Partially Ordered Sets with Interfaces: A Novel Algebraic Approach for Concurrency

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Abstract

A partially ordered set, or poset for short, is a set (of events) together with a partial order on it. Formally, a poset P is defined as an ordered pair $(E_P, \preceq)$, where E_P is called the ground set of P and $\preceq$ is the partial order on E_P. A poset of events can be used to model both sequential and parallel behaviour. For instance, we can represent two events that can execute in parallel by a two-element poset with no order between the two events and their sequential execution by a two-element poset with the two events ordered. The series-parallel posets are restricted class of posets that are generated from single event posets by a finite number of sequential and parallel compositions. The isomorphic class of Σ-labelled series-parallel posets over a finite alphabet Σ have been extensively studied as an algebraic model of concurrency.

We investigate posets with interfaces, or iposets for short, as a concurrency model, and explore their language theoretic properties. In this model, an interface is defined as the image of a monomorphism on the minimal and maximal events of the poset. Compared to series-parallel posets, iposets admit richer algebraic properties. Further, we define axioms of domain operations for iposet languages, which lead to a simple and natural algebraic approach to modal logic based on equational reasoning. Such algebraic approaches to modal operators are known to be suitable for automated reasoning, for example, using tools like Isabelle.

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Contents

Chapter 1

Introduction

Pomsets [13] are a widely studied model of true concurrency. Pomset labels the vertices of an underlying poset by letters to capture actions taking place at particular events. A pomset over an alphabet Σ is defined as an isomorphic class of Σ-labelled posets, and those defined by single vertex posets are called singleton pomsets. Pomsets are generalisation of words over an alphabet in that letters may be partially ordered rather than totally ordered. This generalisation of pomsets produces two kinds of words, linear words and commutative words. Linear words can be generated from singleton pomsets by using a finite number of non-commutative sequential operations whereas commutative words by using a finite number of commutative parallel operations. This behavioural justification for the linear and parallel compositions of actions based on sequential and parallel operations project pomsets a natural model of concurrency.

Pomset languages are sets of pomsets, and the class of pomset languages generated by a finite number of sequential and parallel operations over singleton pomsets are called series-parallel (SP) or N-free pomsets [13, 32]. SP pomsets have been extensively studied in the literature [1, 3, 17, 23, 24, 25] for a language-theoretic model of concurrency. A pomset with a four element set {a, b, c, d} defined by the following

$$a \preceq c, a \preceq d \text{ and } b \preceq d$$

non-trivial partial order relation is called a N-pomset as shown in Figure 1.1. By definition, N-pomset does not belong to the class of SP pomsets. However, N-pomset is not free in the broader spectrum of concurrency problems. It has a natural appearance in various graph structures such as zigzag posets [35]

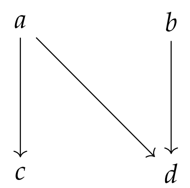


FIGURE 1.1: N-pomset

and concurrency problems such as the bounded-buffer problem [22] as shown in the Figure 1.2. The algebraic treatment of such N-pomset and their graph structures [11] which are natural in concurrency modelling and computational science [2, 5] is the primary goal of this thesis.

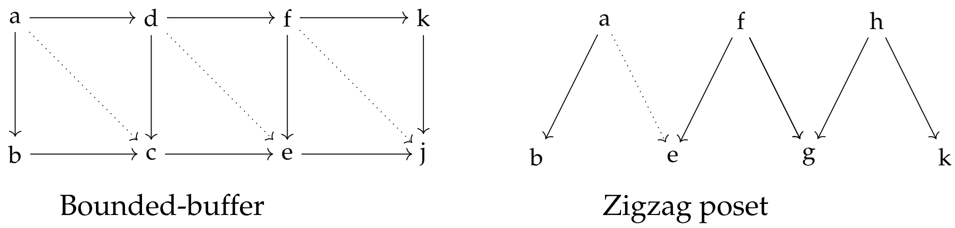


FIGURE 1.2: Graphs that contain N-pomset structures

Furthermore, subsets of SP pomsets that are closed under Gischer's subsumption order [13] are called downward-closed SP pomsets. The downward-closed SP pomsets generate a free language model of Concurrent Kleene Algebra [17, 24] and confirm that the original axioms of Concurrent Kleene algebra [14] are complete for the verification task of concurrent programs. Moreover, the modal operators [8] derived from domain definitions of the language model [7, 9] asymmetrically support computer-enhanced automated verification of programs. Therefore, we are concerned with the domain definitions of such language model of Concurrent Kleene algebra that produce an algebraic approach to modal operators, which is the second goal of the thesis.

To address these stated goals, we present a language-theoretic model of posets with interfaces, named iposets. Interfaces are an image of monomorphisms on minimal and maximal events of posets. The image of monomorphisms on minimal events of posets define the source interfaces of iposets. Similarly, the image of monomorphisms on maximal events of posets define the target interfaces of iposets. This definition of interfaces surprisingly generates a rich series-parallel compositionality of iposets compared to SP posets. The languages of iposets support definition of domain operations based on the relational semantics of their source and target interfaces.

We now outline the remainder of this chapter. We shortly state our motivation for posets with interfaces and summarise the chapter by presenting contributions and the organisation of the thesis.

1.1 Motivation for the posets with interfaces

We propose a language-theoretic model of posets with interfaces for modelling concurrency inspired by Pratt's [32, 33] and Gischer's [13] foundational work on partially ordered multisets, pomsets. Our motivation for the language-theoretic model of iposets lies in the following facts.

1.1.1 Partial order

Pratt [32, Theorem 1] advocates our motivation for partial orders. The partially ordered sets can be expressed as a set of its linearizations. The linearizability of partially ordered sets implies that the existing linear order model of computation can be extended to the partial orders by a adding shuffle operator in their associated language theory as shown by Gischer [13]. Linearization gives partial orders the same representational ability as those of linear orders with concurrency.

Further, partial order models of concurrency confer partial orders as a subset of linear orders that adopts the causal structure of events in an interval order rather than a complete linear order of time. The interval order representation of events implies that partial orders are more natural to concurrency modelling compared to the linear order model of computation. For instance, take an example of orthocurrence [34] composition of pomsets which clearly break downs the linear order compositionality of computation. A similar argument can be made for an interleaving model of pomsets unless pomsets are a singleton.

1.1.2 Domain

The domain operations [7, 9] on language semirings produce a straightforward algebraic approach to modal logic that enables computer enhanced automated analysis and verification of programs [8]. The Boolean algebra extensions of language semirings which model guards in the language definition such as Kleene Algebra with tests [21], Synchronous Kleene algebra (with tests) [36] and Concurrent Kleene algebra with tests [15] are not efficient for automated reasoning compared to the domain operations on language semirings. Therefore, we are concerned with such domain definitions of language models that enables automated verification of concurrent programs.

1.2 The contributions of this thesis

Chapter 2 of the thesis is dedicated to an overview of Kleene Algebra [19]. We focus on domain operations [7, 9] of language semirings for modal logic extensions based on the equational theory of Kleene algebra.

The contribution of the thesis starts with the presentation of posets with interfaces in Chapter 3. We define posets with interfaces including their series and parallel compositions in Section 3.3. The main contributions of the thesis are presented in Chapter 4. We present algebraic results of iposets theory that include an equational theory of iposets language and their structured theory under subsumption order. Further, we axiomatise domain operations for iposets languages and derive the corresponding modal operators in Chapter 5. We give concluding remarks on iposets theory in Chapter 6.

1.3 The organisation of this thesis

Readers may notice that the thesis presents specialised literature in algebra and order theory inclined towards an algebraic approach to modal logics for automated verification of programs. We expect that the readers are familiar with the basics of order theory and Kleene algebra. Therefore, we will keep the rest of the chapters carefully brief with precise technical details and theoretical presentation.

Chapter 2 presents Kleene algebra and its language theoretic model. We begin with a complete set of Kleene algebra axioms and brief remarks on Kleene algebra completeness of language models. Further, it presents axioms of Kleene algebra with domain [9] based on language semirings.

Chapter 3 presents definition of posets with interfaces and their series and parallel compositions. We begin with the definition of posets, and their sequential and parallel compositions. The subsequent section presents a basic categorical [26] overview of morphisms theory along with the horizontal and vertical compositions corresponding to the series and parallel compositions of posets.

Chapter 4 presents an algebraic theory of iposets that includes axioms of iposets algebra and languages including the order structure of iposets under subsumption.

Chapter 5 presents axioms of domain operations for iposet languages and their algebraic approach to the modal operators.

Chapter 6 presents concluding remarks on the iposets theory.

Chapter 2

Background

In this chapter, we present theoretical results of Kleene algebra [19] and construction of its language theoretic models. In remark, we will show how a Kleene algebra completeness result for a language model can be derived. Here, we use some examples to illustrate applications of Kleene algebra. The brief exposition of Kleene language semirings and their domains axioms presented in this chapter serve as cornerstone for an understanding of development in language theory in Chapter 4 and their domain axiomatization in Chapter 5.

2.1 Kleene algebra

Definition 1. *A* semiring *is a structure* $(S, +, \cdot, 0, 1)$ *such that* $(S, +, 0)$ *is a commutative monoid,* $(S, \cdot, 1)$ *is a monoid and these two monoids communicate through distributive law i.e. multiplication is left and right distributive with respect to addition, for all* $a, b, c \in S$*;*

$$a(b + c) = ab + ac \quad \textit{and} \quad (a + b)c = ac + bc.$$

Zero 0 is an annihilator with respect to multiplication.

$$a0 = 0 = 0a.$$

If 0 = 1 then $a = a1 = a0 = 0$ for all $a \in S$, i.e., semiring S is trivial. Therefore, we always assume $0 \neq 1$.

Definition 2. *A semiring* S *is* idempotent *iff addition* $a + a = a$ *holds for all* $a \in S$*. A* dioid *is an idempotent semiring.*

A partial ordering relation $\leq$ over a dioid S forms a semilattice with 0 as least element and addition as join,

$$a \leq b \Leftrightarrow a + b = b$$

for all $a, b \in S$. A addition and multiplication are monotonic with respect to the partial order $\leq$.

Definition 3. *An elements of a dioid S are called* subidentities *if*

$$a \leq 1, \textit{for any } a \in S.$$

Let $sid(S)$ be the set of subidentities of dioid S. The subidentities of a dioid forms a dioid where multiplication produce a lower bound operation. For example, for any $a, b \in sid(S)$ then ab forms a lower bound of a and b

$$a = a1 \geq ab \leq 1b = b.$$

Multiplicative idempotent $sid(S)$ of a dioid S form a bounded distributive lattice. For example, for any $a, b, c \in sid(S)$ with $c \leq a$ and $c \leq b$ then $c = cc \leq ab$ where ab forms greatest lower bound of a and b. Similarly,

$$a(b + c) = ab + ac$$
$$a + bc = (a + b)(a + c)$$

holds by the dioid distributive law.

Definition 4 (Kleene algebra)**.** *A* Kleene algebra *is a structure $(K, +\cdot, *, 0, 1)$, dioid expanded by a star operation such that $(K, +, \cdot, 0, 1)$ is a dioid and, for all $a, b \in K$ satisfies following equations and equational implications*

$$1 + aa^* \leq a^* \tag{2.1}$$
$$1 + a^*a \leq a^* \tag{2.2}$$
$$b + ax \leq x \rightarrow a^*b \leq x \tag{2.3}$$
$$b + xa \leq x \rightarrow ba^* \leq x \tag{2.4}$$

Axioms (2.5)-(2.6) are equivalent to the star induction axioms (2.3)-(2.4). They explain that the $*$ behaves similar to asterate operator of formal language theory.

$$ab \leq b \rightarrow a^*b \leq b \tag{2.5}$$
$$ba \leq b \rightarrow b^*a \leq b \tag{2.6}$$

The $\leq$ refers to partial order on K, and $+$ operation forms a lower bound with respect to the partial order $\leq$ on K

$$a \leq b \leftrightarrow a + b = b.$$

Remark 1. *All the operators of Kleene algebra are monotone with respect to $\leq$, i.e., if $a \leq b$ then $ac \leq bc$, $a + c \leq b + c$ and $a^* \leq b^*$ for any $a, b, c \in K$.*

Axioms (2.1)-(2.4) model the $*$ operation similar to the reflexive transitive closure operator of relational algebra. Axioms (2.1)-(2.2) and (2.3)-(2.4) are $*$

unfold and induction axioms. The implications (2.3)-(2.4) denote left-hand substitution of a^*b and ba^* for x, i.e., a^*b and ba^* are the least pre-fixed point of the following monotone functional map,

$$x \mapsto b + ax \quad \text{and} \quad x \mapsto b + xa.$$

A Kleene algebra is * -continuous if it satisfies the infinitary condition

$$ab^*c = \bigcup_{n \geq 0} ab^n c \tag{2.7}$$

where,

$$b^0 \stackrel{\text{def}}{=} 1$$
$$b^{n+1} \stackrel{\text{def}}{=} bb^n.$$

Equation (2.7) is conjunction of infinitely many equational axioms

$$ab^n c \leq ab^*c, \quad n \geq 0$$

and infinitary Horn formula

$$\bigwedge_{n \geq 0} ab^n c \leq x \;\rightarrow\; ab^*c \leq x. \tag{2.8}$$

The $*$-continuity (2.7) implies (2.3)-(2.6) in the presence of other axioms above, and strictly strong in the sense that there exist Kleene algebras that are not $*$-continuous [20]. The rest important theorems of kleene algebra are

$$(ab)^*a = a(ba)^* \tag{2.9}$$
$$ax = bx \rightarrow a^*x = xb^* \tag{2.10}$$
$$(a+b)^* = a^*(ba^*)^* \tag{2.11}$$
$$a^{-1}b^*a = (a^{-1}ba)^* \quad \text{iff } a \in K \text{ with } a^{-1}. \tag{2.12}$$

Remark 2 (Kleene algebra completeness of a language model)**.**

Let Σ^* be the set of finite words over finite alphabet Σ. Then, the structure

$$\mathrm{LAN}(\Sigma) = (2^{\Sigma^*}, \cup, ., \emptyset, \{\varepsilon\})$$

where

2^{Σ^*} denotes the set of languages over Σ,
$\cup$ denotes set union,
uv denotes concatenation of u and v such that $L_1.L_2 = \{uv \mid u \in L_1 \wedge v \in L_2\}$,
$\emptyset$ denotes the empty language, and
ε denotes the empty word.

