec{r}(x_i)}\ . \tag{8}$$

The non-informative prior weight W will normally be set to $W = 2$ when a uniform distribution over a binary frame is assumed. Selecting a larger value for W will result in new observations having less influence over the Dirichlet distribution, and can in fact represent specific *a priori* information provided by a domain expert. It can be noted that it would be unnatural to require a uniform distribution over arbitrary large frames because it would make the sensitivity to new evidence arbitrarily small.

The mapping between a subjective opinion and a Dirichlet PDF is described below.

Theorem 1 Evidence Notation Equivalence
Let $\omega_X = (\vec{b}_X, u_X, \vec{a}_X)$ be an opinion expressed in belief notation, and $\omega = (\vec{r}, \vec{a})$ be an opinion expressed in evidence notation, both over the same frame X. Then the following equivalence holds:

For $u_X \neq 0$:

$$\begin{cases} b_X(x_i) = \frac{r(x_i)}{W + \sum_{i=1}^{k} r(x_i)} \\ \\ u_X = \frac{W}{W + \sum_{i=1}^{k} r(x_i)} \end{cases} \Leftrightarrow \begin{cases} r(x_i) = \frac{W b_X(x_i)}{u_X} \\ \\ 1 = u_X + \sum_{i=1}^{k} b_X(x_i) \end{cases} \tag{9}$$

For $u_X = 0$:

$$\begin{cases} b_X(x_i) = p(x_i) \\ \\ u_X = 0 \end{cases} \Leftrightarrow \begin{cases} r(x_i) = p(x_i) \sum_{i=1}^{k} r(x_i) = p(x_i)\infty \\ \\ 1 = \sum_{i=1}^{k} m(x_i) \end{cases} \tag{10}$$

This theorem can be derived by assuming that corresponding subjective opinions and Dirichlet PDFs have equal probability expectation values [8].

In the case where $u_X = 0$ a few additional comments can be made. If $p(x_i) = 1$ for a particular proposition x_i, then $r(x_i) = \infty$ and all the other evidence parameters are finite. If $p(x_i) = 1/k$ for all $i = 1 \ldots k$, then all the evidence parameters are all equally infinite. As already mentioned, the prior non-informative weight W is a constant that is normally set to $W = 2$.

3 Fusion of Multinomial Opinions

In many situations there will be multiple sources of evidence, and fusion can be used to combine evidence from different sources.

In order to provide an interpretation of fusion in subjective logic it is useful to consider a process that is observed by two sensors. A distinction can be made between two cases.

1. The two sensors observe the process during disjoint time periods. In this case the observations are independent, and it is natural to simply add the observations from the two sensors, and the resulting fusion is called *cumulative fusion*.

2. The two sensors observe the process during the same time period. In this case the observations are dependent, and it is natural to take the average of the observations by the two sensors, and the resulting fusion is called *averaging fusion*.

3.1 Cumulative Fusion

Assume a frame X containing k elements. Then assume two observers A and B who have independent opinions over the frame X. This can for example result from having observed the outcomes of a process over two separate time periods.

Let the two observers' respective opinions be expressed as $\omega_X^A = (\vec{b}_X^A, u_X^A, \vec{a}_X^A)$ and $\omega_X^B = (\vec{b}_X^B, u_X^B, \vec{a}_X^B)$.

The cumulative fusion of these two bodies of evidence is denoted as $\omega_X^{A\diamond B} = \omega_X^A \oplus \omega_X^B$. The symbol "$\diamond$" denotes the merging of two observers A and B who hold independent opinions about the frame X into a single imaginary observer denoted as $A \diamond B$. The mathematical expressions for cumulative fusion is described below.

Definition 1 The Cumulative Fusion Operator

Let ω_X^A and ω_X^B be opinions respectively held by agents A and B over the same frame $X = \{x_i \mid i = 1, \cdots, k\}$. The opinion $\omega_X^{A\diamond B} = (\vec{b}_X^{A\diamond B}, u_X^{A\diamond B}, \vec{a}_X^{A\diamond B})$ is the cumulatively fused opinion of ω_X^A and ω_X^B. The opinion components are expressed as:

Case I: *For* $u_X^A \neq 0 \;\vee\; u_X^B \neq 0$:

$$\begin{cases} b_{x_i}^{A\diamond B} = \frac{b_{x_i}^A u_X^B + b_{x_i}^B u_X^A}{u_X^A + u_X^B - u_X^A u_X^B} \\ u_X^{A\diamond B} = \frac{u_X^A u_X^B}{u_X^A + u_X^B - u_X^A u_X^B} \end{cases} \tag{11}$$

