Multi-Agent Preference Combination using Subjective Logic *

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Abstract. Situations where agents with different preferences try to agree on a single choice occur frequently. This must not be confused with fusion of evidence from different agents to determine the most likely correct hypothesis or actual event. Multi-agent preference combination assumes that each agent has already made up her mind, and is about determining the most acceptable decision or choice for the group of agents. This paper formalises and expresses preferences for a state variable in the form of subjective opinions over a frame, and then applies the belief constraint operator of subjective logic as a method for merging preferences of multiple agents into a single preference for the whole group. The model is expressive and flexible, and produces perfectly intuitive results.

1 Introduction

In situations where two or more agents need to make a selection among alternatives their preferences can be combined to derive the selection that best satisfies all agents. For example, person *A* might say: *"I like broccoli, but I dislike celery"* and person *B* might say: *"I like both of them"*. Assume that person *A* and person *B* are cooking a meal together, and they want to decide whether to include a particular ingredient, then inclusion of broccoli is obvious because both like it. The inclusion of celery however is unclear because *A* and *B* have opposite preferences. In this case, cultural norms would play a role, as e.g. politeness, or the relative status or authority of *A* and *B*. If the preferences had been expressed as hard constraints, i.e. if *A* said *"For me celery is out of the question"* and *B* said *"For me celery is mandatory"* then it would seem that they simply can not cook the meal together.

In addition to having both positive and negative preferences, it is natural to also express indifference, stating that we neither have a positive nor a negative preference over a specific object. By continuing the above cooking example, person *A* might say: *"I'm indifferent to carrots"* and person *B* might say: *"I like carrots"*. Then the inclusion of carrots seems natural because *B* likes it and *A* is indifferent, i.e. the indifference of *A* lets *B* decide.

In this paper, we investigate how subjective opinions can be used to express preferences in general. In particular we analyse the applicability of the belief constraint operator, which in fact is an extension of Dempster's rule [11], for combining preferences of multiple agents about the same choice variable. The intuitive motivation behind our study is that preference can be represented as belief and that indifference can

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be represented as uncertainty/uncommitted belief. Positive and negative preferences are considered as symmetric concepts, so they can be represented in the same way and combined using the same operator. A totally uncertain opinion has no influence and thereby represents the neutral element.

Our study focuses on multi-agent preferences over a single variable represented as the possible states in a frame. In future research we plane to extend our study by analysing multi-agent preferences over multiple variables, i.e. over multiple frames.

2 Related Work

The work presented here extends the fundamental idea of bipolar preferences wherein agents can express positive and/or negative preferences for a particular choice. Publications that focus on this principle include [1, 2]. In [2] the soft constraint formalism based on semirings is used to model negative preferences, and a separate algebraic structure is used to model positive preferences. To model bipolar problems these two structures are linked and the highest negative preference is set to coincide with the lowest positive preference to model indifference. A combination operator is defined between positive and negative preferences to model preference compensation. In [1] uncertainty is modelled by the presence of uncontrollable variables. This means that the value of such variables is decided by ”Nature” or by some other agent. A solution is then only assigned to controllable variables, not to uncontrollable ones. A typical example of uncontrollable variable, in the context of satellite scheduling, is a variable representing the time when clouds will disappear. Although the value for such uncontrollable variables can not be chosen directly, the plausibility of the values in their domains can be expressed. The plausibility information, which is not bipolar, is expressed by probability distributions.

Possibility theory applied to preference combination has also be investigated e.g. in [3, 10]. The main idea in [10] is to represent preferences (or respective certainty degrees) as a possibility distribution over labelings (choices). Such a distribution then induces a possibility measure and a necessity measures over constraints. With this formalism constraints can be expressed as bounds on possibility or necessity defining a set of possibility distribution among labelings. One can then define a set of ”most possible” labelings satisfying these bounds. The main idea in [3] formalizes the notion of possibilistic constraint satisfaction problems (CSP) that allows the modeling of uncertainly satisfied constraints. Necessity-valued constraints then express the respective certainty degrees of each constrain.

3 Subjective Logic Basics

Our model follows the same ideas as those of the models mentioned in Sec.2 above, namely to express and combine positive and negative preferences as well as indifference/uncertainty. However, the model of subjective logic is quite different from the models of the previous approaches. Subjective opinions simultaneously express positive and negative preferences as well as indifference/uncertainty, thereby avoiding the complexity of integrating multiple formalisms to express the various aspects of preferences. A subjective opinion expresses preferences over possible states of a frame, which constitutes a multi-polar preference model. Preference combination based on subjective logic can be interpreted as a form of majority voting where the weight of each agent's vote is inversely proportional to the indifference/uncertainty of that agent's preference. A totally indifferent/uncertain opinion then carries no weight and represents the neutral element. Subjective logic thus provides the basis for a very general preference combination model.

A subjective opinion is a composite function that consists of belief masses, uncommitted belief mass (uncertainty) and base rates, and that can also indicate the belief source or owner. The main idea behind our study is to interpret belief mass as preference, and uncommitted belief mass as indifference. Base rates can be interpreted as average preferences in the population.