LAN(Σ) is a language dioid. The language dioid LAN(Σ) can be extended to Kleene algebra by

$$L^* = \{w_1 w_2 .....w_n \mid n \geq 0 \wedge w_i \in L\},$$

called Kleene algebra over Σ. The set that can be obtained from finite subsets of Σ^* by a finite number of regular operations $(\cup, ., *)$ are regular events of Σ^*. The equational theory of regular subsets is called algebra of regular events of Σ^*. Let T be the finitely generated term algebra from the generator $\Sigma \cup \{0, 1\}$. The syntactic terms t of T can be generated by the following grammar, for $a \in \Sigma$

$$t ::= a \mid 0 \mid 1 \mid t + t \mid t.t \mid t^*.$$

The natural homomorphism h from term algebra $s, t \in T$ over the Kleene algebra generated by Σ onto the algebra of regular events over Σ^* is given by

$$\begin{aligned}
h(a) &= \{a\} \\
h(1) &= 1_h \text{ iff } \{1_h = \{\varepsilon\}\} \\
h(0) &= \emptyset \\
h(s+t) &= h(s) \cup h(t) \\
h(s.t) &= h(s).h(t) \\
h(s^*) &= \bigcup_{n \geq 0} h(s)^n.
\end{aligned}$$

A subset L of Σ^* is a rational language if $L = h(t)$ for some Kleene algebra term t. The subalgebra $h_\Sigma = \{h(t) : t \in T(\Sigma)\}$ is the algebra of rational languages, and h_Σ is isomorphic to the free Kleene algebra over Σ.

Remark 3 (Regular languages)**.**

A subset L of Σ^* is a regular language if it is accepted by a finite automaton, and the Kleene theorem states that a subsets of Σ^* is a rational language iff it is a regular language. A subset L of Σ^* is a recognizable language if it is recognized by a finite-index congruence algebra [27], and the Myhill-Nerode theorem states that a language is regular iff it is recognizable.

Now, we present some examples of Kleene algebra.

Example 1. Consider the structure over a set Σ

$$\mathrm{R}(\Sigma) = (2^{\Sigma \times \Sigma}, \cup, \circ, \emptyset, i),$$

where

$2^{\Sigma \times \Sigma}$ denotes the set of binary relation $\mathcal{R}$ over Σ,

$\circ$ denotes relational product,

$\emptyset$ denotes the empty relation, and

$$i \text{ denotes the identity relation } i = \{(a,a)|a \in \Sigma\}.$$

R(Σ) is called a relational dioid over Σ. The $R(\Sigma)$ relational dioid can be extended to a Kleene algebra over Σ by defining $\mathcal{R}^*$ for all relations $\mathcal{R} \in R(\Sigma)$

$$\mathcal{R}^* = \bigcup_{x \geq 0} R^x \text{ with } \mathcal{R}^0 = i \text{ and } \mathcal{R}^{x+1} = \mathcal{R} \circ \mathcal{R}^x.$$

Example 2. Let Σ be a set of vertices of a graph. Then, subsets of Σ^* can be seen as the set of possible paths in graph, and ε as the empty path in graph. The fusion of two paths can be defined as, for $u, v \in \Sigma^*$ and $x, y \in \Sigma$

$$(u.x) \circledcirc (y.v) = \begin{cases} u.x.v & \text{if } x = y \\ \text{undefined} & \text{otherwise} \end{cases}$$

describing the gluing of paths at a common end points. This gluing $\circledcirc$ operation can be extended to the subsets of Σ^* by

$$U \circledcirc V = \{u \circledcirc v | u \in U \wedge v \in V \text{ and } u \circledcirc v \text{ defined}\}.$$

The structure $P(\Sigma) = (2^{\Sigma^*}, \cup, \circledcirc, \Sigma \cup \{\varepsilon\})$ defines a dioid, called path dioid. The path diod $P(\Sigma)$ can be extended to a Kleene algebra by a following procedure like relation dioid in Example 2.

2.2 Kleene algebra with tests

Definition 5 (Test semiring)**.** *A* test semiring *is a dioid S in which a boolean algebra B is embedded by a map $h : B \to S$ such that, for any $a, b \in S$*

$$\begin{aligned} h(0) &= 0 \\ h(1) &= 1 \\ h(a \sqcup b) &= h(a) + h(b) \\ h(a \sqcap b) &= h(a).h(b) \end{aligned}$$

where the operators $+, ., 0$ and 1 of the dioid S defines join, meet, falsity and truth on the element of B.

Definition 6 (KAT)**.** *A* Kleene algebra with tests *(KAT) is a Kleene algebra with an embedded Boolean subalgebra. It is a two-sorted structure*

$$(K, B, +, \cdot, *, -, 0, 1)$$

where $-$ is a unary operator defined only on element of B, such that

$$\begin{aligned} &B \subseteq K, \\ &(K, +, \cdot, *, 0, 1) \textit{ is a Kleene algebra, and} \end{aligned}$$

$(B, +, \cdot, -, 0, 1)$ *is a Boolean subalgebra.*

The other operators $+, ., 0$ and 1 in the KAT structure play two different roles; they refers to choice, composition, fail and skip when applied to arbitrary element of K and give meaning of join, meet, falsity and truth on the element of B as described in the Definition 5 above.

Definition 7 (Boolean algebra)**.** *Boolean algebra B admits a boolean negation operator $^-$ defined on the element of B and satisfies following axioms in addition to the Kleene algebra axioms, for any $b, c \in B$*

$$b \leq 1 \tag{2.13}$$
$$b + \bar{b} = 1 \tag{2.14}$$
$$b\bar{b} = 0 \tag{2.15}$$
$$bc = cb \tag{2.16}$$

The meet operator $\cdot$ acts as conjunction when applied to elements of B whereas join operator $+$ acts as disjunction. Conjunction implies a test ab succeeds iff both a and b succeed. Similarly, disjunction implies a test $a + b$ succeeds iff either one of a and b succeeds.

The axiom $b \leq 1$ for all $b \in B$, intuitively implies that the test semiring can be obtained from Kleene algebra semiring K [6] by $test(K)$ of $sid(K)$ with greatest element 1 and least element 0 such that

$$test(K) = \{b \in K | b \leq 1\}. \tag{2.17}$$

Although $test(K)$ in Equation (2.17) over Klene semiring K seems plausible in relational algebra, test(K) is not always extendable to a Boolean algebra [21, Theorem 4]. For instance, modelling of identity relation corresponding to the pre-post or input-out conditions where complement of such identity relation will be undecidable is a counterexample to Cohen's construction of Boolean algebras [6]. Therefore, elements of boolean algebra in KAT, i.e. $\{0, 1\}$, are explicitly defined to be a simple predicate that are easily decidable.

2.2.1 Language model of Kleene algebra with tests

KAT is defined over a free Kleene algebra K on generator Σ with an embedded Boolean subalgebra B. The Σ and B are finite disjoint set of actions and tests.

Let $T_{\Sigma,B}$ denote the set of all terms over $\Sigma \cup B$. Similarly, T_Σ denotes all K terms over Σ that represent actions and T_B denotes all Boolean term over B that represent tests. The language theoretic model of KAT is based on the idea of guarded terms $T_{\Sigma,B}$ that can be obtained by inserting terms T_B among the terms T_Σ.

Definition 8 (Guarded string). *A* guarded string *(GS) over $\alpha_i \in B$ and $p_i \in \Sigma$ can be expressed by $(1_G\Sigma)^*1_G$ such as*

$$\{\alpha_0 p_1 \alpha_1 p_2 ... p_n \alpha_n \ \geq 0\} = (\Sigma \cup B)^*,$$

where 1_G denotes the multiplicative identity over T_B. If $U, V \subseteq GS$ then

$$U \circledcirc V \stackrel{def}{=} \{u \circledcirc v | u \in U, v \in V\},$$

where the partial binary operation $\circledcirc$ on GS can be defined as

$$(ux) \circledcirc (yv) = \begin{cases} uxv & \text{if } x = y \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The operation $\circledcirc$ is similar to string concatenation except that two intermediate atoms are glued into one if the terminal atoms of the first string is the same as the initial atom of the second string. The identity of glued product is 1_G.

Now, we present remarks on KAT language model and their completeness result.

Remark 4. Consider the structure over Σ and B

$$A_{\Sigma,B} = (2^{GS}, 2^{1_G}, \cup, \circledcirc, ^-, \emptyset, 1_G)$$

where,

Σ and B are finite disjoint sets of actions and tests,

2^{1_G} is a set-theoretic Boolean algebra.

The structure $A_{\Sigma,B}$ can be extended to free $*$-continuous Kleene algebra with tests by defining $*$ (infinite union) over GS. The infinite union over $U, V, S \in GS$ can be expressed as

$$\begin{aligned} U \circledcirc S^* \circledcirc V =& U \circledcirc (\bigcup_{n\geq 0} S^n) \circledcirc V \\ & \bigcup_{n\geq 0} U \circledcirc S^n \circledcirc V. \end{aligned}$$

Let $\mathcal{A}$ be subalgebra of $A_{\Sigma,B}$ generated by following GS element

$$\begin{aligned} &\{\alpha p \beta \mid \alpha, \beta \in 1_G\}, \quad p \in \Sigma \\ &\{\alpha \in 1_G \mid \alpha \leq b\}, \quad b \in B. \end{aligned}$$

Then, $\mathcal{A}$ is the smallest subalgebra of $A_{\Sigma,B}$ containing set Σ and B, and closed under operations of $A_{\Sigma,B}$.

Remark 5. The Kleene algebra with tests over actions Σ and tests B is the homomorphic mapping onto guarded terms $T_{\Sigma,B}$ generated by $\Sigma \cup B$. The homomorphism

$$h : T_{\Sigma,B} \longrightarrow A_{\Sigma,B}$$

can be defined inductively as follows,

$$\begin{aligned}
h(p) &= \{\alpha p \beta \mid \alpha, \beta \in 1_G\}, p \in \Sigma \\
h(b) &= \{\alpha \in 1_G \mid \alpha \leq b\}, b \in B \\
h(p+q) &= h(p) \cup h(q) \\
h(pq) &= h(p) \circledcirc h(q) \\
h(1) &= 1_G \\
h(0) &= \emptyset \\
h(\bar{b}) &= 1_G - h(b) \\
h(p^*) &= h(p)^*.
\end{aligned}$$

A subset L of $(\Sigma \cup B)^*$ is a rational guarded language if $L = h(t)$ for some guarded term $t \in T_{\Sigma,B}$. The subalgebra $h_{\Sigma,B} = \{h(t) : t \in T_{\Sigma,B}\}$ is the the algebra of rational guarded languages, and $h_{\Sigma,B}$ is isomorphic to the free Kleene algebra with tests over $\Sigma \cup B$.

Example 3 (KAT over relational structure)**.**

Consider a relational structure over set $\Sigma \cup B$ where Σ and B denote set of actions and tests as given in Definition 8 above

$$\mathrm{R}_{\Sigma,B} = (2^{GS \times GS}, 2^{1_G}, \cup, \circ, \emptyset, 1^G)$$

where,

$2^{GS \times GS}$ denotes the set of binary relation $\mathcal{R}$ over GS,

$\circ$ denotes relational product,

$\emptyset$ denotes the empty relation,

1^G denotes the identity relation, and

$(2^{1_G}, \cup, \circ, \emptyset, 1^G)$ substructure is a Boolean algebra of subsets of 1^G.

The relational dioid $R_{\Sigma,B}$ can be extended to Kleene algebra by defining $\mathcal{R}^*$ for all relations $\mathcal{R} \in R_{\Sigma,B}$. Therefore, all $R_{\Sigma,B}$ relational algebra with tests are free Kleene algebra with tests over $\Sigma \cup B$.

2.3 Kleene algebra with domain

KAT subsumes propositional Hoare logic. However, it is not rich enough to admit the structure theory of modalities that occurs in popular formalism such as dynamic and temporal logic. Kleene algebra with domain (KAD) close this gap between KAT including Kleene algebra and various modal logics by defining an algebraic approach to the modal logics based on equational reasoning. KAD generates modal operators based on the axioms of domain operations and provides partial correctness semantics of programs [29] in wp style and encodings of Hoare logic.

Definition 9 (Test semiring)**.** *A* test semiring *is a composite structure* (S, B)*, where* B *is a Boolean algebra embedded into a dioid* S *such that join and meet operations correspond to addition and multiplication. The* B *contains only a subset of elements in* S *that are below* 1*, and the Boolean operations are restrictions of the semiring operations to* B*. Since the elements of* B *defines tests on* S*, they can be expressed as* $test(S)$*. Similarly, a test semiring is a KAT if the semirings is also a Kleene algebra, where all tests* $p \in B$ *satisfy* $p^* = 1$*.*

The domain [1] of an action denotes the set of states from which action can execute. The domain operations in KAD are initially axiomatised based on test semiring [7] instead dioid [9] because of its simplicity and naturalness. One reason is the domain elements in test semirings essentially possess a boolean structure that gratifies multiplicative idempotent property of least left preservers and greatest left annihilator, which is not straightforward in the dioid without a test. Another reason is an algebraic extension of domain based on the dioid such as quantiles and relation algebras are not language equivalence to the KAT. The test semirings in KAT is entirely focused on first-order approach which is essential and straightforward for modelling the usual constructs of sequential programming.