Case II: *For* $u_X^A = 0 \;\wedge\; u_X^B = 0$:

$$\begin{cases} b_{x_i}^{A\diamond B} = \gamma\, b_{x_i}^A + (1-\gamma) b_{x_i}^B \\ u_X^{A\diamond B} = 0 \end{cases} \tag{12}$$

$$\textit{where} \quad \gamma = \lim_{\substack{u_X^A \to 0 \\ u_X^B \to 0}} \frac{u_X^B}{u_X^A + u_X^B}$$

The opinion $\omega_X^{A\diamond B}$ represents the fusion of independent opinions of observers A and B about the same frame X.

The cumulative fusion operator is equivalent to *a posteriori* updating of Dirichlet distributions. Its derivation is based on the addition of Dirichlet parameters combined with the bijective mapping between multinomial opinions and the Dirichlet distribution as described in Theorem 1.

It can be verified that the cumulative fusion operator is commutative, associative and non-idempotent. In Case II of Def.1, the associativity depends on the preservation of relative weights of intermediate results, which requires the additional weight variable γ. In this case, the cumulative operator is equivalent to the weighted average of probabilities.

The cumulative fusion operator represents a generalisation of the consensus operator [2, 4] which emerges directly from Def.1 by assuming a binary frame.

3.2 Averaging Fusion

Assume a frame X containing k elements. Then, assume two observers A and B who have dependent opinions over the frame X. This can for example result from observing the outcomes of a process over the same time periods.

Let the two observers' respective opinions be expressed as $\omega_X^A = (\vec{b}_X^A, u_X^A, \vec{a}_X^A)$ and $\omega_X^B = (\vec{b}_X^B, u_X^B, \vec{a}_X^B)$.

The averaging fusion of these two bodies of evidence is denoted as $\omega_X^{A\underline{\diamond}B} = \omega_X^A \underline{\oplus} \omega_X^B$. The symbol "$\underline{\diamond}$" denotes the merging of two observers A and B who hold dependent opinions about the frame X into a single imaginary observer denoted as $A\underline{\diamond}B$. The mathematical expressions for averaging fusion is described below.

Definition 2 The Averaging Fusion Operator

Let ω_X^A and ω_X^B be opinions respectively held by agents A and B over the same frame $X = \{x_i \mid i = 1, \cdots, k\}$. The opinion $\omega_X^{A\underline{\diamond}B}$ is the averaged opinion of ω_X^A and ω_X^B. Then the opinion components are expressed as:

Case I: *For* $u_X^A \neq 0 \;\vee\; u_X^B \neq 0$:

$$\begin{cases} b_{x_i}^{A\underline{\diamond}B} = \frac{b_{x_i}^A u_X^B + b_{x_i}^B u_X^A}{u_X^A + u_X^B} \\ u_X^{A\underline{\diamond}B} = \frac{2 u_X^A u_X^B}{u_X^A + u_X^B} \end{cases} \tag{13}$$

Case II: *For* $u_X^A = 0 \;\wedge\; u_X^B = 0$:

$$\begin{cases} b_{x_i}^{A\diamond B} = \gamma\, b_{x_i}^A + (1-\gamma) b_{x_i}^B \\ u_X^{A\diamond B} = 0 \end{cases} \tag{14}$$

$$\textit{where} \quad \gamma = \lim_{\substack{u_X^A \to 0 \\ u_X^B \to 0}} \frac{u_X^B}{u_X^A + u_X^B}$$

The opinion $\omega_X^{A\underline{\diamond}B}$ represents the combination of the dependent opinions of observers A and B about the same frame X.

The averaging fusion operator is equivalent to averaging the evidence of Dirichlet distributions. Its derivation is based on the average of Dirichlet parameters combined the bijective mapping between multinomial opinions and the Dirichlet distribution as described in Theorem 1.

It can be verified that the averaging fusion operator is commutative and idempotent, but not associative.

The averaging fusion operator represents a generalisation of the consensus operator for dependent opinions described in [11].

It can be noted that partially dependent opinions can be fused using a combination of the cumulative and averaging fusion operators [11], but this requires an additional parameter to determine the degree of dependence between the two argument opinions.