3.1 The Reduced Powerset of Frames

A state space of mutually exclusive states is called a *"frame of discernment"* or *"frame"* for short. Let X be a frame of cardinality k. In this study the possible states in the frame represent the preference variable, i.e. agents can express preferences over states in the frame. It is assumed that the goal of the multi-agent preference combination is to select a single state from the frame as the most preferred state for the group of agents.

Belief mass (preference) is distributed over the reduced powerset of the frame denoted as $\mathscr{R}(X)$. More precisely, the reduced powerset $\mathscr{R}(X)$ is defined as:

$$\mathscr{R}(X) = 2^X \setminus \{X, \emptyset\} = \{x_i \mid i = 1 \ldots k,\ x_i \subset X\}\ , \tag{1}$$

which means that all proper subsets of X are elements of $\mathscr{R}(X)$, but X itself is not in $\mathscr{R}(X)$. The emptyset $\emptyset$ is also not considered to be a proper element of $\mathscr{R}(X)$.

An agent can thus express preference for singleton states as well as for subsets containing multiple singletons. Assigning belief mass to a singleton or to a subset is interpreted as positive preference for that singleton or subset, and as negative preference for their complements in the frame. This can be considered as model for expressing multi-polar preferences, and thereby extends the idea of bipolar preferences described in [1, 2]. By not assigning all the belief mass to singletons or subsets the agent can express indifference, i.e. the level of indifference is equal to the amount of uncommitted belief mass.

The cardinality of $\mathscr{R}(X)$ is computed as $\kappa = |\mathscr{R}(X)| = (2^k - 2)$, i.e. the reduced powerset has only $(2^k - 2)$ elements because it is assumed that X and $\emptyset$ are not elements of $\mathscr{R}(X)$. The first k elements of $\mathscr{R}(X)$ have the same index as the corresponding singletons of X. The remaining elements of $\mathscr{R}(X)$ are grouped in classes according to the number j of singletons they contain. The class is then called "class j", meaning that all elements belonging to class j have cardinality j. The actual number of elements belonging to each class is determined by the Choose Function $\mathrm{C}(\kappa, j)$ which dictates the number of ways that j out of κ singletons can be chosen. The Choose Function, equivalent to the binomial coefficient, is defined as:

$$\mathrm{C}(\kappa, j) = \binom{\kappa}{j} = \frac{\kappa!}{(\kappa - j)!\ j!}\ . \tag{2}$$

Within a class each element is indexed after the order of the lowest indexed singletons from X that it contains. For example in case of the frame $X = \{x_1, x_2, x_3, x_4\}$, class 1 has 4 elements, and class 2 has 6 elements, which together makes 10 elements. The first element of class 3 therefore has index 11. Table 1 defines the index and class of all the elements of $\mathscr{R}(X)$ according to this scheme in case of $|X| = 4$.

		Singleton selection per element													
Singletons	x_4				$*$			$*$		$*$	$*$		$*$	$*$	$*$
	x_3			$*$			$*$		$*$		$*$	$*$		$*$	$*$
	x_2		$*$			$*$			$*$	$*$		$*$	$*$		$*$
	x_1	$*$				$*$	$*$	$*$				$*$	$*$	$*$	
Element Index:		1	2	3	4	5	6	7	8	9	10	11	12	13	14
Element Class:		1				2						3			

Table 1. Index and class of elements of $\mathscr{R}(X)$ in case $|X| = 4$.

Class-1 elements are the original singletons from X, i.e. we can state the equivalence $(x_i \in X) \Leftrightarrow (x_i$ is a class-1 element in $\mathscr{R}(X))$. The frame $X = \{x_1, x_2, x_3, x_4\}$ does not figure as an element of $\mathscr{R}(X)$ in Table 1 because excluding X is precisely what makes $\mathscr{R}(X)$ a reduced powerset.

3.2 Belief Distribution over the Reduced Powerset

Subjective logic allows various types of belief mass distributions over a frame X, which in this study is interpreted as a *preference mass distribution*. The distribution vector can be additive (i.e. sum = 1) or sub-additive (i.e. sum $<$ 1), and it can be restricted to elements of X or it can include proper subsets of X. A belief mass on a proper subset of X is equivalent to a belief mass on an element of $\mathscr{R}(X)$. In case of sub-additive belief mass distribution, (i.e. sum $<$ 1) the complement is defined as uncommitted belief mass, which in this study is interpreted as *indifference mass*. An additive belief mass distribution means that there is no uncommitted mass. In general, the belief vector $\vec{b}_X$ specifies the distribution of belief masses over the elements of $\mathscr{R}(X)$, and the uncommitted mass denoted as u_X represents the uncertainty about the probability expectation value, as will be explained below. The sub-additivity of the belief vector and the complement property of the uncommitted mass (uncertainty) are expressed by Eq.(3) and Eq.(4) below:

$$\text{Belief sub-additivity: } \sum_{x_i \in \mathscr{R}(X)} \vec{b}_X(x_i) \leq 1 \; , \;\; \vec{b}_X(x_i) \in [0,1] \tag{3}$$

$$\text{Belief and uncertainty additivity: } u_X + \sum_{x_i \in \mathscr{R}(X)} \vec{b}_X(x_i) = 1 \; , \;\; \vec{b}_X(x_i), u_X \in [0,1] \; . \tag{4}$$

An element $x_i \in \mathscr{R}(X)$ is a *focal element* when its belief mass is non-zero, i.e. when $\vec{b}_X(x_i) > 0$. The frame X is not considered to be a focal element, even when $u_X > 0$.

