Partially Ordered Sets with Interfaces: A Novel Algebraic Approach for Concurrency

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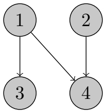
Overview

Introduction

[N-poset]

A poset P with a four element set {1,2,3,4} defined by the $1 \preceq 3, 1 \preceq 4$ *and* $2 \preceq 4$ *non-trivial partial order relation is called a* **N-poset**.

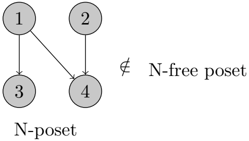


N-poset

Introduction

[N-free poset]

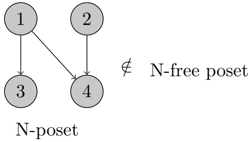
- *A poset is* **"N-free"** *if it does not contain a cover preserving subsets isomorphic to "N".*



- *Given a poset P, following properties are equivalent;*
  1. *P is an N-free poset.* [$P.A.Grilet(1969)^{1,5}$, $C.Heuchenne(1964)^{1,6}$]
  2. *P is Series-parallel graphs or diagraph. [R.J. Duffin (1965)]*
  3. *P is Series-parallel posets. [E.L Lawler (1978), J. Valdes (1979)]*
  4. *P is QSP graphs. [W.H. Cunningham (1980,1982)]*
  5. *P is a Chain-antichain Complete.*
  6. *The Hasse diagram of P is a line-diagraph.*

Introduction

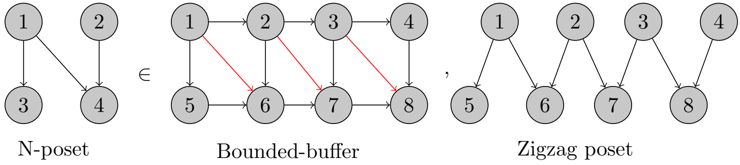
[N-free poset]



- *"N-free" posets language for modeling concurrency*
  1. *Full abstraction for series-parallel pomsets, Luca Aceto (1991).*
  2. *Free shuffle algebras in language varieties, Stephen L. Bloom and Zoltan Esik (1996).*
  3. *Series-parallel languages and the bounded-width property, Kamal Lodaya and Pascal Weil (2000).*
  4. *Completeness theorems for bi-Kleene algebras and series-parallel rational pomset languages, Michael R. Laurence and Georg Struth (2014).*
  5. *Completeness theorems for pomset languages and concurrent Kleene algebras, Michael R. Laurence and Georg Struth (2017).*
  6. *Concurrent Kleene Algebra: Free Model and Completeness, Tobias Kappe et. al. (2017).*

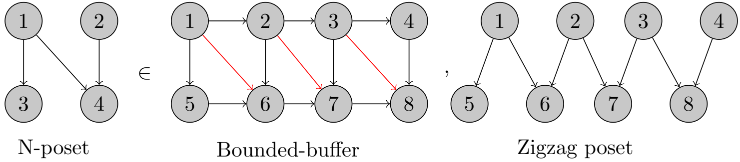
Introduction

[Problem statement]



Introduction

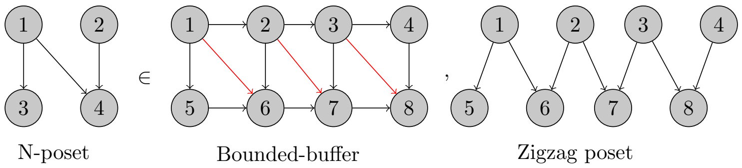
[Research goal]



- Langauge theoretic model $\mathcal{L}(Q)$, for any poset $\mathcal{Q}$ such that **N-poset** $\in \mathcal{Q}$.
- Derive domain and modal operators for $\mathcal{L}(Q)$.

Introduction

[Our approach]



Research Goal:

- Langauge theoretic model $\mathcal{L}$ such that poset $\mathcal{Q} \in \mathcal{L}$, where **N-poset** $\in \mathcal{Q}$.
- Derive domain and modal operators for $\mathcal{L}$.

Approach:

- We propose poset with interfaces, named iposet, $\mathcal{P}$.
- We investigate $\mathcal{L}(P)$ for modeling concurrency.
- We derive domain and modal operators for $\mathcal{L}(P)$.

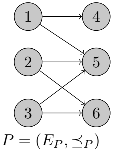
Iposet

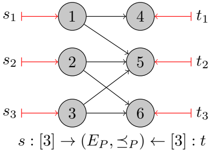
Definition 2. In modelling concurrency, a **poset** *P can be defined as an ordered pair $(E_P, \preceq_P)$, where E_P is called the ground set of P, known as set of events and $\preceq_P$ is the partial order relation on E_P.*

Definition 3. A poset with interfaces, **iposet**, *is a cospan*

$$s : [n] \rightarrow (E_P, \preceq_P) \leftarrow [m] : t$$

of monomorphisms s, t on poset P such that $s[n]$ is the image of minimal and $t[m]$ is the image of maximal events of P.