4 Fission of Multinomial Opinions

The principle of opinion fission is the opposite operation to opinion fusion. This section describes the fission operator corresponding to the cumulative fusion operator that was described in the previous section.

There are in general an infinite number of ways to split an opinion. The principle followed here is to require an auxiliary fission parameter ϕ to determine how the argument opinion shall be split. As such, opinion fission is a binary operator, i.e. it takes two input arguments which are the fission parameter and the opinion to be split.

4.1 Opinion Fission

Assume a frame X containing k elements. Assume that the opinion $\omega_X^C = (\vec{b}, u, \vec{a})$ over X is held by a real or imaginary entity C.

The fission of ω_X^C consists of splitting ω_X^C into two opinions $\omega_X^{C_1}$ and $\omega_X^{C_2}$ assigned to the (real or imaginary) agents C_1 and C_2 so that $\omega_X^C = \omega_X^{C_1} \oplus \omega_X^{C_2}$. The parameter ϕ determines the relative proportion of evidence that each new opinion gets. Fission of ω_X^C results in two opinions denoted as $\phi \oslash \omega_X^C = \omega_X^{C_1}$ and $\overline{\phi} \oslash \omega_X^C = \omega_X^{C_2}$. The mathematical expressions for cumulative fission are constructed as follows.

The mapping of an opinion $\omega_X^C = (\vec{b}, u, \vec{a})$ to Dirichlet evidence parameters $\text{Dir}(\vec{r}_X^{\,C}, \vec{a}_X^{\,C})$ according to Eq.(9) and Eq.(10), and linear splitting into two parts $\text{Dir}(\vec{r}_X^{\,C_1}, \vec{a}_X^{\,C_1})$ and $\text{Dir}(\vec{r}_X^{\,C_2}, \vec{a}_X^{\,C_2})$ as a function of the fission parameter ϕ produces:

$$\text{Dir}(\vec{r}_X^{\,C_1}, \vec{a}_X^{\,C_1}) : \begin{cases} \vec{r}_X^{\,C_1} = \frac{\phi W \vec{b}}{u} \\ \\ \vec{a}_X^{\,C_1} = \vec{a} \end{cases} \tag{15}$$

$$\text{Dir}(\vec{r}_X^{\,C_2}, \vec{a}_X^{\,C_2}) : \begin{cases} \vec{r}_X^{\,C_2} = \frac{(1-\phi) W \vec{b}}{u} \\ \\ \vec{a}_X^{\,C_2} = \vec{a} \end{cases} \tag{16}$$

where W denotes the non-informative prior weight.

The reverse mapping of these evidence parameters into two separate opinions according to Eq.(9) and Eq.(10) produces the expressions of Def.3 below. As would be expected, the base rate is not affected by fission.

Definition 3 The Fission Operator

Let ω_X^C be an opinion over the frame X. The cumulative fission of ω_X^C based on the fission parameter ϕ where $0 < \phi < 1$ produces two opinions $\omega_X^{C_1}$ and $\omega_X^{C_2}$ defined by:

$$\omega_X^{C_1} : \begin{cases} \vec{b}_X^{\,C_1} = \frac{\phi \vec{b}}{u + \phi \sum_{i=1}^{k} b(x_i)} \\ \\ u_X^{C_1} = \frac{u}{u + \phi \sum_{i=1}^{k} b(x_i)} \\ \\ \vec{a}_X^{\,C_1} = \vec{a} \end{cases} \tag{17}$$

$$\omega_X^{C_2} : \begin{cases} \vec{b}_X^{\,C_2} = \frac{(1-\phi) \vec{b}}{u + (1-\phi) \sum_{i=1}^{k} b(x_i)} \\ \\ u_X^{C_2} = \frac{u}{u + (1-\phi) \sum_{i=1}^{k} b(x_i)} \\ \\ \vec{a}_X^{\,C_2} = \vec{a} \end{cases} \tag{18}$$

By using the symbol '$\oslash$' to designate this operator, we define:

$$\omega_X^{C_1} = \phi \oslash \omega_X^C \tag{19}$$
$$\omega_X^{C_2} = \overline{\phi} \oslash \omega_X^C \tag{20}$$

In case $[C : X]$ represents a trust edge where X represents a target entity, it can also be assumed that the entity X is being split, which leads to the same mathematical expression as Eq.(17) and Eq.(18), but with the following notation:

$$\omega_{X_1}^C = \phi \oslash \omega_X^C = \omega_X^{C_1} \tag{21}$$
$$\omega_{X_2}^C = \overline{\phi} \oslash \omega_X^C = \omega_X^{C_2} \tag{22}$$

It can be verified that $\omega_X^{C_1} \oplus \omega_X^{C_2} = \omega_X^C$, as expected. In case $\phi = 0$ or $\phi = 1$ one of the resulting opinions will be vacuous, and the other equal to the argument opinion.