3.3 Base Rates over Frames

The concept of base rates is central in the theory of probability, and also in subjective logic. Given a frame of cardinality k, the default base rate of for each singleton in the frame is $1/k$, and the default base rate of a subset consisting of n singletons is n/k. In other words, the default base rate of a subset is equal to the number of singletons in the subset relative to the cardinality of the whole frame. A subset also has default *relative base rates* with respect to every other fully or partly overlapping subset of the frame.

In practical situations base rates are normally different from the default values. When modelling preferences, base rates can express average preferences in the population. The base rate function is a vector denoted as $\vec{a}$ so that $\vec{a}(x_i)$ represents the base rate of element $x_i \in X$. The base rate function is formally defined below.

Definition 1 (Base Rate Function). Let X be a frame of cardinality k, and let $\vec{a}_X$ be the function from X to $[0,1]^k$ satisfying:

$$\vec{a}_X(\emptyset) = 0, \quad \vec{a}_X(x_i) \in [0,1] \quad \text{and} \quad \sum_{i=1}^{k} \vec{a}_X(x_i) = 1 \ . \tag{5}$$

Then $\vec{a}_X$ is a base rate distribution over X.

Two different observers can share the same base rate vectors. However, it is obvious that two different observers can also assign different base rates to the same frame, in addition to assigning different beliefs to the frame. This naturally reflects different views, analyses and interpretations of the same situation by different observers. Base rates can thus be partly objective and partly subjective.

Events that can be repeated many times are typically frequentist in nature, meaning that the base rates for these often can be derived from statistical observations. For events that can only happen once, the analyst must often extract base rates from subjective intuition or from analyzing the nature of the phenomenon at hand and any other relevant evidence. However, when no specific base rate information is known, the default base rate of the singletons in a frame must be defined to be equally partitioned between them. More specifically, when there are k singletons in the frame, the default base rate of each element is $1/k$. For this study, the base rates are interpreted as average preferences in the population.

The usefulness of base rate function emerges from its application as the basis for probability projection. Because belief mass can be assigned to any subset of the frame it is necessary to also represent the base rates of such subsets. This is defined below.

Definition 2 (Subset Base Rates). Let X be a frame of cardinality k, and let $\mathscr{R}(X) = 2^X \setminus \{X, \emptyset\}$ be its reduced powerset of cardinality $\kappa = (2^k - 2)$. Assume that a base rate function $\vec{a}_X$ is defined over X according to Def.1. Then the base rates of the elements of the reduced powerset $\mathscr{R}(X)$ are expressed according to the powerset base rate function $\vec{a}_{\mathscr{R}(X)}$ from $\mathscr{R}(X)$ to $[0,1]^\kappa$ expressed below:

$$\vec{a}_{\mathscr{R}(X)}(\emptyset) = 0 \quad \text{and} \quad \vec{a}_{\mathscr{R}(X)}(x_i) = \sum_{\substack{x_j \in X \\ x_j \subseteq x_i}} \vec{a}_X(x_j)\ , \ \ \forall x_i \in \mathscr{R}(X)\ . \tag{6}$$

Note that $x_j \in X$ means that x_j is a singleton in X, so that the subset base rate in Eq.(6) is the sum of base rates on singletons $x_j \in x_i$. Trivially, it can be seen that when $x_i \in X$ then $\vec{a}_{\mathscr{R}(X)}(x_i) \equiv \vec{a}_X(x_i)$, meaning that $\vec{a}_{\mathscr{R}(X)}$ simply is an extension of $\vec{a}_X$. Because of this strong correspondence between $\vec{a}_{\mathscr{R}(X)}$ and $\vec{a}_X$ we will simply denote both base rate functions as $\vec{a}_X$. Because belief masses can be assigned to fully or partially overlapping subsets of the frame it is necessary to also derive relative base rates of subsets as a function of the degree of overlap with each other. This is defined below.

Definition 3 (Relative Base Rates). Assume frame X of cardinality k where $\mathscr{R}(X)$ is its reduced powerset of cardinality $\kappa = (2^k - 2)$. Assume the base rate function $\vec{a}_X$ defined over X according to Def.2. Then the base rates of an element x_i relative to an element x_j is expressed by the relative base rate function $\vec{a}_X(x_i/x_j)$ expressed below:

$$\vec{a}_X(x_i/x_j) = \frac{\vec{a}_X(x_i \cap x_j)}{\vec{a}_X(x_j)}\ , \ \ \forall\, x_i, x_j \in \mathscr{R}(X)\ . \tag{7}$$

4 Opinion Classes

An opinion is a composite function consisting of the belief mass vector $\vec{b}_X$, uncommitted belief mass u_X and the base rate vector $\vec{a}_X$, and can also indicate ownership whenever required. A subjective opinion is normally denoted as ω_X^A where A is the opinion owner, also called the subject, and X is the target frame to which the opinion applies [5]. An alternative notation is $\omega(A:X)$. There can be different classes of opinions, of which *hyper opinions* are the most general. *Multinomial opinions* and *binomial opinions* are specific sub-types. More specific opinion classes are DH opinion (Dogmatic Hyper), UB Opinion (Uncertain Binomial) etc. The six main opinion classes defined in this way are listed in Table 2 below, and are described in more detail in the next sections.