Iposet

[Concatenation]

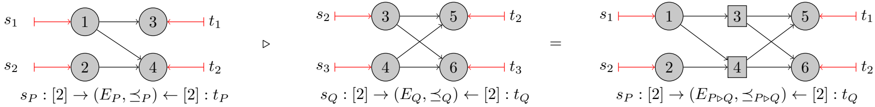
We defined **concatenation** $\rhd$ of iposets whose interfaces agree.

Definition 4. *The* **concatenation** $\rhd$ *of iposets P and Q such that*

$$s_P : [n] \to (E_P, \preceq_P) \leftarrow [m] : t_P \text{ and } s_Q : [m] \to (E_Q, \preceq_Q) \leftarrow [k] : t_Q$$

is an iposet $P \rhd Q := s_P : [n] \to (E_{P\rhd Q}, \preceq_{P\rhd Q}) \leftarrow [k] : t_Q$, *where*

$$E_{P\rhd Q} = (E_P \sqcup E_Q)_{/\, t_P(i) = s_Q(i)}$$
$$\preceq_{P\rhd Q} = \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q).$$



Iposet

[Concatenation]

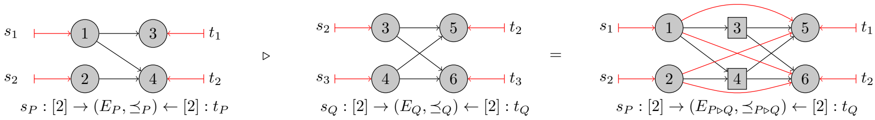
We defined **concatenation** $\triangleright$ of iposets whose interfaces agree.

Definition 4. The **concatenation** $\triangleright$ *of iposets P and Q such that*

$$s_P : [n] \to (E_P, \preceq_P) \leftarrow [m] : t_P \text{ and } s_Q : [m] \to (E_Q, \preceq_Q) \leftarrow [k] : t_Q$$

is an iposet $P \triangleright Q := s_P : [n] \to (E_{P \triangleright Q}, \preceq_{P \triangleright Q}) \leftarrow [k] : t_Q$, *where*

$$E_{P \triangleright Q} = (E_P \sqcup E_Q)_{/ t_P(i) = s_Q(i)}$$
$$\preceq_{P \triangleright Q} = \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q).$$



Iposet

[Parallel product]

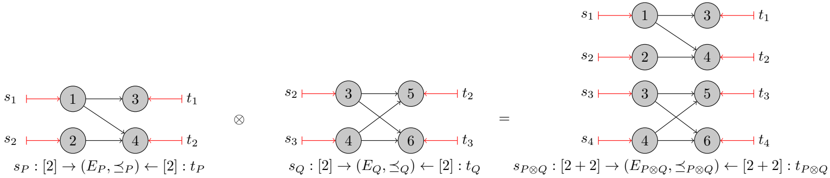
Definition 5. *The* **parallel product** $\otimes$ *of iposets P and Q such that*

$$s_P : [n] \to (E_P, \preceq_P) \leftarrow [m] : t_P \text{ and } s_Q : [m] \to (E_Q, \preceq_Q) \leftarrow [k] : t_Q$$

is an iposet $P \otimes Q := s : [n+m] \to (E_{P\otimes Q}, \preceq_{P\otimes Q}) \leftarrow [m+k] : t$, *where*

$$E_{P\otimes Q} = E_P \sqcup E_Q,$$
$$\preceq_{P\otimes Q} = \preceq_P \cup \preceq_Q \quad \text{and}$$

$$s_{P\otimes Q}(i) = \begin{cases} s_P(i) & \text{if } i \le n \\ s_Q(i-n) & \text{if } i > n \end{cases}, \quad t_{P\otimes Q}(i) = \begin{cases} t_P(i) & \text{if } i \le m \\ s_Q(i-m) & \text{if } i > m. \end{cases}$$



Iposet

[Subsumption]

Definition 6. *The* **subsumption** *order* $Q \leq P$ *on iposets*

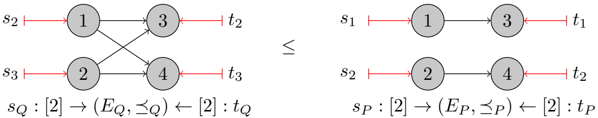
$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } Q = (E_Q, \preceq_Q, s_Q, t_Q)$$

is defined if there exists bijection $h : E_P \to E_Q$ *such that*

$$x \preceq_P y \implies h(x) \preceq_Q h(y) \text{ for all } x, y \in P$$

along with the source and target interface bijections

$$h : s_P[n_P] \to s_Q[n_Q] \text{ and } h : t_P[m_P] \to t_Q[m_Q].$$



- *"Implementation"* $\leq$ *"Specification"*

Iposet

[Isomorphism]