4.2 Other Types of Opinion Fission

4.2.1 Fission of Average

Assume a frame X containing k elements. Then assume that the opinion $\omega_X^A = (\vec{b}, u, \vec{a})$ over X is held by a real or imaginary entity A.

Average fission of ω_X^A consists of splitting ω_X^A into two opinions $\omega_X^{A_1}$ and $\omega_X^{A_2}$ assigned to the (real or imaginary) agents A_1 and A_2 so that $\omega_X^A = \omega_X^{A_1} \underline{\oplus} \omega_X^{A_2}$.

It turns out that averaging fission of an opinion trivially produces two opinions that are equal to the argument opinion. This is because the average fusion of two equal opinions necessarily produces the same opinion. It would be meaningless to define this operator formally because it is trivial, and because it does not provide a useful model for any interesting practical situation.

4.2.2 Unfusion of Opinions

Assume a frame X containing k elements. Then assume two observers A and B whose opinions have been fused into $\omega_X^{A\diamond B} = \omega_X^C = (\vec{b}_X^C, u_X^C, \vec{a}_X^C)$, and assume that entity B's contributing opinion $\omega_X^B = (\vec{b}_X^B, u_X^B, \vec{a}_X^B)$ is known.

The unfusion of these two bodies of evidence is denoted as $\omega_X^{C\overline{\diamond}B} = \omega_X^A = \omega_X^C \ominus \omega_X^B$, which represents entity A's contributing opinion. This is different from, but still related to fission. Unfusion is described in [7].

5 Trust Network Canonicalisation through Opinion Fission

This section describes an example of applying opinion fission to transform a trust network of dependent trust paths into a trust network of independent trust paths.

Trust networks can be modelled with subjective logic, where a trust relationship between two nodes is represented as a binomial opinion. Binomial opinions are expressed as $\omega_x^A = (b, d, u, a)$ where d denotes disbelief in statement x. When the statement for example says x : *"'David is honest and reliable"'*, then the opinion can be interpreted as evaluation trust in David. More specifically, the trust target is David, and the trust scope is σ : *"To be honest and reliable"*, so that $x \equiv D(\sigma)$. The opinion can be denoted with explicit attributes as $\omega_{D(\sigma)}^A$, but the trust scope can be omitted when it can be implicitly assumed.

As an example, let us assume that Alice needs to get her car serviced, and that she asks Bob to recommend a good car mechanic. When Bob recommends David, Alice would like to get a second opinion, so she asks Claire for her opinion about David. The trust scope in this case can be expressed as σ : *"To be a competent car mechanic"*. This situation is illustrated in Fig. 3 below where the indexes on arrows indicates the order in which they are formed.

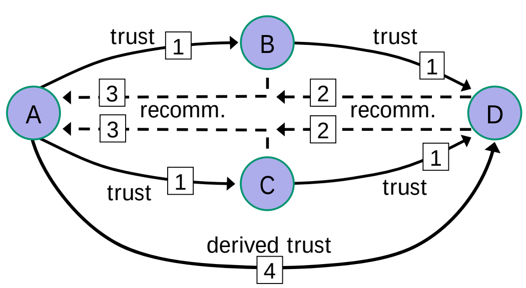

Figure 3: Deriving trust from parallel transitive chains

When trust and referrals are expressed as subjective opinions, each transitive trust path Alice→Bob→David, and Alice→Claire→David can be computed with the *transitivity operator*[2], where the idea is that the referrals from Bob and Claire are discounted as a function Alice's trust in Bob and Claire respectively. Finally the two paths can be combined using the cumulative fusion operator.