	Binomial Binary frame Focal element $x \in X$	**Multinomial** n-ary frame Focal elements $x \in X$	**Hyper** n-ary frame Focal elements $x \in \mathscr{R}(X)$
Uncertain $u > 0$	**UB opinion** Beta pdf	**UM opinion** Dirichlet pdf over X	**UH opinion** Dirichlet pdf over $\mathscr{R}(X)$
Dogmatic $u = 0$	**DB opinion** Scalar probability	**DM opinion** Probabilities on X	**DH opinion** Probabilities on $\mathscr{R}(X)$

Table 2. Opinion classes with equivalent probabilistic representations

The propositions of a frame are assumed to be exhaustive and mutually disjoint. For binary frames the opinion is binomial. In case the frame is larger than binary, and only singletons of X (i.e. class-1 elements of $\mathscr{R}(X)$) are focal elements, then the opinion is multinomial. In case the frame is larger than binary and there are focal elements of any class of $\mathscr{R}(X)$), then it is a *hyper opinion*. In case of uncommitted belief mass, i.e. $u_X > 0$, it is called an *uncertain opinion* which expresses degrees of indifference. In case $u_X = 0$ it is called a *dogmatic opinion* which represents dogmatic preferences.

The six entries in Table 2 also mention the equivalent probability representation of opinions, e.g. as Beta pdf, Dirichlet pdf or as a distribution of scalar probabilities over elements of X or $\mathscr{R}(X)$ [8]. This offers a frequentist interpretation of subjective opinions and preferences, and provides a method for deriving opinions and preferences from statistical data. Alternatively it is possible to map subjective opinions and preferences to Beta pdfs or Dirichlet pdfs, for further processing and analysis within classical statistical frameworks. The detailed description of the equivalence between opinions and probability density functions is outside the scope of this paper.

4.1 Binomial Opinions

A special notation is used for representing opinions over binary frames. A general *n*-ary frame X can be considered binary when seen as a binary partitioning consisting of one of its proper subsets x and the complement $\bar{x}$.

Definition 4 (Binomial Opinion). *Let $X = \{x, \bar{x}\}$ be either a binary frame or a binary partitioning of an n-ary frame. A binomial opinion about the truth of state x is the ordered quadruple $\omega_x = (b, d, u, a)$ where:*

b : **belief** *belief mass in support of x being true (preference for x),*
d : **disbelief** *belief mass in support of x being false (negative preference for x),*
u : **uncertainty** *the amount of uncommitted belief mass (indifference about x),*
a : **base rate** *the* a priori *probability of x (average preference for x).*

These components satisfy $b + d + u = 1$ and $b, d, u, a \in [0, 1]$. The characteristics of various binomial opinion classes are listed below. A binomial opinion:

where $b = 1$ is equivalent to binary logic TRUE (hard positive constraint),
where $d = 1$ is equivalent to binary logic FALSE (hard negative constraint),
where $b + d = 1$ is equivalent to a traditional probability (preference),
where $b + d < 1$ expresses degrees of uncertainty (indifference), and
where $b + d = 0$ expresses total uncertainty (indifference).

Binomial opinions can be represented on an equilateral triangle as shown in Fig.1.

A point inside the triangle represents a (b, d, u) triple. The belief, disbelief, and uncertainty-axes run from one edge to the opposite vertex indicated by the b_x axis, d_x axis and u_x axis labels. The base rate[1], is shown as a point on the base line, and the probability expectation, E_x, is formed by projecting the opinion point onto the base, parallel to the base rate director line. The opinion $\omega_x = (0.2,\ 0.5,\ 0.3,\ 0.6)$ with expectation value $\mathrm{E}_x = 0.38$ is shown as an example.

[1] Also called *relative atomicity* in case of default base rates [5]