Definition 7. *The* **isomorphism** $Q = P$ *on iposets*

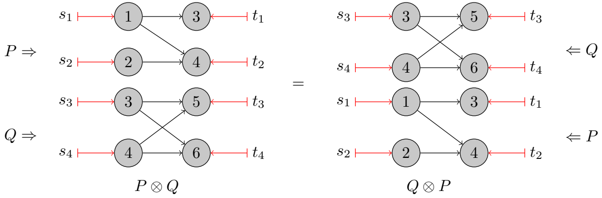
$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } Q = (E_Q, \preceq_Q, s_Q, t_Q)$$

is defined if there exists bijection $h : E_P \to E_Q$ *such that*

$$x \preceq_P y \iff h(x) \preceq_Q h(y) \text{ for all } x, y \in P$$

along with the source and target interface bijections

$$h : s_P[n_P] \to s_Q[n_Q] \text{ and } h : t_P[m_P] \to t_Q[m_Q].$$



• An **ipomset** over Σ, a finite set of alphabet, is an isomorphic class of Σ-labelled iposets.

Iposet algebra

We investigate the equational theory of iposets close to the algebraic results of **Concurrent Kleene Algebra** *[1].*

[1]. T. Hoare, B. Moller, G. Struth, I. Wehrman, "Concurrent Kleene algebra and its foundations". The Journal of Logic and Algebraic Programming,2011

Iposet algebra

[Ordered bisemigroup structure]

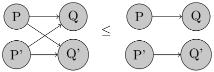
Let $\mathbf{P}$ *be the set of iposets. For* $P, P', Q, Q', R \in \mathbf{P}$*,*

Proposition 1. $(\mathbf{P}, \triangleright, \otimes)$ *forms an ordered bisemigroup that satisfies following axioms of concurrent semigroup [1, Definition 6.6]*

$$P \triangleright (Q \triangleright R) = (P \triangleright Q) \triangleright R \tag{1}$$

$$P \otimes (Q \otimes R) = (P \otimes Q) \otimes R \tag{2}$$

$$(P \otimes P') \triangleright (Q \otimes Q') \leq (P \triangleright Q) \otimes (P' \triangleright Q') \tag{3}$$



Lemma 1. *The ordered bisemigroup* $(\mathbf{P}, \triangleright, \otimes)$ *entail the following axioms*

$$P \triangleright Q \leq P \otimes Q \qquad \text{if } t_P = s_Q = 0 \tag{4}$$

$$(P \otimes Q) \triangleright R \leq P \otimes (Q \triangleright R) \qquad \text{if } t_P = 0 \tag{5}$$

$$P \triangleright (Q \otimes R) \leq (P \triangleright Q) \otimes R \qquad \text{if } s_R = 0 \tag{6}$$

- *Concurrent Kleene algebra [1, Lemma 6.8].*

Iposet languages

Let $\mathcal{P}$ denote the set of all isomorphic class of iposets. We use notations

$$id_1 = (E_{id_1}, \preceq_{id_1}, s_{id_1}, t_{id_1}) \quad and \quad id_0 = (E_{id_0}, \preceq_{id_0}, s_{id_0}, t_{id_0})$$

for an identity and an empty iposet respectively. An iposet language over $\mathcal{P}$ denotes a subset of $\mathcal{P}$, i.e., an element of $2^{\mathcal{P}}$.

Proposition 2. $(\mathcal{P}, \rhd, \otimes, id_n, id_0)$ *forms a double monoid, for $n > 0$. The double monoid is a composite of two ordered monoids*

$$(\mathcal{P}, \rhd, id_n) \text{ and } (\mathcal{P}, \otimes, id_0)$$

such that, for $P \in \mathcal{P}$

$$P \rhd id_n = id_n \rhd P = P \quad , \quad P \otimes id_0 = id_0 \otimes P = P.$$

Iposet languages

[Double monoid structure]

Proposition 3. $(2^{\mathcal{P}}, \rhd, \otimes, 1_{\rhd}, 1_{\otimes})$ *forms a double monoid over* $2^{\mathcal{P}}$*.*

- $1_{\otimes} = \{id_0\}$ denotes set containing only the empty iposet, *i.e.,*ϵ string,
- $1_{\rhd} = \{id_n\}$ for $n > 0$ denotes set containing only the identity iposets.