Definition 10 (Domain over test semiring)**.** *The* domain operation $d : S \to B$ over test semiring (S, B) *satisfies following axioms, for all* $x, y \in S$ *and* $p \in B$

$$x \leq d(x)x \tag{2.18}$$

$$d(px) \leq p \tag{2.19}$$

$$d(xd(y)) \leq d(xy) \tag{2.20}$$

It is clear that these axioms force the embedded Boolean algebra B to be maximal, *i.e.,* the complemented subidentity of S must be in B. The Boolean algebra B in this definition of the domain is not free, B is embedded into S

[1] Let h be a function from X to Y, then the inverse image of a set $Z \subseteq Y$ under h is the subset of X defined by

$$h^{-1}(Z) = \{x \in X \mid h(x) \in Z\}$$

The inverse image of $y \in Y$, denoted by $h^{-1}(y)$, is called the fiber over y. Lifting definition of inverse image to binary mapping $h : 2^\Sigma \to 2^\Sigma$ for $Z \subseteq \Sigma$ over the power set of Boolean algebra 2^Σ, where $\Sigma = \{x, y, ...\}$ represent a finite set of states. $h^{-1}(z)$ represent the set of states which are connected to Z by h, called the domain of h over Z. Similarly, an antidomain of h over Z defined by

$$\bar{h}^{-1}(Z) = \{x \in \Sigma \mid h(x) \notin Z\}$$

forming a subalgebra which yields a domain operation of type $d : S \to B$.

Corollary 1. *Let S be a dioid and B is a Boolean algebra. Then $d(S)$ contains the greatest Boolean subalgebra of S bounded by 0 and 1, i.e., $B \subseteq d(S)$.*

However, the property "x is a domain element" can be expressed internally in the language of semirings as observed by Cohen [6]. The Cohen observation implies that the domain elements can be expressed internally without typing constrained. The following example explains the internal [9] definition of domain element without the typing constrained.

Example 4. The domain $d(R)$ of a binary relation $R \subseteq K \times K$ is an endomorphism on the relation semiring $2^{K \times K}$ given by

$$d(R) = \{(a,a) \in K \times K : (a,b) \in R \text{ for some } b \in K\}.$$

Here $d(R)$ encodes a subset of the identity relation on K, more explicitly it can be defined as

$$d(R) = R \circ U \cap I_K,$$

where U and I_K denotes the universal and identity relation over K. $R \circ U$ models all the inputs-outputs relation R over K, and an intersection with I_K projects $R \circ U$ to the set of pairs (a,a) with $(a,b) \in R$. Further, this observation can be simplified over $P \subseteq 2^K$ under $R \subseteq K \times K$ in relation semirings by

$$\begin{aligned}\langle R \rangle P &= \{(a,a) \in K \times K : (a,b) \in R \text{ for some } b \in P\} \\ &= d(R \circ P).\end{aligned}$$

The structure $(I_K, (\langle R \rangle : R \subseteq K \times K))$ with endomorphism $\langle R \rangle : I_K \to I_K$ on the identity relation I_K over K forms a Boolean algebra similar to Jonsson and Tarski [16], and supports standard modal semantics based on kripke frames and labelled transition system. Logically, $d(R \circ P)$ demonstrate a mapping $d : K \to 2^B$ analogous to KAT but without typing constraints, where B is a Boolean algebra. To be precise, the set $d(R \circ P)$ is an inverse image of P under R.

Definition 11 (Domain over dioid)**.** *The domain operation $d : S \to S$ over a dioid S satisfies following axioms, for all $x, y \in S$*

$$d(0) = 0 \tag{2.21}$$

$$d(x) + 1 = 1 \tag{2.22}$$

$$x + d(x)x = d(x)x \tag{2.23}$$

$$d(xy) = d(xd(y)) \tag{2.24}$$

$$d(x + y) = d(x) + d(y) \tag{2.25}$$

By Axiom 2.21 domain is strict, Axiom 2.22 domain elements are subidentities, by Axiom 2.23 and Axiom 2.24 domain elements are left preservers and local, and by Axiom 2.25 domain is additive. By duality, it requires that the opposite of all these axioms of domain semirings should hold in codomain semirings [7].

The elements $x, y \in d(S)$ of some domain semiring $d(S)$ are called Boolean elements if they meet these three conditions

$$
\begin{aligned}
x + y &= 1, \\
xy &= 0 \quad \text{and} \\
yx &= 0.
\end{aligned}
$$

These axioms implies x and y are complement of each other. Let B_S denote the set of all complemented elements in S, then by definition $(B_S, +, ., 0, 1)$ forms a Boolean semiring correspond to dioid S. By duality, this result of domain semirings can be translated into codomain semirings. However, this duality does not hold in the domain definition over dioid due to the absence of Boolean complements as pointed out by Kozen and Smith [21], consequently, that leads to the failure of Galois connection [29]. Thus, antidomain is introduced [9] in order to reconcile a notion of complements in domain definitions over diod .

Definition 12 (Antidomain). *The antidomain operation $a : S \to S$ over a dioid S satisfies following axioms, for all $x, y \in S$*

$$a(x) \leq 1 \tag{2.26}$$
$$a(x)x = 0 \tag{2.27}$$
$$a(0) = 1 \tag{2.28}$$
$$a(xy) = a(xd(y)) \tag{2.29}$$
$$a(x + y) = a(x)a(y) \tag{2.30}$$

Antidomain of an action denotes the set of states from which it cannot execute, and naturally produce the complement of domain operations. Axioms of antidomain can be obtained by complimenting domain axioms.

Definition 13 (Boolean semiring). *A dioid S is a* Boolean semiring *iff it can be extended by a domain and an antidomain operation $a : S \to S$ that satisfies*

$$
\begin{aligned}
d(x) + a(x) &= 1 \\
d(x) &= a^2(x) \quad \textit{and} \\
d(x)a(x) &= 0
\end{aligned}
$$

These axioms of Boolean semiring state that domain semirings including

Boolean semirings, are entirely recoverable from the definition of antidomains over dioid by stipulating domain and antidomain elements complement of each other. This definition of antidomains $a(S)$ over dioid S defines KAD as a composite structure $(S, a(S))$, where $a(s)$ defines $test(S)$ on dioid S equivalent to the domain operations over test semiring in Definition 10 but without the typing constraints.

Chapter 3

Posets with interfaces

In this chapter, we begin with a brief exposition of partial order relations and partially ordered sets, posets. We explain posets and their structure with some examples and remarks including definitions for their series and parallel compositions. In subsequent section, we shortly present categorical theory of morphisms and 2-category with vertical and horizontal compositions defining interchange law. We keep exposition brief and precise, unless we introduce some non-standard notions or the new definitions, for a good understanding of development in the succeeding Section 3.3.

3.1 Posets

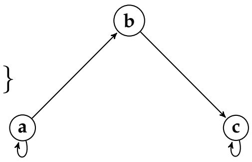
Definition 14 (Relations)**.** *Let P be a finite set of vertices. If $R \subseteq P \times P$ then R is a* binary relation *on P, denoted by $(u, v) \in R$. R is a* preorder or quasi-order relation *if it is reflexive and transitive. R is an* equivalence relation *if it is reflexive, symmetric, and transitive. R is a* partial order relation*, denoted by $\preceq$, if it is reflexive, anti-symmetric, and transitive. Similarly, R is a linearly order relation if it is reflexive, anti-symmetric, transitive and comparable, i.e., a partial order relation with requirement that each u, v in P with $u \neq v$, either $(u, v) \in R$ or $(v, u) \in R$.*

Example 5 (Binary Relation)**.** Let have a look at binary relations $R \subseteq P \times P$ in Figure 3.1. We can find four different ways to express $R:$ as a set, as a bipartite graph, as a directed graph and as an incidence matrix. Each of them have a different context, and yet are completely equivalent. For instance, the bipartite view is appropriate when we are interested in matching and incidence matrix is useful when we are interested in translating one context into another.

Let $\sim$ be an equivalence relation on finite set P defined by rule such that $u \sim v$ iff $(u, v), (v, u) \in R$. Moreover, if $u \sim u'$ and $v \sim v'$ then $(u, v) \in R$ iff $(u', v') \in R$. Equivalence relations $\sim$ on a set P generate a quotient set $P/_\sim$ defined by

$$P/_\sim = \{[u] \sim v \in P\}$$

$$\begin{array}{cccc} & a & b & c \\ a & & & \\ b & & & \\ c & & & \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \{(a,a),(a,b),(b,c),(c,c)\}$$



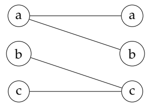


FIGURE 3.1: Different representation of binary relations

where $P/_\sim$ denotes set of equivalent classes and $[u]$ is the equivalence class of u in P. It explains that every equivalence relations yield partitions, and every partition yields an equivalence relation in a set.

Example 6 (Equivalence relation and partition)**.** Let $h : P \to Q$ be a function, and f denotes the collection of fibers of h which can be defined as follows

$$f = \{h^{-1}(\{h(x)\}) \mid x \in P\}.$$

In short, f is the collection of inverse image of elements of the range of h.

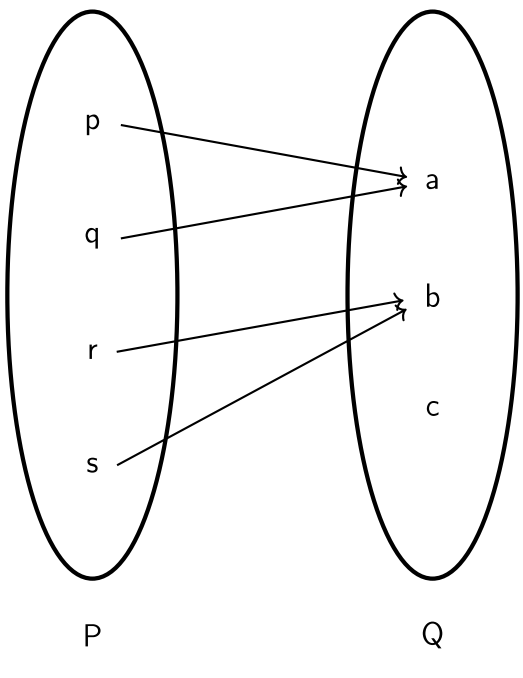


FIGURE 3.2: Partition

Suppose, $h : \{p,q,r,s\} \to \{a,b,c\}$ such that $h(p) = h(q) = a$ and $h(r) = h(r) = b$. Then, $h^{-1}\{h(p)\} = h^{-1}\{h(q)\} = h^{-1}(\{a\}) = \{p,q\}$ and $h^{-1}\{h(r)\} = h^{-1}\{h(s)\} = h^{-1}(\{b\}) = \{r,s\}$. So, $f = \{\{p,q\},\{r,s\}\}$ is a partition of $P = \{p,q,r,s\}$. It is immediate that every equivalence relation yields a partition and every partition yields an equivalence relation.

Now, we can define the equivalence classes $u \sim v$ as follows

$$[u]_\sim = \{v \in P \mid v \sim u\} = \{v \in P \mid h(v) = h(u)\} = h^{-1}(\{h(u)\}),$$

where $[u]_\sim$ denotes an equivalence class in P, *i.e.*, the set of all the elements of P that are related to u. This implies that fibers of h are precisely $\sim$ equivalence classes.

Remark 6 (Partial order relation)**.** $P/_\sim$ over a finite set P are order isomorphic class of preorders. Preorders are more general than equivalence relations and (non-strict) partial orders, both of which are special cases of a preorder. An anti-symmetric preorder is a partial order and a symmetric preorder is an equivalence relation. This shows how 'preorder or quasi-order' generalised; it can be made into a partial order by taking a quotient of equivalent relations. For example, if R is a quasi-order relation on a set P then the $\sim$ equivalence relation on P, which is a canonical quotient $P/_\sim$ of P, descends R to a partial order relation on $P/_\sim$.

Definition 15 (Poset)**.** *A poset P is a finite set of vertices E_P, whose elements are called event, together with a partial order relation $\preceq \subseteq E_P \times E_P$.*

Definition 16 (Parallel product)**.** *Given two posets $P = (E_P, \preceq_P)$ and $Q = (E_Q, \preceq_Q)$ such that $|E_P| \cap |E_Q| = \emptyset$, their* parallel product $\otimes$ *is the poset*

$$P \otimes Q = (E_P \sqcup E_Q, \preceq_{P \otimes Q})$$

where $u, v \in (E_P \sqcup E_Q)$ such that

$$u \preceq_{P \otimes Q} v \text{ iff } u \preceq_P v \text{ or } u \preceq_Q v.$$

Here, $\sqcup$ denotes the disjoint union (or coproduct) of sets. Every discrete poset $[n]$ is parallel product of singleton posets

$$[n] = [n-1] \otimes [1],$$

where posets with single vertex $|E_P| = 1$ are called singleton posets. Note that the parallel product of isomorphic posets are isomorphic, and that $[n + m]$ is isomorphic to $[n] \otimes [m]$. For each n, m, we introduce a ("standard") isomorphism $\phi_{n,m} : [n+m] \to [n] \otimes [m]$ given by the equation

$$\phi_{n,m}(i) = \begin{cases} i_{[n]} & \text{if } i \leq n\,, \\ (i-n)_{[m]} & \text{if } i > n\,. \end{cases} \tag{3.1}$$

Definition 17 (Sequential product)**.** *Given two posets $P = (E_P, \preceq_P)$ and $Q =$*

$(E_Q, \preceq_Q)$ *such that* $|E_P| \cap |E_Q| = \emptyset$, *their* sequential (or concatenation) product *is the poset*

$$PQ = (E_P \sqcup E_Q, \preceq_{PQ})$$

where $u, v \in (E_P \sqcup E_Q)$ *such that*

$$u \preceq_{PQ} v \text{ iff } u \preceq_P v \text{ or } u \preceq_Q v \text{ or } u \in E_P \text{ and } v \in E_Q.$$

Definition 18. *An element* u *of a poset* $P = (E_P, \preceq_P)$ *is called* maximal event *iff* $\nexists v \in E_P$ *such that* $u \preceq_P v$,

$$\{u \in E_P \mid u \preceq_P v \text{ such that } \nexists v \in E_P\}.$$

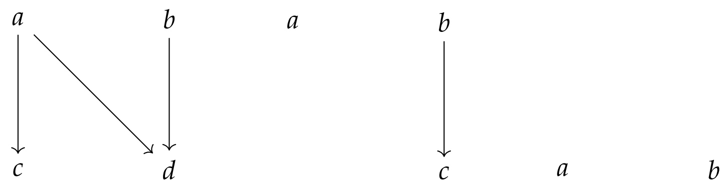
The u *is called top or maximum element of* P *if* $v \preceq_P u$ *for all* $v \in E_P$. *Dually,* u *is* minimal event *iff* $\nexists v \in E_P$ *such that* $v \preceq_P u$,

$$\{u \in E_P \mid v \preceq_P u \text{ such that } \nexists v \in E_P\}.$$

The u *is called the bottom or minimum element of* P *if* $u \preceq_P v$ *for all* $v \in E_P$.