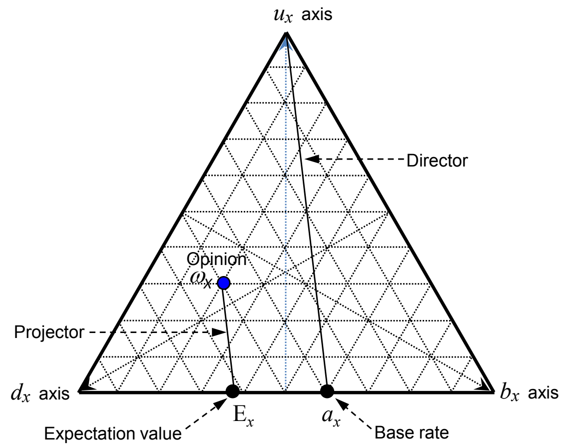

Fig. 1. Opinion triangle with example opinion

Binomial opinions with $u \geq 0$ are called UB opinions (Uncertain Binomial), whereas binomial opinions with $u = 0$ are called DB opinions (Dogmatic Binomial). DB opinions are equivalent to classical probabilities. In case the opinion point is located at one of the three vertices in the triangle, i.e. with $b = 1$, $d = 1$ or $u = 1$, the reasoning with such opinions becomes a form of three-valued logic that is an extension of Kleene logic [4]. Because the three-valued arguments of Kleene logic do not contain base rates, probability expectation values can not be derived from Kleene logic arguments. The conjunction of multiple Kleene logic arguments is therefore incompatible with multiplication of probabilities or opinions [7], and is inconsistent in general because the conjunction of an infinity of UNKNOWN arguments produces UNKNOWN in Kleene logic, whereas realistically it should converge towards FALSE. Eq.(8) defines the probability projection of a binomial opinion on proposition x:

$$\mathrm{E}_x = b + au \, . \tag{8}$$

In case the opinion point is located at the left or right bottom vertex in the triangle, i.e. with $d = 1$ or $b = 1$ and $u = 0$, the opinion is equivalent to boolean TRUE or FALSE, and is called an AB (Absolute Binomial) opinion. Reasoning with AB opinions is an extension of reasoning within binary logic.

4.2 Multinomial Opinions

An opinion on a frame X that is larger than binary and where the set of focal elements is restricted to class-1 elements is called a multinomial opinion. The uncommitted belief mass, which can be interpreted as uncertainty mass on the frame X, represents indifference in the present model. A UM (Uncertain Multinomial) opinion has $u_X > 0$, and a DM (Dogmatic Multinomial) has $u_X = 0$. Multinomial opinions on ternary frames can be presented as a point inside a tetrahedron, as shown in Fig.2.

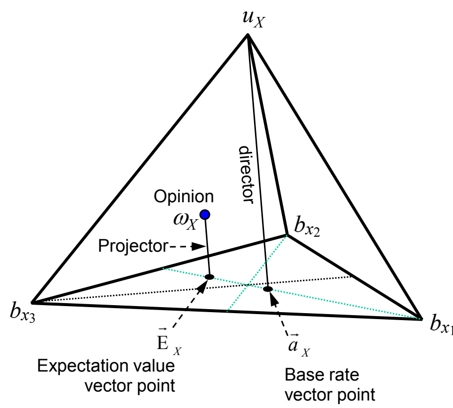

Fig. 2. Opinion tetrahedron with example opinion

The vertical elevation of the opinion point inside the tetrahedron represents the uncertainty mass in Fig.2. The distances from each of the three triangular side planes to the opinion point represents the respective belief mass values. The base rate vector $\vec{a}_X$ is indicated as a point on the base plane. The line that joins the tetrahedron summit and the base rate vector point represents the director. The probability expectation vector point is geometrically determined by drawing a projection from the opinion point parallel to the director onto the base plane.

A multinomial opinion thus contains $(2k+1)$ parameters. However, given Eq.(4) and Eq.(5), multinomial opinions only have $(2k-1)$ degrees of freedom.

In general, the triangle and tetrahedron belong to the *simplex* family of geometrical shapes. Multinomial opinions on frames of cardinality k can in general be represented as a point in a simplex of dimension $(k+1)$. The probability projection of multinomial opinions is expressed by Eq.(9) below:

$$\vec{\mathrm{E}}_X(x_i) = \vec{b}_X(x_i) + \vec{a}_X(x_i)\, u_X \ , \quad \forall\, x_i \in X \ . \tag{9}$$

The probability projection of multinomial opinions expressed by Eq.(9) is a generalisation of the probability projection of binomial opinions expressed by Eq.(8).

4.3 Hyper Opinions

An opinion on a frame X of cardinality $k > 2$ where any element $x \in \mathscr{R}(X)$ can be a focal element is called a hyper opinion. The special characteristic if this opinion class is thus that possible focal elements $x \in \mathscr{R}(X)$ can be overlapping subsets of the frame X. The frame X itself can have uncertainty mass assigned to it, but is not considered as a focal element. Definition 5 below not only defines hyper opinions, but also represents a general definition of subjective of opinions. In case $u_X \neq 0$ it is called a UH opinion (uncertain hyper opinion), and in case $u_X = 0$ it is called a DH opinion (dogmatic hyper opinion).

Definition 5. Hyper Opinion
Assume X be to a frame where $\mathscr{R}(X)$ denotes its reduced powerset. Let $\vec{b}_X$ be a belief vector over the elements of $\mathscr{R}(X)$, let u_X be the complementary uncertainty mass, and let $\vec{a}$ be a base rate vector over the frame X, all seen from the viewpoint of the opinion owner A. The composite function $\omega_X^A = (\vec{b}_X, u_X, \vec{a}_X)$ is then A's hyper opinion over X.

The belief vector $\vec{b}_X$ has $(2^k - 2)$ parameters, whereas the base rate vector $\vec{a}_X$ only has k parameters. The uncertainty parameter u_X is a simple scalar. A hyper opinion thus contains $(2^k + k - 1)$ parameters. However, given Eq.(4) and Eq.(5), hyper opinions only have $(2^k + k - 3)$ degrees of freedom.