The double monoid

$$(2^{\mathcal{P}}, \rhd, \otimes, 1_{\rhd}, 1_{\otimes})$$

is a composite of two ordered monoids

$$(2^{\mathcal{P}}, \rhd, 1_{\rhd}) \quad \text{and} \quad (2^{\mathcal{P}}, \otimes, 1_{\otimes})$$

such that, for $P \in 2^{\mathcal{P}}$

$$P \rhd 1_{\rhd} = 1_{\rhd} \rhd P = P \quad , \quad P \otimes 1_{\otimes} = 1_{\otimes} \otimes P = P.$$

Iposet languages

[Bisemiring structure]

Proposition 4. The structure $(2^{\mathcal{P}}, \cup, \triangleright, \otimes, 0, 1_{\triangleright}, 1_{\otimes})$ *forms a bisemiring such that following equations holds, for* $P, Q, R \in 2^{\mathcal{P}}$

$$P \triangleright 0 = 0 \tag{7}$$

$$P \otimes 0 = 0 \tag{8}$$

$$P \triangleright 1_{\triangleright} = P = 1_{\triangleright} \triangleright P \tag{9}$$

$$P \otimes 1_{\otimes} = P = 1_{\otimes} \otimes P \tag{10}$$

$$(P \triangleright Q) \triangleright R = P \triangleright (Q \triangleright R) \tag{11}$$

$$(P \otimes Q) \otimes R = P \otimes (Q \otimes R) \tag{12}$$

$$P \cup 0 = P \tag{13}$$

$$P \triangleright (Q \cup R) = P \triangleright Q \cup R \triangleright Q \tag{14}$$

$$(P \cup Q) \triangleright R = P \triangleright R \cup Q \triangleright R \tag{15}$$

$$P \otimes (Q \cup R) = P \otimes Q \cup P \otimes R \tag{16}$$

$$(P \cup Q) \otimes R = P \otimes R \cup Q \otimes R \tag{17}$$

Here, constant 0 *denotes set of* $\emptyset$ *iposets and operation* $\cup$ *denotes choice.*

• $(2^{\mathcal{P}}, \cup, \triangleright, \otimes, 0, 1_{\triangleright}, 1_{\otimes}) = (2^{\mathcal{P}}, \cup, \triangleright, 0, 1_{\triangleright})$ and $(2^{\mathcal{P}}, \cup, \otimes, 0, 1_{\otimes})$

Structured theory of iposets

• *We define a hierarchy for iposets; generated by a finite number of series and parallel compositions over singleton iposets.*

• *We compare the hierarchy of iposets with the hierarchy of SP posets.*

Structured theory of iposets

[Iposet hierarchy]

Definition 9. *The class of singleton iposets $\mathcal{S}$,*

$$\begin{aligned} S = \{ & [0] \to [1] \leftarrow [0], \\ & [1] \to [1] \leftarrow [1], \\ & [0] \to [1] \leftarrow [1], \\ & [1] \to [1] \leftarrow [0] \}. \end{aligned}$$

Then, iposet hierarchy is given by

$$\mathcal{C}_0 = \mathcal{D}_0 = \mathcal{S}$$

$$\mathcal{C}_{2n+1} = \mathcal{C}_{2n}^{\otimes} \qquad \mathcal{D}_{2n+1} = \mathcal{D}_{2n}^{\triangleright}$$

$$\mathcal{C}_{2n+2} = \mathcal{C}_{2n+1}^{\triangleright} \qquad \mathcal{D}_{2n+2} = \mathcal{D}_{2n+1}^{\otimes}$$

Lemma 2. *For all n, we have*

1. $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ *and* $\mathcal{C}_n \subseteq \mathcal{D}_{n+1}$.
2. $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ *and* $\mathcal{D}_n \subseteq \mathcal{C}_{n+1}$.
3. $\mathcal{C}_n \cup \mathcal{D}_n \subseteq \mathcal{C}_{n+1} \cap \mathcal{D}_{n+1}$.

Definition 10. *The class of singleton poset $\mathcal{S}_0$,*

$$\mathcal{S}_0 = \{ [0] \to [1] \leftarrow [0] \}$$

Then, SP-poset hierarchy is given by

$$\mathcal{T}_0 = \mathcal{U}_0 = \mathcal{S}_0$$

$$\mathcal{T}_{2n+1} = \mathcal{T}_{2n}^{\otimes} \qquad \mathcal{U}_{2n+1} = \mathcal{U}_{2n}^{\triangleright}$$

$$\mathcal{T}_{2n+2} = \mathcal{T}_{2n+1}^{\triangleright} \qquad \mathcal{U}_{2n+2} = \mathcal{U}_{2n+1}^{\otimes}$$

Corollary 1. *For all n, we have*

1. $\mathcal{T}_n \cup \mathcal{U}_n = \mathcal{T}_{n+1} \cap \mathcal{U}_{n+1}$

Structured theory of iposets

[Iposet hierarchy vs. SP hierarchy]

Lemma 3. *For all n we have,*

1. $\mathcal{D}_{2n+1} \otimes \mathcal{D}_{2n} \subseteq \mathcal{D}_{2n+2}$, and $\mathcal{D}_{2n} \otimes \mathcal{D}_{2n+1} \subseteq \mathcal{D}_{2n+2}$.
2. $\mathcal{C}_{2n+1} \rhd \mathcal{C}_{2n} \subseteq \mathcal{C}_{2n+2}$, and $\mathcal{C}_{2n} \rhd \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n+2}$.