The transitivity operator is used to compute trust along a chain of trust edges. Assume two agents A and B where A has referral trust in B, denoted by ω_B^A, for the purpose of judging the functional or referral trustworthiness of C. In addition B has functional or referral trust in C, denoted by ω_C^B. Agent A can then derive her trust in C by discounting B's trust in C with A's trust in B, denoted by $\omega_C^{A:B}$. Transitivity is denoted by the symbol '$\otimes$', and defined as:

$$\omega_C^{A:B} = \omega_B^A \otimes \omega_C^B \text{ where } \begin{cases} b_C^{A:B} = b_B^A b_C^B \\ d_C^{A:B} = b_B^A d_C^B \\ u_C^{A:B} = d_B^A + u_B^A + b_B^A u_C^B \\ a_C^{A:B} = a_C^B \ . \end{cases} \tag{23}$$

The effect of transitivity is a general increase in uncertainty, and not necessarily an increase in disbelief [3].

The operators for modelling trust networks are used in subjective logic [2, 6], and semantic constraints must be satisfied in order for the computational transitive trust derivation to be meaningful [9].

A trust relationship between A and B is denoted as [A,B]. The transitivity of two arcs is denoted as ":" and the fusion of two parallel paths is denoted as "$\diamond$". The trust graph of Fig.3 can then be expressed as:

$$[A, D] = ([A, B] : [B, D]) \diamond ([A, C] : [C, D]) \tag{24}$$

The transitivity operator for opinions is denoted as "$\otimes$" and the fusion operator as "$\oplus$". The computational trust expression corresponding to Eq.(24) is then:

$$\omega_D^A = (\omega_B^A \otimes \omega_D^B) \oplus (\omega_C^A \otimes \omega_D^C) \tag{25}$$

The existence of a dependent edge in a trust graph is recognised by multiple instances of the same edge in the trust network expression. Edge splitting is a method to isolate independent trust edges. This is achieved by splitting a given dependent edge into as many different edges as there are different instances of the same edge in the trust network expression. Edge splitting is achieved by splitting one of the nodes in the dependent edge into different nodes so that each independent edge is connected to a different node.

A general directed trust graph is based on directed trust edges between pairs of nodes. It is desirable not to put any restrictions on the possible trust edges except that they should not be cyclic. This means that the set of possible trust paths from a given source X to a given target Y can contain dependent paths. An example of a trust network with dependent paths is shown on the left-hand side of Fig.4, and the result of edge splitting is shown on the right-hand side.

Below we will show how opinion fission can be used for practical edge splitting.

[2]Also called the discounting operator

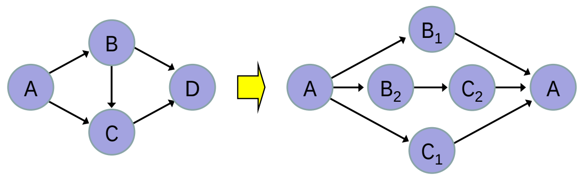

Figure 4: Edge splitting of dependent trust network

The non-canonical expression for the left-hand side trust network of Fig.4 is:

$$\begin{aligned}[A,D] = \quad & ([A,B]:[B,D]) \\ & \diamond ([A,C]:[C,D]) \\ & \diamond ([A,B]:[B,C]:[C,D])\end{aligned} \tag{26}$$

In this expression the edges $[A,B]$ and $[C,D]$ appear twice. Edge splitting in this example consists of splitting the node B into B_1 and B_2, and the node C into C_1 and C_2. This produces the right-hand side trust network in Fig.4 with canonical expression:

$$\begin{aligned}[A,D] = \quad & ([A,B_1]:[B_1,D]) \\ & \diamond ([A,C_1]:[C_1,D]) \\ & \diamond ([A,B_2]:[B_2,C_2]:[C_2,D])\end{aligned} \tag{27}$$

Edge splitting must be translated into opinion splitting in order to apply subjective logic. The principle for opinions splitting is to separate the opinion on the dependent edge into two independent opinions that when cumulatively fused produce the original opinion. This is opinion fission as described in Sec.4, and depends on the fission factor ϕ that determines the proportion of evidence assigned to each independent opinion part. The binomial expressions for the fission of a trust opinion such as $\omega_D^C = (b,d,u,a)$ is expressed as:

$$\omega_D^{C_1} : \begin{cases} b_D^{C_1} = \frac{\phi b}{\phi(b+d)+u} \\ d_D^{C_1} = \frac{\phi d}{\phi(b+d)+u} \\ u_D^{C_1} = \frac{u}{\phi(b+d)+u} \\ a_D^{C_1} = a \end{cases} \tag{28}$$

$$\omega_D^{C_2} : \begin{cases} b_D^{C_2} = \frac{(1-\phi)b}{(1-\phi)(b+d)+u} \\ d_D^{C_2} = \frac{(1-\phi)d}{(1-\phi)(b+d)+u} \\ u_D^{C_2} = \frac{u}{(1-\phi)(b+d)+u} \\ a_D^{C_2} = a \end{cases} \tag{29}$$