Hyper opinions represent the most general class of opinions. It is challenging to design meaningful visualisations of hyper opinions because belief masses are distributed over the reduced powerset with partly overlapping elements. It can be seen that for a frame X of cardinality $k = 2$ a multinomial and a hyper opinion both have 3 degrees of freedom which is the same as for binomial opinions. Thus both multinomial and hyper opinions collapse to binomial opinions in case of binary frames.

The integration of the base rates in opinions allows the probability projection to be independent from the internal structure of the frame. The probability expectation of hyper opinions is a vector expressed as a function of the belief vector, the uncertainty mass and the base rate vector.

Definition 6 (Probability Projection of Hyper Opinions).
Assume X to be a frame of cardinality k where $\mathscr{R}(X)$ is its reduced powerset of cardinality $\kappa = (2^k - 2)$. Let $\omega_X = (\vec{b}_X,\, u_X,\, \vec{a}_X)$ be a hyper opinion on X. The probability projection of hyper opinions is defined by the vector $\vec{\mathrm{E}}_X$ from $\mathscr{R}(X)$ to $[0,1]^\kappa$ expressed as:

$$\vec{\mathrm{E}}_X(x_i) = \sum_{x_j \in \mathscr{R}(X)} \vec{a}_X(x_i/x_j)\, \vec{b}_X(x_j) \;+\; \vec{a}_X(x_i)\, u_X \;, \quad \forall\, x_i \in \mathscr{R}(X)\,. \tag{10}$$

5 The Belief Constraint Operator

The belief constraint operator described here is an extension of Dempster's rule which in Dempster-Shafer belief theory is often presented as a method for fusing evidence from different sources [11]. Many authors have however demonstrated that Dempster's rule is not an appropriate operator for evidence fusion [12], and that it is better suited as a method for combining constraints [8], which is also our view.

Assume two opinions ω_X^A and ω_X^B over the frame X. The superscripts A and B are attributes that identify the respective belief sources or belief owners. These two opinions can be mathematically merged using the belief constraint operator denoted as "$\odot$" which in formulas is written as: $\omega_X^{A\&B} = \omega_X^A \odot \omega_X^B$. Belief source combination denoted with "&" thus represents opinion combination with "$\odot$". The algebraic expression of the belief constraint operator for subjective opinions is defined next.

Definition 7 (Belief Constraint Operator).

$$\omega_X^{A\&B} = \omega_X^A \odot \omega_X^B = \begin{cases} \vec{b}^{A\&B}(x_i) = \frac{Har(x_i)}{(1-Con)}, & \forall\, x_i \in \mathscr{R}(X),\, x_i \neq \emptyset \\ u_X^{A\&B} \quad\quad = \frac{u_X^A u_X^B}{(1-Con)} & \\ \vec{a}^{A\&B}(x_i) = \frac{\vec{a}^A(x_i)(1-u_X^A)+\vec{a}^B(x_i)(1-u_X^B)}{2-u_X^A-u_X^B}, & \forall\, x_i \in X,\, x_i \neq \emptyset \end{cases} \quad (11)$$

The term $Har(x_i)$ represents the degree of *Harmony*, or in other words overlapping belief mass, on x_i. The term *Con* represents the degree of belief *Conflict*, or in other words non-overlapping belief mass, between ω_X^A and ω_X^B. These are defined below:

$$\begin{aligned} Har(x_i) &= \vec{b}^A(x_i)u_X^B + \vec{b}^B(x_i)u_X^A + \textstyle\sum_{y\cap z=x_i} \vec{b}^A(y)\vec{b}^B(z), \qquad \forall\, x_i \in \mathscr{R}(X)\;. \\ Con &= \textstyle\sum_{y\cap z=\emptyset} \vec{b}^A(y)\vec{b}^B(z)\;. \end{aligned} \quad (12)$$

The purpose of the divisor $(1-Con)$ in Eq.(11) is to normalise the derived belief mass, or in other words to ensure belief mass and uncertainty mass additivity. The use of the belief constraint operator is mathematically possible only if ω^A and ω^B are not totally conflicting, i.e., if $Con \neq 1$.

The belief constraint operator is commutative and non-idempotent. Associativity is preserved when the base rate is equal for all agents. Associativity in case of different base rates requires that all preference opinions be combined in a single operation which would require a generalisation of Def.7 for multiple agents, i.e. for multiple input arguments, which is relatively trivial. A totally indifferent opinion acts as the neutral element for belief constraint, formally expressed as:

$$\text{IF } (\omega_X^A \text{ is totally indifferent, i.e. with } u_X^A = 1) \text{ THEN } (\omega_X^A \odot \omega_X^B = \omega_X^B)\;. \quad (13)$$

Having a neutral element in the form of the totally indifferent opinion is very useful when modelling situations of preference combination.