Corollary 2. *For all n we have,*

1. $\mathcal{U}_{2n+1} \otimes \mathcal{U}_{2n} \subseteq \mathcal{U}_{2n+2}$, and $\mathcal{U}_{2n} \otimes \mathcal{U}_{2n+1} \subseteq \mathcal{U}_{2n+2}$.
2. $\mathcal{T}_{2n+1} \rhd \mathcal{T}_{2n} \subseteq \mathcal{T}_{2n+2}$, and $\mathcal{T}_{2n} \rhd \mathcal{T}_{2n+1} \subseteq \mathcal{T}_{2n+2}$.

Lemma 4. *For all $n \geq 1$ we have,*

1. $\mathcal{D}_{2n} \rhd \mathcal{D}_{2n-1} \subseteq \mathcal{D}_{2n+1}$ and $\mathcal{D}_{2n-1} \rhd \mathcal{D}_{2n} \subseteq \mathcal{D}_{2n+1}$.
2. $\mathcal{C}_{2n} \otimes \mathcal{C}_{2n-1} \subseteq \mathcal{C}_{2n+1}$ and $\mathcal{C}_{2n-1} \otimes \mathcal{C}_{2n} \subseteq \mathcal{C}_{2n+1}$.

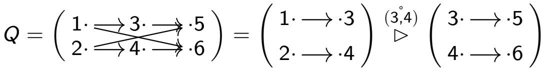
Corollary 3. *For all $n \geq 1$ we have,*

1. $\mathcal{U}_{2n} \rhd U_{2n-1} \subseteq \mathcal{U}_{2n+1}$, and $U_{2n-1} \rhd \mathcal{U}_{2n} \subseteq \mathcal{U}_{2n+1}$.
2. $\mathcal{T}_{2n} \otimes T_{2n-1} \subseteq \mathcal{T}_{2n+1}$, and $T_{2n-1} \otimes \mathcal{T}_{2n} \subseteq \mathcal{T}_{2n+1}$.

- *We have,*

$$\mathcal{S}_0 \subsetneq \mathcal{S} \implies \mathcal{T}_n \subsetneq \mathcal{C}_n \text{ and } \mathcal{U}_n \subsetneq \mathcal{D}_n \text{ For all } n.$$

- *For instance, $Q \in \mathcal{C}_n$ and $Q \notin \mathcal{T}_n$, where Q preserves sub-structures isomorphic to N-poset.*



Structured theory of iposets

[Iposet hierarchy conjectured]

- *The non-collapsing hierarchy, conjectured.*

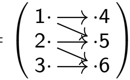
Conjecture 1. Let P_1, P_2, Q_1, Q_2 be iposets such that $P_1 \otimes P_2 = Q_1 \rhd Q_2$, then one of the following is true

1. $P_1 = \emptyset$ or $P_2 = \emptyset$.
2. $Q_1 = id_n$ or $Q_2 = id_n$ for some $n \in \mathbb{N}$.

Conjecture 2. $P_n \in \mathcal{C}_{2n} \setminus \mathcal{C}_{2n-1}$ for all $n \geq 1$.

Corollary 4. $\mathcal{C}_{2n-1} \subsetneq \mathcal{C}_{2n}$ for all $n \geq 1$, hence the $\mathcal{C}_n$ hierarchy does not collapse.

- *The incomplete hierarchy, conjectured.*

Proposition 5. Let $P =$ . Then for all $n \geq 0$, $P \notin C_n$.

The theory of iposets under subsumption

We investigate algebraic results of iposets under subsumption order.

• *Downwards-closed pomset languages can be used to reason about concurrent programs with a refinement order, for instance, downwards-closed SP pomsets (Laurence and Struth, 2017).*

• *[In computation] The language theory under subsumption order is one of the important techniques for cutting down the search space in an automated theorem prover.*

The theory of iposets under subsumption

[Uniqueness]

Lemma 5. **[Uniqueness]** *Let P be an iposet, then exactly one of the following case holds for P*

1. *P is an empty iposet, or*
2. *P is an singleton iposet, or*
3. *There exists non-empty non-identity iposets P_0 and P_1 such that $P = P_0 \rhd P_1$. or,*
4. *There exists non-empty non-identity iposets P'_0 and P'_1 such that $P = P'_0 \otimes P'_1$. or,*
5. *P is prime iposet, i.e., $P \notin C_n$ for all $n \geq 0$.*

Definition 12. The iposets P is said to be prime when it can not be uniquely decomposable into either $\rhd_i$ or $\otimes_i$ -product of i-reducible n non-empty non-identity iposets.