When deriving trust values from the canonicalised trust network of Eq.(26) we are interested in knowing its certainty level as compared with a simplified network, as described in [9]. The computational trust expression of Eq.(27) is:

$$\begin{aligned}\omega_D^A = \quad & (\omega_{B_1}^A \otimes \omega_D^B) \\ & \oplus (\omega_{B_2}^A \otimes \omega_C^B \otimes \omega_D^{C_2}) \\ & \oplus (\omega_C^A \otimes \omega_D^{C_1})\end{aligned} \tag{30}$$

We are interested in the expression for the uncertainty of ω_D^A corresponding to trust expression of Eq.(27). Since edge splitting introduces parameters for splitting opinions, the uncertainty will be a function of these parameters. By using Eq.(23) the expressions for the uncertainty in the trust paths of Eq.(27) can be derived as:

$$\begin{aligned} u_D^{A:B_1} &= d_{B_1}^A + u_{B_1}^A + b_{B_1}^A u_D^{B_1} \\ u_D^{A:C_1} &= d_{C_1}^A + u_{C_1}^A + b_{C_1}^A u_D^{C_1} \\ u_D^{A:B_2:C_2} &= b_{B_2}^A d_{C_2}^{B_2} + d_{B_2}^A + u_{B_2}^A \\ &\quad + b_{B_2}^A u_D^{B_2} \quad + b_{B_2}^A b_{C_2}^{B_2} u_D^{C_2} \end{aligned} \tag{31}$$

By using Eq.(11) and Eq.(31), the expression for the uncertainty in the trust network of Eq.(27) can be derived as:

$$u_D^A = \frac{u_D^{A:B_1} u_D^{A:C_1} u_D^{A:B_2:C_2}}{u_D^{A:B_1} u_D^{A:C_1} + u_D^{A:B_1} u_D^{A:B_2:C_2} + u_D^{A:C_1} u_D^{A:B_2:C_2} - 2 u_D^{A:B_1} u_D^{A:C_1} u_D^{A:B_2:C_2}} \tag{32}$$

By using Eq.(28), Eq.(31) and Eq.(32), the uncertainty value of the derived trust ω_D^A according to the edge splitting principle can be computed. This value depends on the edge opinions and on the two fission parameters ϕ_B^A and ϕ_D^C.

As an example the opinion values will be set to:

$$\omega_B^A = \omega_D^B = \omega_C^A = \omega_D^C = \omega_C^B = (0.9, 0.0, 0.1, 0.5) \tag{33}$$

The computed trust values for the two possible simplified graphs are:

$$(\omega_B^A \otimes \omega_D^B) \oplus (\omega_C^A \otimes \omega_D^C) = (0.895, 0, 0.105, 0.5) \tag{34}$$

$$\omega_B^A \otimes \omega_C^B \otimes \omega_D^C = (0.729, 0, 0.271, 0.5) \tag{35}$$

The uncertainty level u_D^A when combining these two graphs through edge splitting as a function of ϕ_B^A and ϕ_D^C is shown in Fig.5

The result of combining dependent parallel paths is that the produced uncertainty is lower than it should be because too much evidence is taken into account during opinion fusion. The result of removing arbitrary edges from a network in order to avoid dependence is the risk that too little evidence is taken into account so that the uncertainty is higher than it should be. The ideal solution is to split the dependent paths into separate independent paths in such a way that the uncertainty is minimized. This precisely avoids the risk of producing too low uncertainty by including dependent edges, as well as the risk of producing too high uncertainty by removing genuine trust edges.