It is possible that a poset might have one or more or no maximal event at all. A empty poset has no maximal event. However, a non-empty finite poset always at has at least one maximal element that can be found by choosing any element u in set and replacing it by an element v iff $u \preceq v$, and repeating until $u \npreceq v$. The process terminates after a finite step due to irreflexive and transitive laws. Dually, minimal elements.

Example 7. Figure 3.3 illustrates the structure of poset P, Q and R. By Definition 18, the set of vertices $(a, b) \in P$ and $(a, b) \in Q$ denote minimal and $(c, d) \in P$ and $(a, c) \in Q$ denote maximal events. However, $(a, b) \in R$ denote both minimal and maximal events. R is a discrete poset by definition given by Equation (3.1).



P Q R

FIGURE 3.3: Poset

Remark 7. Here, one should clearly distinguish between the concept of maximum (resp. minimum) and maximal (resp. minimal) events in a poset. Consider a poset P with a set of events $\{a, b, c, d\}$ with ordering relation $a \prec c, a \prec d, b \prec c$ and $b \prec d$ that consists of maximal (resp. minimal) events but no maximum (resp. minimum) event.

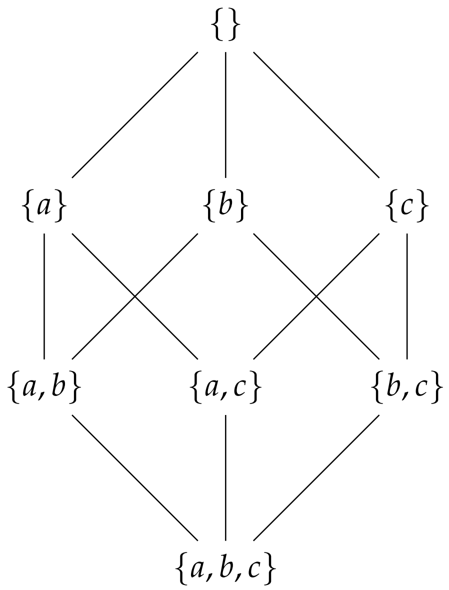


FIGURE 3.4: Chains and antichains

For any events u, v in a poset $(E_P, \preceq_P)$ are called comparable events iff there exists $u \preceq_P v$ or $v \preceq_P u$, else incomparable *i.e.*, neither $u \preceq_P v$ nor $v \preceq_P u$. Chains in poset $(E_P, \preceq_P)$ are subsets Q of P in which each pair of events are comparable, i.e., Q are totally ordered subset of P. Similarly, antichains in P are subsets Q of P in which each pair of events are incomparable, i.e., there is no order relation between each pair of events in Q. Now, it is immediate that a poset P can be expressed as a set of subsets Q of P based on chains and antichains. The depth of a poset is the largest cardinality of the chains, and width is the largest cardinality of the antichains.

Example 8. Illustrated by Figure 3.2, horizontal collection of set of events such as $\{\{a\}, \{b\}, \{c\}\}$ and $\{\{a, b\}, \{a, c\}, \{b, c\}\}$ produce a set of antichains whereas the vertical collection of set of events such as $\{\{a\}, \{a, b\}, \{a, b, c\}\}$ and $\{\{a\}, \{a, c\}, \{a, b, c\}\}$ produce a set of chains.

Definition 19 (Pomset)**.** *A poset $P = (E_P, \preceq_P)$ with a labeling $l : E_P \to \Sigma$, is called a* Σ-labelled *poset. A* morphism $h : P \to Q$ *of* Σ*-labelled posets is a function which preserves the ordering and labelling, i.e., for all* $u, v \in E_P$

$$u \preceq_P v \text{ implies } h(l_u) \preceq_Q h(l_v).$$

Similarly, an isomorphism $h : P \to Q$ *of* Σ*-labelled posets is a bijective morphism* $h : E_P \to E_Q$ *such that, for all* $u, v \in E_P$

$$u \preceq_P v \text{ iff } h(l_u) \preceq_Q h(l_v) \text{ and } l_P(u) = h^{-1}(l_Q(u)).$$

A pomset *over* Σ *is an isomorphic class of* Σ*-labelled posets.*

Remark 8 (Isomorphism)**.** A bijective morphism in totally ordered set defines an isomorphism, however, this is not true in poset. For example, consider two posets P and Q with following order structures

$$u \preceq_P v, u \preceq_Q u \text{ and } v \preceq_Q v$$

then mapping $h : E_Q \to E_P$ defines bijective morphism but not an isomorphism. Here, one should make a clear distinction between the concept of bijective morphism and isomorphism in posets. The bijective morphism $h : E_Q \to E_P$ says that Q subsumes P, denoted by $P \leqslant Q$, whereas isomorphism $Q = P$ says that the Q and P are equivalent posets *i.e.,* $P \equiv Q$. Therefore, a bijective morphism in posets is an isomorphism if and only if its inverse is also a morphism.

3.2 Category theory

A category X represents a class of objects [26]. Each disjoint pair of objects (x, y) in X can be expressed as set $X(x, y)$ of morphisms. Morphisms can be defined as a function $f : x \to y$ with an arrow connecting source x and target y. Similarly, composition of morphisms is given by

$$\begin{aligned} X(x,y) \times X(y,z) &= X(x,z) \\ \langle g, f \rangle &= g.f \end{aligned}$$

The composition of morphisms is defined to be associative. The unique morphism $1_x : x \to x$ known as identity morphisms are units with respect to the composition

$$X[x,x] \times X[x,y] = X[x,y].$$

The types of $f : x \to y$ morphisms are defined as follows

Definition 20 (Isomorphism)**.** *f is an* isomorphism *if there is a morphism $g : y \to x$ such that $g.f = 1_x$ and $f.g = 1_y$.*

An isomorphism f is also called an invertible morphism and morphism g of the definition is called the inverse, denoted as f^{-1}. The set of invertible elements $x \in X$ of morphism $X(x, x)$ forms a group under composition, and the group is called the automorphism group of $x \in X$. Therefore, a groupoid is a category where every morphism is an isomorphism.

Definition 21 (Monomorphism)**.** *f is a* monomorphism *if there are morphisms $g : z \to x$ and $h : z \to x$ such that $\forall g, h : f.h = f.g \implies g = h$.*

$$\begin{tikzcd} z \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & x \arrow[r, "f"] & y \end{tikzcd}$$

Definition 22 (Epimorphism)**.** *f is a* epimorphism *if there are morphisms $g : y \to z$ and $h : y \to z$ such that $\forall g, h : h.f = g.f \implies g = h$.*

$$x \xrightarrow{f} y \underset{h}{\overset{g}{\rightrightarrows}} z$$

An object s of category X is called an initial iff for each $x \in X$ there exist a unique morphism $m : s \to x$. Dually, an object t of category X is called a terminal if for each $x \in X$ there exist a unique morphism $m : x \to t$. Similarly, if s and s' (respectively t and t') are both initial (resp. terminal) objects, then there is exist a unique isomorphsim $i : s \to s'$ (resp. $i : t \to t'$). In category of groups, a singleton group is both initial and terminal. All objects in singleton groups are groups and morphism $f : x \to y$ is group homomorphism. If a category has two initial (resp. terminal) objects then these are necessarily isomorphic.

Definition 23 (Opposite category)**.** *Given a category X, the* opposite category *X^{op} is the category with same objects and identities as satisfying morphisms $X^{op}(y, x) = X(x, y)$ and composition $X^{op}\langle g.f\rangle = X\langle f.g\rangle$ being reversed to X.*

Definition 24 (Subcategory)**.** *A* subcategory *of a category X is a category X' whose objects and morphisms form subsets of the objects and morphisms of X such that source and target of composition in X' agree with those in X.*

X' is defined to be full subcategory of X if $X(x, y) = X'(x, y)$ for all $x, y \in X$, and strictly full subcategory of X if it is a full subcategory such that given $x \in X$ there exist isomorphic $x \in X'$ (i.e. $X' \subset X$ where inclusion functor is fully faithful $X' \subseteq X$).

Definition 25 (Functor)**.** *A functor $F : X \to Y$ between two category X and Y, for all $x, y \in X$, is given by*

$$F : X(x, y) \to Y(F(x), F(y))$$

Functore should be compatible with composition $F(g.f) = F(g).F(f)$ for a composable morphism pairs $\langle g, f\rangle$ of X and $F(1_x) = 1_{F(x)}$. F is faithful *if for any $x, y \in X$ the map $F : X(x, y) \to Y(F(x), F(y))$ is injective,* fully faithful *if these map are all bijective and surjecitve, i.e., if for any $y \in Y$ there exist an $x \in X$ such that $F(x)$ is isomorphic to $y \in Y$.*

Definition 26 (2-category)**.** *A* 2-category *[4] X consist of $X(x, y)$ such that $f : x \to y$ and $g : x \to y$ are called horizontal morphisms, and morphisms between horizontal morphisms $u : f \to g$ are called vertical-morphims in $X(x, y)$. $I(f)$ denotes identity morphisms $I : f \to f$ in $X(x, y)$.*

Here, $v \circ u : f \to g$ defines vertical composition on vertical morphisms

$$u : f \to h \text{ and } v : h \to g \text{ in } X(x, y).$$

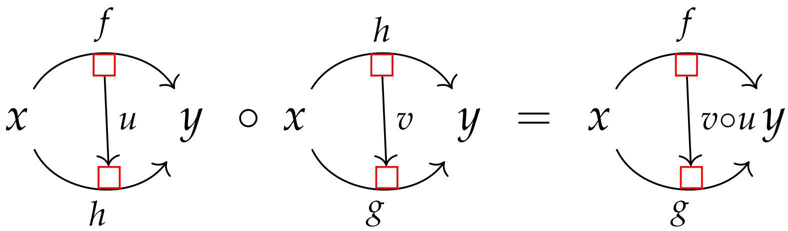


FIGURE 3.5: Vertical composition

Similarly, $v.u : f'.f \to g'.g$ in $X(x,z)$ defines horizontal composition on vertical morphisms $u : f \to g$ in $X(x,y)$ and $v : f' \to g'$ in $X(y,z)$.

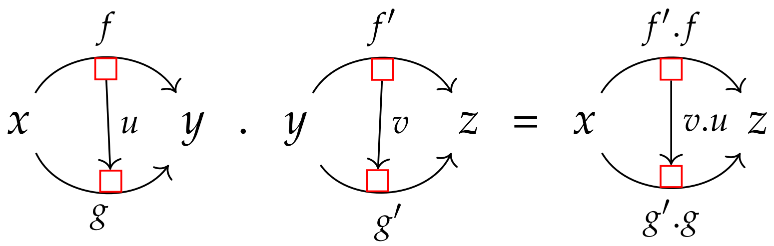


FIGURE 3.6: Horizontal composition

A 2-category can be expressed by triple such as

$$c = (u, f, g)$$

Here f and g are horizontal morphisms and u represents veritcal morphism $f \to g$ in $X(x,y)$. Then,

$$(u, f, g) \,.\, (v, f', g') = (v.u, f'.f, g'.g)$$

construct the horizontal composition shown in Figure3.6 where $(u, f, g) : x \to y$ and $(v, f', g') : y \to z$. Similarly,

$$(u, f, h) \circ (v, h, g) = (u \circ v, f, g)$$

construct the composition shown in the Figure 3.5 on the vertical morphisms where (u, f, h) and (v, h, g) are vertical morphisms in $X(x,y)$. Identity horizontal morphisms $f : x \to x$ correspond to identity vertical morphism $I_f : f \to f$. This means when morphisms $f : n \to X$ and vertical morphism $(u, h, g) : X \to Y$, then $f.u$ is vertical mapping such that

$$I_f.u : h.f \to g.f$$

Definition 27 (Interchange law). *If $u : f \to h$ and $v : h \to g$ are vertical morphisms in $X(x,y)$, and simirlarly, $r : f' \to h'$ and $s : h' \to g'$ in $X(y,z)$, then*

$$(s \circ r) \,.\, (v \circ u) = (s.v) \circ (r.u) \tag{3.2}$$

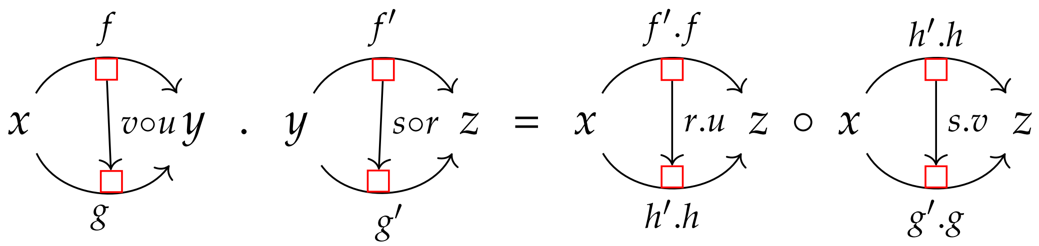


FIGURE 3.7: Interchange law

defines interchange law. Moreover, for $f' : y \to z$ in $Y(y,z)$, and $u : f \to g$ in $X(x,y)$ such that $f : x \to y$ and $g : x \to y$

$$I_f.I'_f = I_{f'.f} \tag{3.3}$$

$$I(1_x).u = u \tag{3.4}$$

$$u.I(1_y) = u \tag{3.5}$$

Remark 9. *The interchange Equation (3.2) becomes inequality over weak class of languages, known as weak exchange law.*

3.3 Posets with interfaces

Definition 28 (Span). *A* span *between objects x and y in any category X is defined by a diagram of type*

$$x \xleftarrow{g} z \xrightarrow{f} y$$

for some $z \in X$. A span is a generalization of the notion of relation between two objects of a category. The defined span is just a morphism $f : x \to y$ if $g = 1$ or $g : y \to x$ if $f = 1$. A span in the opposite category X^{op} is called a cospan in X.