The theory of iposets under subsumption

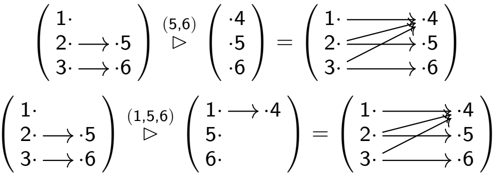
[Unique $\rhd$ factorization]

Lemma 6. Let P, Q, U, V be iposets such that $P \rhd Q \leqslant U \rhd V$. There exists an iposets R such that either $P \leqslant U \rhd R$ and $R \rhd Q \leqslant V$ or $P \rhd R \leqslant U$ and $Q \leqslant R \rhd V$.

Lemma 7. Let P, Q, U, V be iposets such that $P \rhd Q = U \rhd V$. There exists an iposets R such that either $P = U \rhd R$ and $R \rhd Q = V$ or $P \rhd R = U$ and $Q = R \rhd V$.

Lemma 8. **[Unique $\rhd$ Factorization]** *Let P be an iposets such that $U_1 \rhd U_2 \ldots U_n$ and $V_1 \rhd V_2 \ldots V_m$ denotes the Sequential factorization of P for some $n, m \in \mathbb{N}$, then*

$$U_1 \rhd U_2 \ldots U_n = V_1 \rhd V_2 \ldots V_m.$$



The theory of iposets under subsumption

[Unique $\otimes$ factorization]

Lemma 9. Let P, Q, U, V be iposets such that $P \otimes Q \leqslant U \otimes V$. Then, there exist iposets U_0, U_1, V_0, V_1 such that

$$U_0 \otimes U_1 \leqslant U, V_0 \otimes V_1 \leqslant V, P \leqslant U_0 \otimes V_0, \text{ and } Q \leqslant U_1 \otimes V_1.$$

Lemma 10. Let P, Q, U, V be iposets such that $P \otimes Q = U \otimes V$. Then, there exist iposets U_0, U_1, V_0, V_1 such that

$$U_0 \otimes U_1 = U, V_0 \otimes V_1 = V, P = U_0 \otimes V_0, \text{ and } Q = U_1 \otimes V_1.$$

Lemma 11. **[Unique $\otimes$ Factorization]** *Let P be an iposet such that $U_1 \otimes U_2 \dots U_n$ and $V_1 \otimes V_2 \dots V_m$ denotes the parallel factorization of P for some $n, m \in \mathbb{N}$, then*

$$U_1 \otimes U_2 \dots U_n = V_1 \otimes V_2 \dots V_m.$$

The theory of iposets under subsumption

[Interpolation]

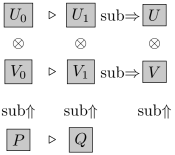
Lemma 12. [Levi] Let P and Q be iposets, and let $W_0, W_1, ..., W_{n-1}$ with $n > 0$ be non-empty iposets such that $P \rhd Q \leqslant W_0 \rhd W_1 \rhd ... \rhd W_{n-1}$. Then, there exists an $m < n$ and iposets U, V such that

$$U \rhd V = W_m, P \leqslant W_0 \rhd W_1 \rhd ... \rhd W_{m-1} \rhd U \text{ and}$$

$$V \leqslant V \rhd W_{m+1} \rhd W_{m+2} \rhd ... \rhd W_{n-1}.$$

Lemma 13. **[Interpolation]** *Let P, Q, U, V be iposets such that $P \rhd Q \leqslant U \otimes V$. Then, there exist iposets U_0, U_1, V_0, V_1 such that*

$$U_0 \rhd U_1 \leqslant U, V_0 \rhd V_1 \leqslant V, P \leqslant U_0 \otimes V_0, \text{ and } Q \leqslant U_1 \otimes V_1.$$



Domain and modal operators for iposet languages

We present axioms of domain operations [2] for iposets languages and derive their corresponding modal operators.

[2]. Jules Desharnais, and Georg Struth. "Internal axioms for domain semirings." Science of Computer Programming 76.3 (2011): 181-203.

Domain and modal operators for iposet languages

[Relation vs. Iposet]

Theorem 1. For any relation $R \subseteq \mathbb{N} \times \mathbb{N}$ *and iposets* $PL, QL \in 2^{\mathcal{P}}$,

$$\mathcal{R}(PL \triangleright QL) = \mathcal{R}(PL) \circ \mathcal{R}(QL), \tag{18}$$

$$\mathcal{L}(R \circ R') = \mathcal{L}(R) \triangleright \mathcal{L}(R'). \tag{19}$$