6 Examples

6.1 Expressing Preferences with Subjective Opinions

Preferences can be expressed e.g. as soft or hard constraints, qualitative or quantitative, ordered or partially ordered etc. It is possible to specify a mapping between qualitative verbal tags and subjective opinions which enables easy solicitation of preferences [9]. Table 3 describes examples of how preferences can be expressed.

All the preference types of Table 3 can be interpreted in terms of subjective opinions, and further combined by considering them as constraints expressed by different agents. The examples that comprise two binary frames could also have been modelled with a quaternary product frame with a corresponding 4-nomial product opinion. In fact

Example & Type	Opinion Expression	
"Ingredient x is mandatory"	Binary frame	$X=\{x,\overline{x}\}$
Hard positive	Binomial opinion	$\omega_x : (1,0,0,\frac{1}{2})$
"Ingredient x is totally out of the question"	Binary frame	$X=\{x,\overline{x}\}$
Hard negative	Binomial opinion	$\omega_x : (0,\ 1,\ 0,\ \frac{1}{2})$
"My preference rating for x is 3 out of 10	Binary frame	$X=\{x,\overline{x}\}$
Quantitative	Binomial opinion	$\omega_x : (0.3,\ 0.7,\ 0.0,\ \frac{1}{2})$
"I prefer x or y, but z is also acceptable"	Ternary frame	$\Theta=\{x,y,z\}$
Qualitative	Trinomial opinion	$\omega_\Theta : (b(x,y)=0.6,\ b(z)=0.3,$
		$u=0.1,\ a(x,y,z)=\frac{1}{3})$
"I like x, but I like y even more"	Two binary frames	$X=\{x,\overline{x}\}$ and $Y=\{y,\overline{y}\}$
Positive rank	Binomial opinions	$\omega_x : (0.6,\ 0.3,\ 0.1,\ \frac{1}{2}),$
		$\omega_y : (0.7,\ 0.2,\ 0.1,\ \frac{1}{2})$
"I don't like x, and I dislike y even more"	Two binary frames	$X=\{x,\overline{x}\}$ and $Y=\{y,\overline{y}\}$
Negative rank	Binomial opinions	$\omega_x : (0.3,\ 0.6,\ 0.1,\ \frac{1}{2}),$
		$\omega_y : (0.2,\ 0.7,\ 0.1,\ \frac{1}{2})$
"I'm indifferent about x, y and z"	Ternary frame	$\Theta=\{x,y,z\}$
Neutral	Trinomial opinion	$\omega_\Theta : (u_\Theta=1.0,\ a(x,y,z)=\frac{1}{3})$
"I'm indifferent but most people prefer x"	Ternary frame	$\Theta=\{x,y,z\}$
Neutral with bias	Trinomial opinion	$\omega_\Theta : (u_\Theta=1.0,\ a(x)=0.6,$
		$a(y)=0.2, a(z)=0.2)$

Table 3. Example preferences and corresponding subjective opinions

product opinions over product frames could be a method of simultaneously considering preferences over multiple variables, and this will be the topic of future research.

Default base rates are specified in all but the last example which indicates total indifference but with a bias which expresses the average preference in the population. Base rates are useful in many situations, such as for default reasoning. Base rates only have an influence in case of significant indifference or uncertainty.

6.2 Going to the Cinema, 1st Attempt

Assume three friends, Alice, Bob and Clark, who want to see a film together at the cinema one evening, and that the only films showing are *Black Dust* (*BD*), *Grey Matter* (*GM*) and *White Powder* (*WP*), represented as the ternary frame $\Theta=\{BD,\ GM,\ WP\}$. Assume that the friends express their preferences in the form of the opinions of Table 4.

Alice and Bob have strong and conflicting preferences. Clark, who only does not want to watch *Black Dust*, and who is indifferent about the two other films, is not sure whether he wants to come along, so Table 4 shows the results of applying the preference combination operator, first without him, and then including in the party.

By applying the belief constraint operator Alice and Bob conclude that the only film they are both interested in seeing is *Grey Matter*. Including Clark in the party does not change that result because he is indifferent to *Grey Matter* and *White Powder* anyway, he just does not want to watch the film *Black Dust*.

		Preferences of:			Results of preference combinations:	
		Alice	Bob	Clark	(Alice & Bob)	(Alice & Bob & Clark)
		ω_Θ^A	ω_Θ^B	ω_Θ^C	$\omega_\Theta^{A\&B}$	$\omega_\Theta^{A\&B\&C}$
$b(BD)$	=	0.99	0.00	0.00	0.00	0.00
$b(GM)$	=	0.01	0.01	0.00	1.00	1.00
$b(WP)$	=	0.00	0.99	0.00	0.00	0.00
$b(GM \cup WP)$	=	0.00	0.00	1.00	0.00	0.00

Table 4. Combination of film preferences

The belief mass values of Alice and Bob in the above example are in fact equal to those of Zadeh's example [12] which was used to demonstrate the unsuitability of Dempster's rule for fusing beliefs because it produces counterintuitive results. Zadeh's example describes a medical case where two medical doctors express their opinions about possible diagnoses, which typically should have been modelled with the averaging fusion operator [6], not with Dempster's rule. In order to select the appropriate operator it is crucial to fully understand the nature of the situation to be modelled. The failure to understand that Dempster's rule does not represent an operator for cumulative or averaging belief fusion, combined with the unavailability of the general cumulative and averaging belief fusion operators for many years (1976[11]-2010[6]), has often led to inappropriate applications of Dempster's rule to cases of belief fusion [8]. However, when specifying the same numerical values as in [12] in a case of preference combination such as the example above, the belief constraint operator which is a simple extension of Dempster's rule is very suitable and produces perfectly intuitive results.