Definition 29 (Cospan). *A* cospan *between objects x and y in any category X is defined by a diagram of type*

$$x \xrightarrow{g} z \xleftarrow{f} y$$

for some $z \in X$.

Definition 30 (iposet). *A poset with interfaces (iposet) is a cospan*

$$s : [n] \to (P, \preceq) \leftarrow [m] : t$$

of monomorphisms s, t on poset P such that s[n] is the set of minimal and t[m] is the set of maximal events in P. Monomorphims s, t on poset P are categorical generalization of injective mappings with $n, m \in \mathbb{N}$ such that $s : [n] \to (P, \preceq)$ defines the source and $t : [m] \to (P, \preceq)$ defines the target of morphisms (s, P, t) .

The Definition 30 defines iposet as a set of morphisms (s, P, t) on poset P. We use notation

$$(s, P, t) : n \to m$$

to denote an iposet on poset P. The isomorphisms $f : P \to P'$ between iposets such that

$$(s, P, t) : n \to m \text{ and } (s', P', t') : n \to m$$

is defined iff there exists

$$f \circ P = P', f \circ s = s' \text{ and } f \circ t = t'$$

as illustrated by Figure 3.8. Isomorphic iposets define the equivalence class of iposets. The isomorphic iposets respect the order isomorphism over equivalence class. This definition of isomorphisms can be lifted to the level of sets of iposets. We allow more varieties in our iposets theory by clearly detaching isomorphic iposets raised from symmetry over an iposet.

$$\begin{tikzcd}
 & P \arrow[dd, "f"] & \\
{[n]} \arrow[ur, hook, "s"] \arrow[dr, hook, "s'"'] & & {[m]} \arrow[ul, hook', "t"'] \arrow[dl, hook', "t'"] \\
 & P' &
\end{tikzcd}$$

FIGURE 3.8: Isomorphic iposets

Definition 31. *An event $x \in P$ in an iposet such that $(s, P, t) : m \to n$ is called*

external *iff* $x \in s([m]) \cup t([n])$ *, otherwise* internal .

Definition 32 (Identity iposet)**.** *The $(s, [n], t) : n \to n$ morphisms on discrete poset $[n]$ defines an* identity iposet. *Similarly, $(s, [n], t) : 0 \to n$ defines left sided identity iposet and $(s, [n], t) : n \to 0$ defines right sided identity iposet depending on the interfaces of a discrete iposet.*

Example 9. Let $P = (E_P, \preceq_P, l_P)$ be a labelled poset given by following set of actions

$$E_P = \{a, b, c, d, e, f, g, h, i\}$$

with temporal precedence $\preceq_P$

$$a < c < e, b < c, b < d, g < h.$$

Consider an iposet $(s, P, t) : 3 \to 3$ on poset P such that

$$s_1 = a, s_2 = f, s_3 = g \text{ and } t_1 = e, t_2 = d, t_3 = i$$

denote the source and target interfaces illustrated by Figure 3.9.

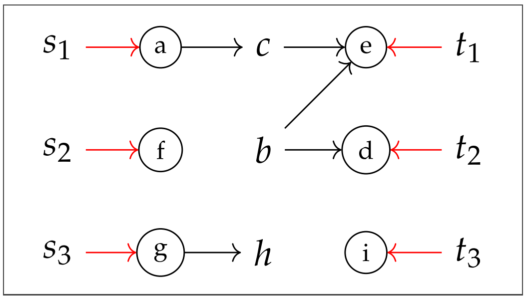


FIGURE 3.9: An iposet

The nodes with a directed arrow represent events and their temporal precedence from left to right in the iposet. The node inside the circle denotes the external events and without circle denotes the internal events in the iposet. The image of the source (resp. target) interfaces of iposet are illustrated by arrows pointing external events in iposet. The s_i with $i \leq 3$ and t_i with $i \leq 3$ are monotonic function with i increasing from top to bottom; denote particular i^{th} monotone mapping on the external events of P. The morphisms $(s, P, t) : 3 \to 3$ on poset P denotes a set of iposets shown in Figure 3.10 and 3.9. We see them as variety of iposets $(s, P, t) : 3 \to 3$ instead equivalent symmetries.

Symmetry shuffles the events of underlying iposets along with their interface injections, which is contrary to the definition of an iposet. It is clear that interfaces of an iposet are an image of injective and monotonic mappings on the external events of poset, in an increasing order from top to bottom, as presented in Figure 3.9. It defines interfaces as tuples denoting a sequence of events.

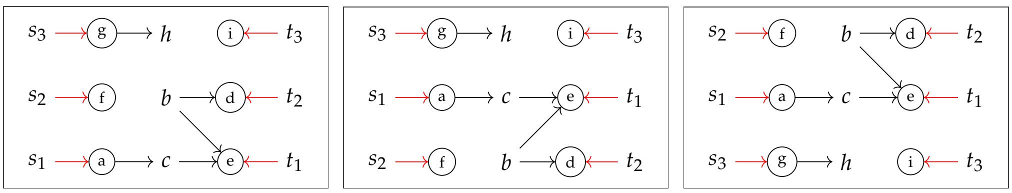


FIGURE 3.10: Symmetries of labelled iposet from Figure 3.9

Therefore, if we reset the interfaces of an iposet in Figure 3.10 according to the definition of iposets, they produce a set of labelled iposets given in Figure 3.11.

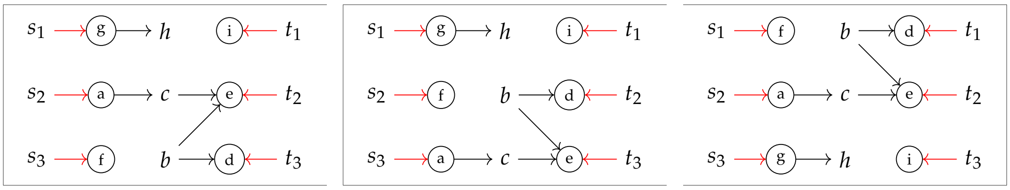


FIGURE 3.11: Iposets

Remark 10 (Variety of iposet)**.** The iposets $(s, P, t) : n \to m$ on poset P of Example 9 generate following class of labelled iposets listed below. Suppose, $P_n = \{a, b, f, g, i\}$ and $P_m = \{e, d, f, h, i\}$ denote set of minimal and maximal events of P. Then,

- $(s, P, t) : 0 \to 0$ denotes an unique class of iposets with zero source and target interface. The $s : 0 \to P_n$ and $t : 0 \to P_m$ denote zero monomorphism on set P_n and P_m of P. The iposets with zero source and target interface denote class of standard posets [13].
- $(s, P, t) : 1 \to 1$ generates 25 variety of an iposet. For instance, $(s, P, t) : 1 \to 1$ with $s_1 = a$ and $t_1 = e$ denotes an iposet which is different from $(s, P, t) : 1 \to 1$ with $s_1 = b$ and $t_1 = f$.
- Similarly, $(s, P, t) : 2 \to 2$, $(s, P, t) : 3 \to 3$, $(s, P, t) : 4 \to 4$ and $(s, P, t) : 5 \to 5$ denote different classes or types of iposets.

Remark 11. The set of iposets $(s, P, t) : n \to m$ over poset P given in Remark 10 can be generalized as a set of subtyped iposets generated from poset P. For example, $(s, P, t) : 1 \to 1$ represents a subtype of typed P, i.e., $(s, P, t) : n \to m$. Typed P means typed with set of source and target interfaces. The $(1, P, 1)$ with $s_1 = a$ and $t_1 = e$ represents a particular subtype of P. In this blend, iposets can be defined as a classification of categorically typed poset.

Definition 33 (Concatenation)**.** *The* concatenation $\rhd$ *of iposets P and Q such that*

$$s_P : [n] \to (E_P, \preceq_P) \leftarrow [m] : t_P \textit{ and } s_Q : [m] \to (E_Q, \preceq_Q) \leftarrow [k] : t_Q$$

is an iposet

$$P \rhd Q := s_P : [n] \to (E_{P \rhd Q}, \preceq_{P \rhd Q}) \leftarrow [k] : t_Q,$$

where

$$\begin{aligned} E_{P \rhd Q} =& (E_P \sqcup E_Q)_{/t_P(i) = s_Q(i)} \\ \preceq_{P \rhd Q} =& \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q). \end{aligned}$$

We defined concatenation of iposets whose interfaces agree. The identities $(\mathrm{id}, [n], \mathrm{id}) : n \to n$ define a (small) category with objects $n \in \mathbb{N}$ and morphisms $[n] : n \to n$. We will denote identity iposet by id_n.

Remark 12. The notation $\sqcup$ denotes disjoint union. We interpret the disjoint union in the definition of $\triangleright$ composition as a quotient union of iposets. We take the quotient union of two iposets by quotienting equivalent events defined by their interface agreement.

Example 10 (Sequential composition of iposets)**.** Figure 3.12 explains concatenation of two iposets. The bullets represent events of iposet. The black bullets denote internal events whereas the empty bullets denote external events. The arrows illustrate temporal precedence of events in the iposet from left to right. The $\triangleright$ composition is illustrated by interfaces agreement between first and second iposet. Square bullets represent gluing points in resulting iposet, and additional arrows denote the temporal precedence induced by $\triangleright$ composition.

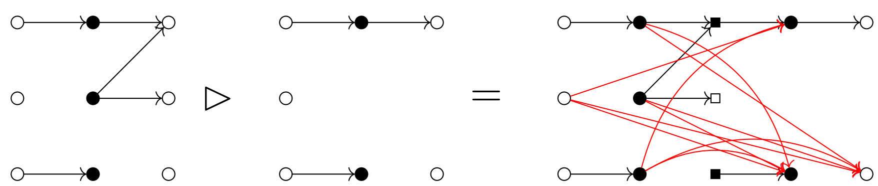


FIGURE 3.12: Concatenation

Definition 34 (Parallel)**.** *The* parallel composition *of iposets P and Q such that*

$$s_P : [n] \to (E_P, \preceq_P) \leftarrow [m] : t_P \text{ and } s_Q : [m] \to (E_Q, \preceq_Q) \leftarrow [k] : t_Q$$

is an iposet

$$P \otimes Q := s : [n+m] \to (E_{P\otimes Q}, \preceq_{P\otimes Q}) \leftarrow [m+k] : t,$$

where

$$\begin{aligned} E_{P\otimes Q} =& E_P \sqcup E_Q \\ \preceq_{P\otimes Q} =& \preceq_P \cup \preceq_Q . \end{aligned}$$

The source $s_{P\otimes Q} = (s_P \otimes s_Q) \circ \phi_{n,m}$ and target $t_{P\otimes Q} = (t_P \otimes t_Q) \circ \phi_{m,k}$ interface injections mapping are given by the Equation (3.1) are

$$s_{P\otimes Q}(i) = \begin{cases} s_P(i) & \text{if } i \le n \\ s_Q(i-n) & \text{if } i > n \end{cases}$$

$$t_{P\otimes Q}(i) = \begin{cases} t_P(i) & \text{if } i \le m \\ s_Q(i-m) & \text{if } i > m. \end{cases}$$

The order relation $\preceq_{P\otimes Q}$ is union of order relation in $\preceq_P$ and $\preceq_Q$. The source $(s_P \otimes s_Q) \circ \phi_{n,m}$ and target $(t_P \otimes t_Q) \circ \phi_{m,k}$ interface are the union of the source and target interfaces of P and Q respectively. By the definition, interfaces are monotone and injective on the external events of individual iposets, they remain monotone and injective over corresponding parallel composition as well.

The concatenation of iposets $(s, P, t) : n \to m$ and $(s, Q, t) : m \to k$ is defined iff their interfaces agree in the order of $\rhd$ composition. This explains that target interface t_P of P must agree to the source interface s_Q of Q. Further, the composition $P \rhd Q$ of iposets

$$(s, P, t) : n \to 0 \text{ and } (s, Q, t) : 0 \to m$$

with zero target $t_P = 0$ and source $t_Q = 0$ interfaces corresponds to the standard sequential composition of posets [13]. The ordering relation on $P \rhd Q$ is obtained from that of $P \otimes Q$ with the additional requirement that every event in E_P must precede every event in $E_Q \setminus s_Q$.

Remark 13. The parallel composition $P \otimes Q$ of two iposets $(s, E_P, t) : n \to m$ and $(s, E_Q, t) : m \to k$ yields an iposet $(s, E_P \sqcup E_Q, t) : (n + m) \to (m + k)$ such that

$$x \preceq_{P \otimes Q} y \text{ iff either } x \preceq_P y \text{ or } x \preceq_Q y$$

Similarly, $P \rhd Q$ yields an iposet $(s, E_{P \rhd Q}, t) : n \to k$ such that $x \preceq_{P \rhd Q} y$ if and only if

$$x \preceq_P y \text{ or } x \preceq_Q y \text{ or there exist}$$

$$x \in E_P \text{ such that } x \notin t_P \text{ and } y \in E_Q \text{ such that } y \notin s_Q.$$

3.3.1 Pomsets with interfaces

Let Σ be a fixed finite set of alphabet. A *pomset* over Σ consists of a poset $(E_P, \preceq_P)$ and a labeling $l : E_P \to \Sigma$. We call a pomset discrete if its underlying poset is discrete. A morphism

$$h : (E_P, \preceq_P, l_P) \to (E_Q, \preceq_Q, l_Q)$$

is a function $h : E_P \to E_Q$ which preserves the ordering and the labeling such that $x \preceq_P y$ implies $h(x) \preceq_Q h(y)$ and $l_Q \circ h = l_P$. Everything we have said about iposet carries over to ipomset.