The conclusion which can be drawn from this in the example above is that the optimal value for the fission parameters are $\phi_B^A = \phi_D^C = 1$ because that is when the overall

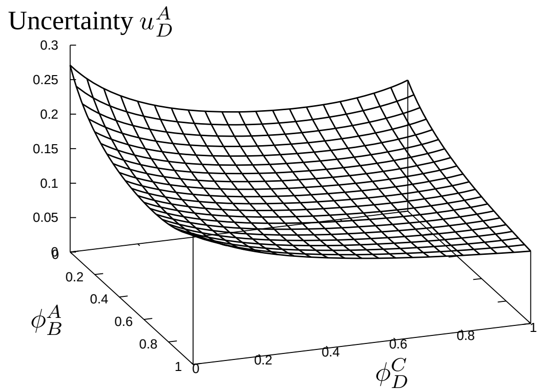

Figure 5: Uncertainty u_D^A as a function of ϕ_B^A and ϕ_D^C

network uncertainty is at its lowest while still avoiding dependent paths. In fact the uncertainty can be evaluated to $u_D^A = 0.105$.

These optimal fission parameters are used when applying Eq.(17). This produces the trust network simplification of Eq.(34) where the edge $[B, C]$ is completely removed from the left-hand side graph of Fig.4.

The fission parameters that produce the highest uncertainty is when $\phi_B^A = \phi_D^C = 0$, resulting in $u_D^A = 0.271$. This also avoids dependent paths but results in the inefficient trust network of Eq.(35) where the edges $[A, C]$ and $[B, D]$, which are the most certain and efficient trust paths, are completely removed from the left-hand side graph of Fig.4. In other words, given the edge opinion values used in this example, $([A, B] : [B, C] : [C, D])$ is the least certain path of the left-hand side graph of Fig.4.

In general, a canonical network derived from a network of dependent paths is when the uncertainty has been minimized while at the same time avoiding dependent paths through edge splitting. Fission of opinions is the operator needed for edge splitting in subjective logic. In brief, opinion fission makes it possible to canonicalise trust networks of dependent paths.

6 Conclusion

The principle of belief fusion is used in numerous applications. The principle of belief fission, which can be considered the inverse of fusion, is less commonly used. However, there are situations where fission can be useful. In this paper we have described the fission operator corresponding to cumulative fusion in subjective logic. Opinion fission can for example be applied for canonicalisation of trust networks with dependent paths,

References

[1] A. Jøsang. Artificial reasoning with subjective logic. In Abhaya Nayak and Maurice Pagnucco, editors, *Proceedings of the 2nd Australian Workshop on Commonsense Reasoning*, Perth, December 1997. Australian Computer Society.

[2] A. Jøsang. A Logic for Uncertain Probabilities. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9(3):279–311, June 2001.

[3] A. Jøsang. Subjective Evidential Reasoning. In *Proceedings of the International Conference on Information Processing and Management of Uncertainty (IPMU2002)*, Annecy, France, July 2002.

[4] A. Jøsang. The Consensus Operator for Combining Beliefs. *Artificial Intelligence Journal*, 142(1–2):157–170, October 2002.

[5] A. Jøsang. Probabilistic Logic Under Uncertainty. In *The Proceedings of Computing: The Australian Theory Symposium (CATS2007), CRPIT Volume 65*, Ballarat, Australia, January 2007.

[6] A. Jøsang. Conditional Reasoning with Subjective Logic. *Journal of Multiple-Valued Logic and Soft Computing*, 15(1):5–38, 2008.

[7] A. Jøsang. Cumulative and Averaging Unfusion of Beliefs. In *The Proceedings of the International Conference on Information Processing and Management of Uncertainty (IPMU2008)*, Malaga, June 2008.

[8] A. Jøsang, J. Diaz, and M. Rifqi. Cumulative and Averaging Fusion of Beliefs (in press). *Information Fusion Journal*, 00(00):00–00, 2009.

[9] A. Jøsang, R. Hayward, and S. Pope. Trust Network Analysis with Subjective Logic. In *Proceedings of the 29^{th} Australasian Computer Science Conference (ACSC2006), CRPIT Volume 48*, Hobart, Australia, January 2006.

[10] A. Jøsang and D. McAnally. Multiplication and Comultiplication of Beliefs. *International Journal of Approximate Reasoning*, 38(1):19–51, 2004.

[11] A. Jøsang, S. Pope, and S. Marsh. Exploring Different Types of Trust Propagation. In *Proceedings of the 4th International Conference on Trust Management (iTrust)*, Pisa, May 2006.

[12] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[13] Florentin Smarandache. An In-Depth Look at Information Fusion Rules & the Unification of Fusion Theories. *Computing Research Repository (CoRR), Cornell University arXiv*, cs.OH/0410033, 2004.

[14] Ph. Smets and R. Kennes. The transferable belief model. *Artificial Intelligence*, 66:191–234, 1994.