- *We call any* $I \subseteq 1_{\triangleright}$ *a subidentity.*

Theorem 2. [Boolean algebra] The set of $I \subseteq 1_{\triangleright}$ *identity iposets generalize Boolean elements such that* $\overline{I} \subseteq 1_{\triangleright} \setminus I$ *denotes Boolean complement of* I*, and satisfies following Boolean axioms*

$$I \triangleright \overline{I} = \emptyset, \tag{20}$$

$$I \cup \overline{I} = 1_{\triangleright}. \tag{21}$$

Domain and modal operators for iposet languages

[Domain definitions]

Definition 13. Domain and Range applied to an iposet P,

$$dom([n] \rightarrow P \leftarrow [m]) \stackrel{def}{=} [n] \rightarrow [n] \leftarrow [n]$$

$$ran([n] \rightarrow P \leftarrow [m]) \stackrel{def}{=} [m] \rightarrow [m] \leftarrow [m]$$

Definition 14. Domain and Range applied to a set of iposets A ,

$$dom(A) \stackrel{def}{=} \{dom(P) \mid P \in A\}$$

$$ran(A) \stackrel{def}{=} \{ran(P) \mid P \in A\}$$

Definition 15. Antidomain and Antirange applied to to a set of iposets A,

$$ant(A) \stackrel{def}{=} 1_{\triangleright} \setminus \{dom(P) \mid P \in A\}$$

$$ar(A) \stackrel{def}{=} 1_{\triangleright} \setminus \{ran(P) \mid P \in A\}$$

Domain and modal operators for iposet languages

[Domain properties]

Lemma 14. Following equalities exist for domain of individual iposet.

$$\begin{aligned} dom(P) \rhd P &= P \\ dom(dom(P)) &= dom(P) \\ dom(P \rhd Q) &= dom(P) \text{ if } ran(P) = dom(Q) \end{aligned}$$

Lemma 15. Following equalities exist for the range of individual iposets.

$$\begin{aligned} P \rhd ran(P) &= P \\ ran(ran(P)) &= ran(P) \\ ran(P \rhd Q) &= ran(Q) \text{ if } ran(P) = dom(Q) \\ ran(dom(P)) &= dom(P) \\ dom(ran(P)) &= ran(P). \end{aligned}$$

Domain and modal operators for iposet languages

[Domain axioms]

Theorem 3. [Domain axioms] For some sets of iposets A, B we have:

$$A \cup dom(A) \rhd A = dom(A) \rhd A \quad (22)$$

$$dom(A \rhd B) = dom(A \rhd dom(B)) \quad (23)$$

$$dom(A) \cup 1_{\rhd} = 1_{\rhd} \quad (24)$$

$$dom(\emptyset) = \emptyset \quad (25)$$

$$dom(A \cup B) = dom(A) \cup dom(B) \quad (26)$$

Corollary 5. [Range axioms] For some sets of iposets A, B we have:

$$A \cup A \rhd ran(A) = A \rhd ran(A) \quad (27)$$

$$ran(A \rhd B) = ran(ran(A) \rhd B) \quad (28)$$

$$ran(A) \cup 1_{\rhd} = 1_{\rhd} \quad (29)$$

$$ran(\emptyset) = \emptyset \quad (30)$$

$$ran(A \cup B) = ran(A) \cup ran(B) \quad (31)$$

Domain and modal operators for iposet languages

[Anti-domain axioms]

Theroem 4. [Anti-domain axioms] For some sets of iposets A, B we have:

$$
\begin{aligned}
ant(A) \rhd A &= \emptyset \\
ant(A \rhd B) &= ant(A \rhd dom(B)) \\
dom(A) \cup ant(A) &= 1_{\rhd} \\
ant(\emptyset) &= 1_{\rhd} \\
dom(A) \rhd ant(A) &= \emptyset \\
ant(A \cup B) &= ant(A) \rhd ant(B)
\end{aligned}
$$

Corollary 6. [Anti-range axioms] For some sets of iposets A, B we have:

$$
\begin{aligned}
ar(A) \cup ran(A) &= 1_{\rhd} \\
A \rhd ar(A) &= \emptyset \\
ar(A \rhd B) \cup ar(ran(A) \rhd B) &= ar(ran(A) \rhd B)
\end{aligned}
$$

Domain and modal operators for iposet languages

[Boolean domain axioms]

Theroem 5. [Boolean domain axioms] The domain and antidomain operations satisfies following axioms, for some sets of iposets A, B and C

$$\begin{aligned} ant(A) \cup dom(A) &= 1_{\triangleright} \\ dom(A) \triangleright (ant(A) \cup dom(B)) &= dom(A) \triangleright dom(B) \\ dom(B) \triangleright (ant(A) \cup dom(B)) &= dom(B) \\ ant(A) \cup (dom(B) \triangleright dom(C)) &= (ant(A) \cup dom(B)) \triangleright (ant(A) \cup dom(C)). \end{aligned}$$

Then, $1_{\triangleright}$ is a Boolean domain semiring and $dom(ant(A)) = ant(A)$.