Definition 35 (Ipomset). *A pomset with interfaces* (ipomset) *is an iposet*

$$s : [n] \to (E_P, \preceq_P) \leftarrow [m] : t$$

together with a labeling function

$$l : E_P \to \Sigma$$

where Σ denotes a finite set of alphabets.

We define *concatenation* of ipomsets whose interfaces agree.

Definition 36 (Concatenation). *The* concatenation $\rhd$ *of ipomsets P and Q such that*

$$s_P : [n] \to (E_P, \preceq_P, l_P) \leftarrow [m] : t_P \text{ and } s_Q : [m] \to (E_Q, \preceq_Q, l_Q) \leftarrow [k] : t_Q$$

is an ipomset

$$P \rhd Q := s_P : [n] \to (E_{P\rhd Q}, \preceq_{P\rhd Q}, l_{P\rhd Q}) \leftarrow [k] : t_Q$$

iff

$$l_P(t_P(i)) = l_Q(s_Q(i)) \quad \text{for all} \quad i \in [m],$$

where

$$E_{P\rhd Q} = (E_P \sqcup E_Q)_{/t_P(i)=s_Q(i)}\,, l_{P\rhd Q}(x) = \begin{cases} l_P(x) & \text{if } x \in E_P \\ l_Q(x) & \text{if } x \in E_Q \end{cases}$$
$$\text{and } \preceq_{P\rhd Q} = \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q).$$

The condition $l_P(t_P(i)) = l_Q(s_Q(i))$ above demands commutativity of the square in the following diagram:

$$\begin{tikzcd}
{[n]} \arrow[dr, "s_P"'] & & {[m]} \arrow[dl, "t_P"] \arrow[dr, "s_Q"'] & & {[k]} \arrow[dl, "t_Q"] \\
& P \arrow[dr, "l_P"'] & & Q \arrow[dl, "l_Q"] & \\
& & \Sigma & &
\end{tikzcd}$$

When this condition is satisfied, we say that the ipomsets in question are composable.

Definition 37 (Parallel). *The* parallel *product of ipomsets P and Q such that*

$$s_P : [n] \to (E_P, \preceq_P, l_P) \leftarrow [m] : t_P \text{ and } s_Q : [m] \to (E_Q, \preceq_Q, l_Q) \leftarrow [k] : t_Q$$

is an ipomset

$$P \otimes Q := s : [n+m] \to (E_{P\otimes Q}, \preceq_{P\otimes Q}, l_{P\otimes Q}) \leftarrow [m+k] : t$$

such that

$$\begin{aligned} E_{P\otimes Q} &= E_P \sqcup E_Q \\ \preceq_{P\otimes Q} &= \preceq_P \cup \preceq_Q . \end{aligned}$$

The labelling on the underlying carrier sets $E_{P\otimes Q}$ is given by the function $l_{P\otimes Q}$

$$l_{P\otimes Q}(x) = \begin{cases} l_P(x) & \textit{if } x \in E_P \\ l_Q(x) & \textit{if } x \in E_Q. \end{cases}$$

Similarly, the source $s_{P\otimes Q}$ and target $t_{P\otimes Q}$ interfaces given by the Equation (3.1) are

$$s_{P\otimes Q}(i) = \begin{cases} s_P(i) & \textit{if } i \leq n \\ s_Q(i-n) & \textit{if } i > n \end{cases}$$
$$t_{P\otimes Q}(i) = \begin{cases} t_P(i) & \textit{if } i \leq m \\ s_Q(i-m) & \textit{if } i > m \end{cases}$$

Note 1. The ordering relation in $\preceq_{P\rhd Q}$ can be obtained from that of $\preceq_{P\otimes Q}$ by requiring that every actions in l_P must precede every actions in $l_Q \setminus s_Q$.

3.4 Summary

In this chapter, we have presented a preorder relation on a finite set and showed how it can be made into a partial ordered relation by taking the quotient of equivalent relations. We presented properties of posets including their series and parallel compositions. We introduced a new definition for the parallel composition of isomorphic posets in Equation (3.1). Further, we took brief note of categorical theory of morphisms in Section 3.3. We explained the construction of interchange law based on horizontal and vertical compositions in Definition 27 followed by an illustration in Figure 3.7.

In Section 3.3, we defined iposet in Definition 30 along with their concatenation and parallel compositions in Definition 33 and 34 respectively. We explained construction of iposets in Remarks 10 and 11 followed by Example 9, thereby, we have shown that the definition of iposets generate more varieties in iposets theory. We summarized the section by defining ipomset along with their sequential and parallel compositions in the Definition 35, 36 and 37 respectively.

Chapter 4

Iposets theory

In this chapter, we present the main results of the thesis. We state and prove algebraic property of iposets including the order structure of iposets under subsumption. We investigate the equational theory of iposets close to the algebraic results of Concurrent Kleene Algebra [14].

4.1 The equational theory of iposets algebra

Let $\mathbf{P}$ be the set of iposets.

Proposition 2. $(\mathbf{P}, \rhd, \otimes)$ *forms an ordered bisemigroup that satisfies following axioms of concurrent semigroup [14, Definition 6.6], for* $P, P', Q, Q', R \in \mathbf{P}$

$$P \rhd (Q \rhd R) = (P \rhd Q) \rhd R \tag{4.1}$$
$$P \otimes (Q \otimes R) = (P \otimes Q) \otimes R \tag{4.2}$$
$$(P \otimes P') \rhd (Q \otimes Q') \leq (P \rhd Q) \otimes (P' \rhd Q') \tag{4.3}$$

Proof. Let P, Q and R be iposets

$$P = (E_P, \preceq_P, s_P, t_P), Q = (E_Q, \preceq_Q, s_Q, t_Q) \text{ and } R = (E_R, \preceq_R, s_R, t_R)$$

To prove the equalities and inequalities of the proposition, we look into three aspects of iposets: sets of interfaces, sets of underlying events and the order relation on those sets of events.

► Proof for equation (4.1):

$$P \rhd (Q \rhd R) = (P \rhd Q) \rhd R.$$

We assume that the composition on both sides of the equation are defined, otherwise the equation is trivially undefined. Therefore, we assume $t_P = s_Q$ and $t_Q = s_R$. Equality (4.1) holds iff there exists equality in all three aspects of iposets on both sides of equation, i.e., the sets of events, sets of interfaces and the order relations.

First, we proceeds with sets of events present on the left-hand side $E_{P \rhd (Q \rhd R)}$

$$\begin{aligned}
E_{P \rhd (Q \rhd R)} &= (E_P \sqcup (E_Q \rhd E_R))_{/t_P(i)=s_{Q \rhd R}(i)} \\
&= (E_P \sqcup (E_Q \sqcup E_R)_{/t_Q(i)=s_R(i)})_{/t_P(i)=s_{Q \rhd R}(i)} \\
&\blacktriangleright \text{ since } s_{Q \rhd R} = s_Q \\
&= (E_P \sqcup (E_Q \sqcup E_R)_{/t_Q(i)=s_R(i)})_{/t_P(i)=s_Q(i)} \\
&\blacktriangleright \text{ combining quotients parts over union} \\
&= (E_P \sqcup E_Q \sqcup E_R)_{/t_Q(i)=s_R(i) \wedge t_P(i)=s_Q(i)}
\end{aligned}$$

which are equal to the events present on the right-hand side $E_{(P \rhd Q) \rhd R}$ given by

$$\begin{aligned}
E_{(P \rhd Q) \rhd R} &= ((E_P \rhd E_Q) \sqcup E_R)_{/t_{P \rhd Q}(i)=s_R(i)} \\
&= ((E_P \sqcup E_Q)_{/t_P(i)=s_Q(i)} \sqcup E_R)_{/t_{P \rhd Q}(i)=s_R(i)} \\
&\blacktriangleright \text{ since } t_{P \rhd Q} = t_Q \\
&= ((E_P \sqcup E_Q)_{/t_P(i)=s_Q(i)} \sqcup E_R)_{/t_Q(i)=s_R(i)} \\
&\blacktriangleright \text{ we can move the quotients parts to obtain} \\
&= (E_P \sqcup E_Q \sqcup E_R)_{/t_P(i)=s_Q(i) \wedge t_Q(i)=s_R(i)} \\
&= E_{P \rhd (Q \rhd R)}
\end{aligned}$$

We fixed notations $s_{P \rhd (Q \rhd R)}$ and $t_{P \rhd (Q \rhd R)}$ to denote the source and target interfaces on left-hand side of the equality

$$s_{P \rhd (Q \rhd R)} = s_P \text{ and } t_{P \rhd (Q \rhd R)} = t_{Q \rhd R} = t_R$$

which are equal to the interfaces $s_{(P \rhd Q) \rhd R}$ and $t_{(P \rhd Q) \rhd R}$ on the right-hand side given by

$$s_{(P \rhd Q) \rhd R} = s_{P \rhd Q} = s_P \text{ and } t_{(P \rhd Q) \rhd R} = t_R$$

Now, we are left with the order relation $\preceq_{P \rhd (Q \rhd R)} = \preceq_{(P \rhd Q) \rhd R}$ part of the equality. By the Definition 33 of $\rhd$, we get following order relation on the left-hand side

$$\begin{aligned}
\preceq_{P \rhd (Q \rhd R)} &= \preceq_P \cup \preceq_{Q \rhd R} \cup (E_P \setminus t_P \times E_{Q \rhd R} \setminus s_{Q \rhd R}) \\
&\blacktriangleright \text{ since } s_{Q \rhd R} = s_Q \\
&= \preceq_P \cup \preceq_{Q \rhd R} \cup (E_P \setminus t_P \times E_{Q \rhd R} \setminus s_Q) \\
&\blacktriangleright \text{ We expand } \preceq_{Q \rhd R} \text{ to obtain} \\
&= \preceq_P \cup (\preceq_Q \cup \preceq_R \cup (E_Q \setminus t_Q \times E_R \setminus s_R)) \cup (E_P \setminus t_P \times E_{Q \rhd R} \setminus s_Q) \\
&= \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_Q \setminus t_Q \times E_R \setminus s_R) \cup (E_P \setminus t_P \times E_{Q \rhd R} \setminus s_Q) \\
&\blacktriangleright \text{ We expand } E_{Q \rhd R} \text{ to obtain}
\end{aligned}$$

$$
\begin{aligned}
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_Q \setminus t_Q \times E_R \setminus s_R) \\
&\qquad\qquad\qquad \cup (E_P \setminus t_P \times (E_Q \cup (E_R \setminus s_R)) \setminus s_Q) \\
&\blacktriangleright \text{ since } s_Q \notin E_R \\
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_Q \setminus t_Q \times E_R \setminus s_R) \\
&\qquad\qquad\qquad \cup (E_P \setminus t_P \times ((E_Q \setminus s_Q) \cup (E_R \setminus s_R))) \\
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_Q \setminus t_Q \times E_R \setminus s_R) \\
&\qquad\qquad\qquad \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup (E_P \setminus t_P \times E_R \setminus s_R) \\
&\blacktriangleright \text{ Rewritting, we obtain} \\
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \\
&\qquad\qquad\qquad \cup (E_Q \setminus t_Q \times E_R \setminus s_R) \cup (E_P \setminus t_P \times E_R \setminus s_R)
\end{aligned}
$$

which is equivalent to the order relation present on the right-hand side

$$
\begin{aligned}
\preceq_{(P \rhd Q) \rhd R} &= \ \preceq_{P \rhd Q} \cup \preceq_R \cup (E_{P \rhd Q} \setminus t_{P \rhd Q} \times E_R \setminus s_R) \\
&\blacktriangleright \text{ since } t_{P \rhd Q} = t_Q \\
&= (\preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q)) \cup \preceq_R \cup (E_{P \rhd Q} \setminus t_Q \times E_R \setminus s_R) \\
&\blacktriangleright \text{ We expand } E_{P \rhd Q} \text{ to get} \\
&= \ \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup \preceq_R \\
&\qquad\qquad \cup (((E_P \setminus t_P) \cup E_Q) \setminus t_Q) \times E_R \setminus s_R) \\
&\blacktriangleright \text{ Since } t_Q \notin E_P \text{, we obtain} \\
&= \ \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup \preceq_R \\
&\qquad\qquad \cup (((E_P \setminus t_P) \cup (E_Q \setminus t_Q)) \times E_R \setminus s_R) \\
&= \ \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup \preceq_R \\
&\qquad\qquad \cup (E_P \setminus t_P \times E_R \setminus s_R) \cup (E_Q \setminus t_Q \times E_R \setminus s_R) \\
&\blacktriangleright \text{ We obtain} \\
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \\
&\qquad\qquad \cup (E_Q \setminus t_Q \times E_R \setminus s_R) \cup (E_P \setminus t_P \times E_R \setminus s_R) \\
&= \ \preceq_{P \rhd (Q \rhd R)}
\end{aligned}
$$

The equality in all three aspects of iposets, above, proves $P \rhd (Q \rhd R) = (P \rhd Q) \rhd R$.