6.3 Going to the Cinema, 2nd Attempt

In this example Alice and Bob soften their strong preference by expressing some indifference in the form of $u = 0.01$, as specified by Table 5. Clark has the same opinion as in the previous example, and is still not sure whether he wants to come along, so Table 5 shows the results without and with his preference included.

		Preferences of:			Results of preference combinations:	
		Alice	Bob	Clark	(Alice & Bob)	(Alice & Bob & Clark)
		ω_Θ^A	ω_Θ^B	ω_Θ^C	$\omega_\Theta^{A\&B}$	$\omega_\Theta^{A\&B\&C}$
$b(BD)$	=	0.98	0.00	0.00	0.490	0.000
$b(GM)$	=	0.01	0.01	0.00	0.015	0.029
$b(WP)$	=	0.00	0.98	0.00	0.490	0.961
$b(GM \cup WP)$	=	0.00	0.00	1.00	0.000	0.010
u	=	0.01	0.01	0.00	0.005	0.000
$a(BD)$	=	0.6	0.6	0.6	0.6	0.6
$a(GM) = a(WP)$	=	0.2	0.2	0.2	0.2	0.2

Table 5. Combination of film preferences with some indifference and with non-default base rates

The effect of adding some indifference is that Alice and Bob should pick film *Black Dust* or *White Powder* because in both cases one of them actually prefers one of the films, and the other finds it acceptable. Neither Alice nor Bob prefers *Gray Matter*, they only find it acceptable, so it turns out not to be a good choice for any of them. When taking into consideration that the base rate $a(BD) = 0.6$ and the base rate $a(WP) = 0.2$, the preference expectation values according to Eq.(10) are such that:

$$\mathrm{E}^{A\&B}(BD) > \mathrm{E}^{A\&B}(WP)\ . \tag{14}$$

More precisely, the preference expectation values according to Eq.(10) are:

$$\mathrm{E}^{A\&B}(BD) = 0.493\ , \qquad \mathrm{E}^{A\&B}(WP) = 0.491\ . \tag{15}$$

Because of the higher base rate, *Black Dust* also has a higher expected preference than *White Powder*, so the rational choice would be to watch *Black Dust*.

However, when including Clark who does not want to watch *Black Dust*, the base rates no longer dictates the result. In this case Eq.(10) produces $\mathrm{E}^{A\&B\&C}(WP) = 0.966$ so the obvious choice is to watch *White Powder*.

6.4 Not Going to the Cinema

Assume now that the Alice and Bob express totally conflicting preferences as specified in Table 6, i.e. Alice expresses a hard preference for *Black Dust* and Bob expresses a hard preference for *White Powder*. Clark still has the same preference as before, i.e he does not want to watch *Black Dust* and is indifferent about the two other films.

	Preferences of:			Results of preference combinations:	
	Alice	Bob	Clark	(Alice & Bob)	(Alice & Bob & Clark)
	ω_Θ^A	ω_Θ^B	ω_Θ^C	$\omega_\Theta^{A\&B}$	$\omega_\Theta^{A\&B\&C}$
$b(BD)$ =	1.00	0.00	0.00	Undefined	Undefined
$b(GM)$ =	0.00	0.00	0.00	Undefined	Undefined
$b(WP)$ =	0.00	1.00	0.00	Undefined	Undefined
$b(GM \cup WP)$ =	0.00	0.00	1.00	Undefined	Undefined

Table 6. Combination of film preferences with hard and conflicting preferences

In this case the belief constraint operator can not be applied because Eq.(11) produces a division by zero. The conclusion is that the friends will not go to the cinema to see a film together. The test for detecting this situation is when *Con* = 1 in Eq.(12). It makes no difference to include Clark in the party because a conflict can not be resolved by including additional preferences. However it would have been possible for Bob and Clark to watch *White Powder* together without Alice.

7 Conclusion

The flexibility of subjective logic makes it simple to express positive and negative preferences within the same framework, as well as indifference/uncertainty. This paper describes how subjective logic can be used to express preferences over a variable represented as the possible states in a frame, and how the belief constraint operator, which is an extension of Dempster's rule, can be applied for combining preferences of multiple agents in order to determine the most preferred choice for the whole group. Because preference can be expressed over arbitrary subsets of the frame this is in fact a multi-polar model for expressing and combining preferences. Even in the case of no overlapping focal elements the belief constraint operator provides a meaningful answer, namely that the preferences are incompatible.

Multi-agent preference combination with subjective logic assumes that individual preferences have been predefined. Future research will focus on applying subjective logic for determining subjective preferences of each agent e.g. in situations with multiple criteria, and on combining preferences from multiple agents over different variables in the form of different frames.

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