Theroem 6. [Domain and $\otimes$ products] For some iposets P, Q we have:

$$dom(P \otimes Q) = dom(P) \otimes dom(Q) = dom(Q \otimes P)$$

- For some sets of iposets A, B we have:

$$dom(A \otimes B) = dom(A) \otimes dom(B)$$

- For some sets of Σ-labelled iposts P, Q, we have:

$$dom(P \otimes Q) \neq dom(Q \otimes P)$$

Domain and modal operators for iposet languages

[Forward modal diamond operator]

Definition 16. We define a **forward modal diamond operator**

$$|-\rangle - : 2^{\mathcal{P}} \times 2^{1_{\triangleright}} \to 2^{1_{\triangleright}}$$

taking one language of iposets and one subidentity and returning another subidentity, as follows

$$|A\rangle I \stackrel{\text{def}}{=} dom(A \triangleright I).$$

Theorem 7. [Forward Modal axioms] For some sets of iposets A, B and some subidentities I, I' we have:

$$
\begin{aligned}
|A \cup B\rangle I &= |A\rangle I \cup |B\rangle I \\
|A\rangle(I \cup I') &= |A\rangle I \cup |A\rangle I' \\
|A \triangleright B\rangle I &= |A\rangle |B\rangle I \\
|A\rangle \emptyset &= \emptyset \\
|1_{\triangleright}\rangle I &= I \\
|I\rangle I' &= I \cap I'
\end{aligned}
$$

Domain and modal operators for iposet languages

[Forward modal box operator]

Definition 17. We define a **forward modal box operator**

$$\neg(|A\rangle I) \stackrel{\text{def}}{=} \neg(dom(A \rhd I)),$$

$$\textit{i.e.,} \quad |A]I \stackrel{\text{def}}{=} ant(A \rhd ant(I)).$$

Corollary 7. [Forward Box axioms] For some sets of iposets A, B and some subidentities I, I' we have:

$$\begin{aligned} |A \cup B]I &= |A]I \cup |B]I \\ |A](I \cup I') &= |A]I \cup |A]I' \\ |A \rhd B]I &= |A]|B]I \\ |A]\emptyset &= \emptyset \\ |\mathbf{1}_{\rhd}]I &= I \\ |I]I' &= I \cap I' \end{aligned}$$

Domain and modal operators for iposet languages

[Backward modal diamond operator]

Definition 18. We define a **backward modal diamond operator**

$$(|A\rangle I)^c \overset{\text{def}}{=} (dom(A \rhd I))^c,$$

$$i.e., \quad \langle A|I \overset{\text{def}}{=} ran(I \rhd A).$$

Corollary 8. [Backward Diamond axioms] For some sets of iposets A, B and some subidentities I, I' we have:

$$\begin{aligned}
\langle A \cup B|I &= \langle A|I \cup \langle B|I \\
\langle A|(I \cup I') &= \langle A|I \cup \langle A|I' \\
\langle A \rhd B|I &= \langle A|\langle B|I \\
\langle A|\emptyset &= \emptyset \\
\langle 1_{\rhd}|I &= I \\
\langle I|I' &= I \cap I'
\end{aligned}$$

Domain and modal operators for iposet languages

[Backward modal box operator]

Definition 19. We define a **backward modal box operator**

$$\neg(|A]I) \stackrel{\text{def}}{=} \neg(ran(I \rhd A)),$$

$$i.e., \quad [A|I = ar(ar(I) \rhd A)$$

Corollary 7. [Backward Box axioms] For some sets of iposets A, B and some subidentities I, I' we have:

$$[A \cup B|I = [A|I \cup [B|I$$
$$[A|(I \cup I') = [A|I \cup [A|I'$$
$$[A \rhd B|I = [A|[B|I$$
$$[A|\emptyset = \emptyset$$
$$[1_{\rhd}|I = I$$
$$[I|I' = I \cap I'$$

Conclusion

1. We derived algebraic results of iposets.
   - $(\mathbf{P}, \rhd, \otimes)$ satisfies (some) axioms of concurrent semigroup.
   - $(2^{\mathcal{P}}, \cup, \rhd, \otimes, 0, 1_{\rhd}, 1_{\otimes})$ iposet language bisemiring.
2. We gave structured theory of iposet language.
   - $\mathcal{T}_n \subsetneq \mathcal{C}_n$
   - $Q \in \mathcal{C}_n$ such that Q is not "N-free"
   - $\mathcal{C}_n$ - complete and incomplete hierarchy
3. We derived algebraic properties of order structure of iposets under subsumption.
4. We axiomatised domain and modal operators for iposets languages.

Future Work

1. Downwards-closed sets of iposets as a language model for concurrency such as Concurrent Kleene Algebra.
2. We also see for an operational model that allows a decision procedure for iposets language equivalence.