▶ Proof of Equation (4.2):

$$P \otimes (Q \otimes R) = (P \otimes Q) \otimes R.$$

First, we proceed with the sets of events present on both sides of the Equation (4.2). By Definition 34 of $\otimes$, we get following set of events present on the left-side $E_{P \otimes (Q \otimes R)}$

$$E_{P \otimes (Q \otimes R)} = E_P \otimes (E_Q \otimes E_R) = E_P \sqcup (E_Q \otimes E_R) = E_P \sqcup (E_Q \sqcup E_R) = E_P \sqcup E_Q \sqcup E_R$$

which is equal to the set of events present at the right-side $E_{(P\otimes Q)\otimes R}$

$$E_{(P\otimes Q)\otimes R} = (E_P \otimes E_Q) \otimes E_R = (E_P \otimes E_Q) \sqcup E_R = (E_P \sqcup E_Q) \sqcup E_R = E_P \sqcup E_Q \sqcup E_R.$$

We proceed with interfaces part of the equality on both side of Equation (4.2). By definition 34$\otimes$, the source $s_{P\otimes(Q\otimes R)}$ and target $t_{P\otimes(Q\otimes R)}$ interface present on the left-hand side of equation are

$$s_{P\otimes(Q\otimes R)} = s_P \sqcup s_{Q\otimes R} = s_P \sqcup (s_Q \sqcup s_R) = s_P \sqcup s_Q \sqcup s_R$$

and

$$t_{P\otimes(Q\otimes R)} = t_P \sqcup t_{Q\otimes R} = t_P \sqcup (t_Q \sqcup t_R) = t_P \sqcup t_Q \sqcup t_R.$$

Notation $\sqcup$ over interfaces denotes non-discriminated union. Similarly, the source $s_{(P\otimes Q)\otimes R}$ and target $t_{(P\otimes Q)\otimes R}$ interfaces present on the right-hand side of equation are

$$s_{(P\otimes Q)\otimes R} = s_{P\otimes Q} \sqcup s_R = (s_P \sqcup s_Q) \sqcup s_R = s_P \sqcup s_Q \sqcup s_R$$

and

$$t_{(P\otimes Q)\otimes R} = t_{P\otimes Q} \sqcup t_R = (t_P \sqcup t_Q) \sqcup t_R = t_P \sqcup t_Q \sqcup t_R,$$

which are equal to the source and target interfaces present on the left-hand side of equation derived above. We now left with the order relations part of the equality. The order relations present at the left hand side $\preceq_{P\otimes(Q\otimes R)}$ is given by

$$\preceq_{P\otimes(Q\otimes R)} = \preceq_P \cup (\preceq_{Q\otimes R}) = \preceq_P \cup (\preceq_Q \cup \preceq_R) = \preceq_P \cup \preceq_Q \cup \preceq_R$$

which is equal to the order relations present at the right-hand side $\preceq_{(P\otimes Q)\otimes R}$

$$\preceq_{(P\otimes Q)\otimes R} = \preceq_{(P\otimes Q)} \cup \preceq_R = (\preceq_P \cup \preceq_Q) \cup \preceq_R = \preceq_P \cup \preceq_Q \cup \preceq_R$$

The equality in all three aspects of iposets, above , proves $P \otimes (Q \otimes R) = (P \otimes Q) \otimes R$.

► Proof of Equation (4.3):

$$(P \otimes P') \rhd (Q \otimes Q') \leq (P \rhd Q) \otimes (P' \rhd Q').$$

Let P, P', Q and Q' be iposets

$$\begin{aligned} P &= (E_P, \preceq_P, s_P, t_P), \\ P' &= (E_{P'}, \preceq_{P'}, s_{P'}, t_{P'}), \\ Q &= (E_Q, \preceq_Q, s_Q, t_Q) \quad \text{and} \\ Q' &= (E_{Q'}, \preceq_{Q'}, s_{Q'}, t_{Q'}). \end{aligned}$$

We consider the compositions on both side of the inequality, given by Equation (4.3), are defined by iposets $\rhd$ and $\otimes$ definitions. We first look at the set of interfaces present on the both side of the inequality. By the Definition 33 and 34 of $\rhd$ and $\otimes$, we get following set of source and target interfaces on the left-hand side of inequality

$$s_{(P\otimes P')\rhd(Q\otimes Q')} = s_{(P\otimes P')} = s_P \sqcup s'_P$$
$$t_{(P\otimes P')\rhd(Q\otimes Q')} = t_{(Q\otimes Q')} = t_Q \sqcup t_{Q'}$$

which are equal to the

$$s_{(P\rhd Q)\otimes(P'\rhd Q')} = s_{P\rhd Q} \sqcup s_{P'\rhd Q'} = s_P \sqcup s_{P'}$$
$$t_{(P\rhd Q)\otimes(P'\rhd Q')} = t_{P\rhd Q} \sqcup t_{P'\rhd Q'} = t_Q \sqcup t_{Q'}$$

source and target interfaces at the right-hand side of inequality. We now proceed for the sets of events present on both side of the inequality. The events on left hand side $E_{(P\otimes P')\rhd(Q\otimes Q')}$ is given by

$$\begin{aligned} E_{(P\otimes P')\rhd(Q\otimes Q')} &= (E_{P\otimes P'} \sqcup E_{Q\otimes Q'})_{/t_{P\otimes P'}(i)=s_{Q\otimes Q'}(i)} \\ &= ((E_P \sqcup E_{P'}) \sqcup (E_Q \sqcup E_{Q'}))_{/t_{P\otimes P'}(i)=s_{Q\otimes Q'}(i)}. \end{aligned}$$

Observing the quotient part of events $t_{P\otimes P'}(i) = s_{Q\otimes Q'}(i)$, $t_{P\otimes P'}(i)$ quotient part denotes the parallel product of the quotient events of the set E_P and E_Q, *i.e.*, $t_P \sqcup t_{P'}(i)$. Similarly, $s_{Q\otimes Q'}(i)$ quotient part denotes the parallel product of the quotient events of the set E_Q and E'_Q, *i.e.*, $s_Q \sqcup s_{Q'}(i)$.

$$E_{(P\otimes P')\rhd(Q\otimes Q')} = (E_P \sqcup E_{P'} \sqcup E_Q \sqcup E_{Q'})_{/t_P\sqcup t_{P'}(i)=s_Q\sqcup s_{Q'}(i)}.$$

If we look back, the composition on the both side of Inequality 4.3 is defined if $P \rhd Q$ and $P' \rhd Q'$ are defined, i.e., if their interfaces agree. This leads composition $(P \otimes P') \rhd (Q \otimes Q')$ on the left side is defined by $t_P = s_Q$ in parallel with $t_{P'} = s_{Q'}$. Then, rewriting quotient part of $E_{(P\otimes P')\rhd(Q\otimes Q')}$

$$E_{(P\otimes P')\rhd(Q\otimes Q')} = (P \sqcup P' \sqcup Q \sqcup Q')_{/t_P(j)=s_Q(j) \text{ and } t_{P'}(k)=s_{Q'}(k)}, \text{s.t. } i = j + k,$$

which are equal to the set of events present on the right-hand side $E_{(P\rhd Q)\otimes(P'\rhd Q')}$ of the inequality below.

$$\begin{aligned} E_{(P\rhd Q)\otimes(P'\rhd Q')} &= E_{P\rhd Q} \sqcup E_{P'\rhd Q'} \\ &= (E_P \sqcup E_Q)_{/t_P(j)=s_Q j)} \sqcup (E_{P'} \sqcup E_{Q'})_{/t_{P'}(k)=s_{Q'}(k)} \\ &\blacktriangleright \text{ summing quotient part over union} \\ &= ((E_P \sqcup E_Q) \sqcup (E_{P'} \sqcup E_{Q'}))_{/t_P(j)=s_Q(j) \text{ and } t_{P'}(k)=s_{Q'}(k)} \\ &\blacktriangleright \text{ using commutativity of set union} \\ &= (E_P \sqcup E_{P'} \sqcup E_Q \sqcup E_{Q'})_{/t_P(i)=s_Q(i) \text{ and } t_{P'}(i)=s_{Q'}(i)}. \end{aligned}$$

Now, we are left with the last part of the proof: the order relation present on the both side of the inequality. The order relation on the left hands side

$\preceq_{(P\otimes P')\rhd(Q\otimes Q')}$ of the inequality is given by

$$\preceq_{(P\otimes P')\rhd(Q\otimes Q')} = \preceq_{(P\otimes P')} \cup \preceq_{(Q\otimes Q')} \cup (E_{P\otimes P'} \setminus t_{(P\otimes P')} \times E_{Q\otimes Q'} \setminus s_{(Q\otimes Q')}).$$

By using definition 4.12 of $\otimes$, the set-minus of the interfaces distributes over the disjoint union of the parallel product

$$\begin{aligned} &= \preceq_{(P\otimes P')} \cup \preceq_{(Q\otimes Q')} \cup ((E_P \sqcup E_{P'}) \setminus t_P \sqcup t_{P'} \times (E_Q \sqcup E_{Q'}) \setminus s_Q \sqcup s_{Q'}) \\ &= \preceq_P \cup \preceq_{P'} \cup \preceq_Q \cup \preceq_{Q'} \cup ((E_P \setminus t_P \cup E_{P'} \setminus t_{P'}) \times (E_Q \setminus s_Q \cup E_{Q'} \setminus s_{Q'})). \end{aligned}$$

Simplifying further,

$$\begin{aligned} = \preceq_P \cup \preceq_{P'} \cup \preceq_Q \cup \preceq_{Q'} &\cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup (E_P \setminus t_P \times E_{Q'} \setminus s_{Q'}) \\ &\cup (E_{P'} \setminus t_{P'} \times E_Q \setminus s_Q) \cup (E_{P'} \setminus t_{P'} \times E_{Q'} \setminus s_{Q'}). \end{aligned}$$

Similarly, the order relations present on the right-hand side $\preceq_{(P\rhd Q)\otimes(P'\rhd Q')}$ of the inequality is given by

$$\begin{aligned} \preceq_{(P\rhd Q)\otimes(P'\rhd Q')} &= \preceq_{(P\rhd Q)} \cup \preceq_{(P'\rhd Q')} \\ &= \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup \preceq_{P'} \cup \preceq_{Q'} \cup (E_{P'} \setminus t_{P'} \times E_{Q'} \setminus s_{Q'}). \end{aligned}$$

Using commutative property of union,

$$= \preceq_P \cup \preceq_{P'} \cup \preceq_Q \cup \preceq_{Q'} \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup (E_{P'} \setminus t_{P'} \times E_{Q'} \setminus s_{Q'})$$

Now comparing the order relations present on the left $\preceq_L = \preceq_{(P\otimes P')\rhd(Q\otimes Q')}$ on the right-hand side $\preceq_R = \preceq_{(P\rhd Q)\otimes(P'\rhd Q')}$ of the inequality.

$$\begin{aligned} \preceq_L = \; &\preceq_A \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup (E_P \setminus t_P \times E_{Q'} \setminus s_{Q'}) \qquad (4.4) \\ &\cup (E_{P'} \setminus t_{P'} \times E_Q \setminus s_Q) \cup (E_{P'} \setminus t_{P'} \times E_{Q'} \setminus s_{Q'}) \\ \preceq_R = \; &\preceq_A \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup (E_{P'} \setminus t_{P'} \times E_{Q'} \setminus s_{Q'}) \qquad (4.5) \end{aligned}$$

Here, we used the notation $\preceq_A$ to denote $\preceq_P \cup \preceq_{P'} \cup \preceq_Q \cup \preceq_{Q'}$ part of order relations. Observing the order relations given by the Equation (4.5), it contains all the ordered pairs (e, f) such that

$$e \in P \setminus t_p \wedge f \in Q \setminus s_Q \quad \text{and} \quad e \in P' \setminus t_{p'} \wedge f \in Q' \setminus s_{Q'}.$$

Similarly, observing the order relations given by the Equation (4.4), it contains all the order pairs (e, f) such that

$$e \in P \setminus t_p \wedge f \in Q \setminus s_Q \quad \text{and} \quad e \in P' \setminus t_{p'} \wedge f \in Q' \setminus s_{Q'},$$

including the additional pairs of orders such that

$$e \in P \setminus t_p \wedge f \in Q' \setminus s_{Q'} \quad \text{and} \quad e \in P' \setminus t_{p'} \wedge f \in Q \setminus s_Q$$

which are absent in Equation (4.5). This clearly shows that relations given by Equation (4.5) on the right-hand side is contained in the relation given by the

Equation (4.4) on the left hand side. This implies the order relation on the right-hands side of the inequality is less restrictive than the left hand side, thus preserves more parallel behaviour. The ordering relation supports the inequality given by Equation (4.3) above. $\square$

Lemma 1. *The ordered bisemigroup* $(\mathbf{P}, \rhd, \otimes)$ *entail the following identities*

$$\begin{aligned} P \rhd Q &\leq P \otimes Q && \textit{if } t_P = s_Q = 0 && (4.6)\\ (P \otimes Q) \rhd R &\leq P \otimes (Q \rhd R) && \textit{if } t_P = 0 && (4.7)\\ P \rhd (Q \otimes R) &\leq (P \rhd Q) \otimes R && \textit{if } s_R = 0 && (4.8)\end{aligned}$$

under the condition on interfaces showing that these laws do not imply the exchange law stated in Equation (4.3).

Proof. Let P, Q and R be iposets

$$P = (E_P, \preceq_P, s_P, t_P), Q = (E_Q, \preceq_Q, s_Q, t_Q) \text{ and } R = (E_R, \preceq_R, s_R, t_R)$$

To prove the inequalities of proposition, we check the three aspects of iposets: sets of interfaces, sets of underlying events and order relation on those sets of events.

► Proof of Equation (4.6):

$$P \rhd Q \ \leq \ P \otimes Q.$$

We assume the composition on the both side of inequality are defined, otherwise the inequality is trivial undefined. Therefore, we assume $t_P = s_Q$. To claim the inequality, we further assume $t_P = s_Q = 0$ to satisfy constrain imposed over interfaces by the definition 43. The Equation (4.6) holds iff there exists equality in terms of sets of events, sets of interfaces and inequality in terms of order relation present on the both side of the inequality.

First, we proceed with the sets of interfaces present on the both side of the inequality. By the Definition 33 of $\rhd$, we get following set of

$$S_{P \rhd Q} = s_P \quad \text{and} \quad t_{P \rhd Q} = t_Q$$

source and target interfaces on the left-hand side, which are equal to the source and target interface present at the right-hand side given by the Definition 34 of $\otimes$

$$s_{P \otimes Q} = s_P \otimes s_Q = s_P \cup s_Q = s_P \cup 0 = s_P$$

$$t_{P \otimes Q} = t_P \otimes t_Q = t_P \cup t_Q = 0 \cup t_Q = t_Q.$$

Similarly, we proceed for sets of events. The sets of events present on the left-hand side $E_{P \rhd Q}$

$$E_{P \rhd Q} = (E_P \sqcup E_Q)_{/t_P(i) = s_Q(i)}$$

$$\begin{aligned}&\blacktriangleright \text{ Since } t_P = s_Q = 0\text{, we get}\\ &= (E_P \sqcup E_Q)_{/0}\\ &= E_P \sqcup E_Q\end{aligned}$$

are equal to the sets of events present on the right-hand side $E_{P\otimes Q}$

$$E_{P\otimes Q} = E_P \sqcup E_Q.$$

Now, we are left with the last part of the proof: the order relation. The order relation present on the left hand side $\preceq_{P\rhd Q}$ is given by

$$\begin{aligned}\preceq_{P\rhd Q} = {}& \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q)\\ &\blacktriangleright \text{ Since } t_P = s_Q = 0\text{, we get}\\ = {}& \preceq_P \cup \preceq_Q \cup (E_P \setminus 0 \times E_Q \setminus 0)\\ = {}& \preceq_P \cup \preceq_Q \cup (E_P \times E_Q)\end{aligned}$$

Similarly, the order relation present at right-hand side $\preceq_{P\otimes Q}$ of the inequality is given by

$$\preceq_{P\otimes Q} = \preceq_P \cup \preceq_Q$$

Observing the order relation $\preceq_{P\rhd Q}$, it is clear that all the ordered pairs (e, f) such that

$$e \in E_P \text{ and } f \in E_Q$$

present on the left hand side, are absent on the right-hand side $\preceq_{P\otimes Q}$. This proves $\preceq_{P\otimes Q}$ is contained in the $\preceq_{P\rhd Q}$, which implies that the order relation at the right-hands side $\preceq_{P\otimes Q}$ is less restrictive than the order relation at the left hand side $\preceq_{P\rhd Q}$ and thus preserves more parallel behaviour. The ordering relation supports the inquality given by Equation (4.6) above. ▶ Proof of Equation (4.7)

$$(P \otimes Q) \rhd R \leq P \otimes (Q \rhd R).$$

The composition on the both side of inequality are defined iff $t_Q = s_R$ and $t_P = 0$. The $t_Q = s_R$ defines $(Q \rhd R)$ which leads to the definedness of $P \otimes (Q \rhd R)$ composition on the right-hand side. Similarly, $t_P = 0$ together with $t_Q = s_R$ defines the $(P \otimes Q) \rhd R$ composition on left-hand side. First, we proceed with the sets of interfaces present on the both side of the inequality. By definition of $\rhd$ and $\otimes$, we get following

$$S_{(P\otimes Q)\rhd R} = s_{P\otimes Q} = s_P \cup s_Q = 0 \cup s_Q = s_Q \quad \text{and} \quad t_{(P\otimes Q)\rhd R} = t_R$$

source and target interfaces at the left-hand side which are equal to the source and target interface present at the right-hand side

$$s_{P\otimes(Q\rhd R)} = s_P \otimes s_{Q\rhd R} = s_P \cup s_Q = 0 \cup s_Q = s_Q \quad \text{and,}$$

$$t_{P\otimes(Q\rhd R)} = t_P \otimes t_{Q\rhd R} = t_P \cup t_R = 0 \cup t_R = t_R.$$

Similarly, we proceed for the sets of events. The set of events present on the left-hand side $E_{(P\otimes Q)\rhd R}$

$$\begin{aligned}
E_{(P\otimes Q)\rhd R} &= (E_{P\otimes Q} \sqcup E_R)_{/t_{P\otimes Q}(i)=s_R(i)} \\
&= ((E_P \sqcup E_Q) \sqcup E_R)_{/t_{P\otimes Q}(i)=s_R(i)} \\
&\blacktriangleright \text{ By } \otimes \text{ def, } t_{P\otimes Q}(i) \text{ denotes } t_P(j) \sqcup t_Q(k) \text{ sets of events such that } i = j+k \\
&= ((E_P \sqcup E_Q) \sqcup E_R)_{/(t_P(j)\sqcup t_Q(k))=s_R(i)} \\
&\blacktriangleright \text{ Since } t_P = 0,\ t_P(j) \sqcup t_Q(k) = t_Q(k) = t_Q(i) \text{ such that } i = 0+k = k \\
&= ((E_P \sqcup E_Q) \sqcup E_R)_{/t_Q(i)=s_R(i)} \\
&= (E_P \sqcup E_Q \sqcup E_R)_{/t_Q(i)=s_R(i)}
\end{aligned}$$

are equal to the set of events present on the right-hand side $E_{P\otimes(Q\rhd R)}$

$$\begin{aligned}
E_{P\otimes(Q\rhd R)} &= E_P \otimes E_{Q\rhd R} \\
&= E_P \sqcup E_{Q\rhd R} \\
&= E_P \sqcup (E_Q \sqcup E_R)_{/t_Q(i)=s_R(i)} \\
&= (E_P \sqcup E_Q \sqcup E_R)_{/t_Q(i)=s_R(i)}.
\end{aligned}$$

We are now left with the last part of the proof: the order relation. The order relation present on the left hand side $(P \otimes Q) \rhd R$ is given by

$$\begin{aligned}
\preceq_{(P\otimes Q)\rhd R} &= \ \preceq_{P\otimes Q} \cup \preceq_R \cup (E_{P\otimes Q} \setminus t_{P\otimes Q} \times E_R \setminus s_R) \\
&= (\preceq_P \cup \preceq_Q) \cup \preceq_R \cup ((E_P \cup E_Q) \setminus (t_P \sqcup t_Q) \times E_R \setminus s_R) \\
&\blacktriangleright \text{ Since } t_P = 0, \text{ we get} \\
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup ((E_P \cup E_Q) \setminus t_Q \times E_R \setminus s_R) \\
&\blacktriangleright \text{ Since } t_Q \notin E_P, \text{ after proper distribution of } t_Q \text{ setminus over } E_P \cup E_Q \\
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup ((E_P \cup E_Q \setminus t_Q) \times E_R \setminus s_R) \\
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_P \times E_R \setminus s_R) \cup (\ E_Q \setminus t_Q \times E_R \setminus s_R).
\end{aligned}$$

Similarly, the order relation present on the right-hand side of the inequality is given by

$$\begin{aligned}
\preceq_{P\otimes(Q\rhd R)} &= \preceq_P \cup \preceq_{Q\rhd R} \\
&= \ \preceq_P \cup \preceq_Q \cup \preceq_R \cup (\ E_Q \setminus t_Q \times E_R \setminus s_R)
\end{aligned}$$

Observing the order relation $\preceq_{(P\otimes Q)\rhd R}$, it is clear that all the ordered pairs (e, f) such that

$$e \in E_P \text{ and } f \in E_R \setminus s_R$$

present on the left hand side, are absent $\preceq_{P\otimes(Q\rhd R)}$ on the right-hand side. This witness $\preceq_{P\otimes(Q\rhd R)}$ is contained in $\preceq_{(P\otimes Q)\rhd R}$ which implies that the order relation at the right-hands side $\preceq_{P\otimes(Q\rhd R)}$ is less restrictive than the order relation at the left hand side $\preceq_{(P\otimes Q)\rhd R}$. The ordering relation supports the Equation (4.7) above.

▶ Proof of Equation (4.8):

$$P \rhd (Q \otimes R) \leq (P \rhd Q) \otimes R.$$

The composition on the both side of inequality is defined iff $t_P = s_Q$ and $s_R = 0$. Therefore, we assume $t_P = s_Q$ and $s_R = 0$. The $t_P = s_Q$ defines $(P \rhd Q)$ which leads to the definedness of $(P \rhd Q) \otimes R$ composition on the right-hand side. Similarly, $s_R = 0$ together with $t_P = s_Q$ defines the $(P \otimes Q) \rhd R$ composition on left hand side.

First, we proceed with the sets of interfaces present on both side of the inequality. By definition of $\rhd$ and $\otimes$, we get following set of

$$S_{P\rhd(Q\otimes R)} = s_P \quad \text{and} \quad t_{P\rhd(Q\otimes R)} = t_{Q\otimes R} = t_Q \cup t_R$$

source and target interfaces on the left-hand side which are equal to the source and target interface present at the right-hand side

$$\begin{aligned} s_{(P\rhd Q)\otimes R} &= s_P \otimes s_R = s_P \cup s_R = s_P \cup 0 = s_P \\ t_{(P\rhd Q)\otimes R} &= t_Q \otimes t_R = t_Q \cup t_R. \end{aligned}$$

Similarly, we proceed for the sets of events. The set of event present on the left-hand side $E_{P\rhd(Q\otimes R)}$

$$\begin{aligned} E_{P\rhd(Q\otimes R)} &= (E_P \sqcup E_{Q\otimes R})_{/t_P(i)=s_{Q\otimes R}(i)} \\ &= (E_P \sqcup (E_Q \sqcup E_R))_{/t_P(i)=s_{Q\otimes R}(i)} \\ &\blacktriangleright \text{ By } \otimes \text{ def, } s_{Q\otimes R}(i) \text{ denotes } s_Q(j) \sqcup t_R(k) \text{ sets of events such that } i = j+k \\ &= (E_P \sqcup (E_Q \sqcup E_R))_{/t_P(i)=(s_Q(j)\sqcup s_R(k))} \\ &\blacktriangleright \text{ Since } s_R = 0,\ s_Q(j) \sqcup s_R(k) = s_Q(j) = s_Q(i) \text{ such that } i = j+0 = j \\ &= (E_P \sqcup (E_Q \sqcup E_R))_{/t_P(i)=s_Q(i)} \\ &= (E_P \sqcup E_Q \sqcup E_R)_{/t_P(i)=s_Q(i)} \end{aligned}$$

are equal to the sets of events present on the right-hand side $E_{(P\rhd Q)\otimes R}$

$$\begin{aligned} E_{(P\rhd Q)\otimes R} &= E_{P\rhd Q} \otimes E_R \\ &= E_{P\rhd Q} \sqcup E_R \\ &= (E_P \sqcup E_Q)_{/t_P(i)=s_Q(i)} \sqcup E_R \\ &= (E_P \sqcup E_Q \sqcup E_R)_{/t_P(i)=s_Q(i)} \end{aligned}$$

Now, we are left with the last part of the proof: the order relation. The order relation present on the left-hand side $P \rhd (Q \otimes R)$ is given by

$$\preceq_{P\rhd(Q\otimes R)} = \ \preceq_P \cup \preceq_{Q\otimes R} \cup (E_P \setminus t_P \times E_{Q\otimes R} \setminus s_{Q\otimes R})$$

$$= \preceq_P \cup (\preceq_Q \cup \preceq_R) \cup (E_P \setminus t_P \times (E_Q \cup E_R) \setminus (s_Q \cup s_R))$$

▶ Since $s_R = 0$, we get

$$= \preceq_P \cup (\preceq_Q \cup \preceq_R) \cup (E_P \setminus t_P \times (E_Q \cup E_R) \setminus (s_Q \cup 0))$$

▶ Since $s_Q \notin E_R$, after distribution of s_Q setminus over $E_Q \cup E_R$

$$= \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_P \setminus t_P \times (E_Q \setminus s_Q \cup E_R))$$
$$= \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup (E_P \setminus t_P \times E_R).$$

Similarly, the order relation present on the right-hand side of the inequality is given by

$$\begin{aligned} \preceq_{(P \rhd Q) \otimes R} &= \preceq_{P \rhd Q} \cup \preceq_R \\ &= \preceq_P \cup \preceq_Q \cup (E_P \setminus t_P \times E_Q \setminus s_Q) \cup \preceq_R \\ &= \preceq_P \cup \preceq_Q \cup \preceq_R \cup (E_P \setminus t_P \times E_Q \setminus s_Q). \end{aligned}$$

Observing the order relation $\preceq_{P \rhd (Q \otimes R)}$, it is clear that all the ordered pairs (e, f) such that

$$e \in E_P \setminus t_P \text{ and } f \in E_R$$

present on the left hand side, are absent on the right-hand side $\preceq_{(P \rhd Q) \otimes R}$. This witness $\preceq_{(P \rhd Q) \otimes R}$ is contained in $\preceq_{P \rhd (Q \otimes R)}$, which implies that the order relation at the right-hands side $\preceq_{(P \rhd Q) \otimes R}$ is less restrictive than the order relation at the left hand side $\preceq_{P \rhd (Q \otimes R)}$. The ordering relation supports the inequality given by Equation (4.8) above. □

4.2 The equational theory of iposet languages

Definition 38. *Let $\mathcal{P}$ denote the set of all isomorphic class of iposets. We use notations*

$$id_1 = (E_{id_1}, \preceq_{id_1}, s_{id_1}, t_{id_1}) \quad \textit{and} \quad id_0 = (E_{id_0}, \preceq_{id_0}, s_{id_0}, t_{id_0})$$

for an identity and empty iposet. An iposet language over $\mathcal{P}$ denotes a subset of $\mathcal{P}$, i.e., an element of $2^{\mathcal{P}}$.

Remark 14. *An* ordered monoid *over $\mathcal{P}$ is a structure $(\mathcal{P}, o, 1)$ such that*

$$P \; o \; 1 = P \textit{ for } P \in \mathcal{P}$$

where 1 denotes an unit iposet with respect to P and operator o. The operator o is isotone with respect to the order relation on $\mathcal{P}$.

Proposition 3. *$(\mathcal{P}, \rhd, \otimes, id_n, id_0)$ forms a double monoid, for $n > 0$. The* double monoid *is a composite of two ordered monoids*

$$(\mathcal{P}, \rhd, id_n) \text{ and } (\mathcal{P}, \otimes, id_0)$$

such that, for $P \in \mathcal{P}$

$$P \rhd id_n = P \tag{4.9}$$

$$P \otimes id_0 = P \tag{4.10}$$

Proof.

► Proof of Equation (4.9):

$$P \rhd \mathrm{id}_n = P.$$

Suppose P be an iposet

$$P = (E_P, \preceq_P, s_P, t_P)$$

By the definition of id_n,

$$\mathrm{id}_n = (E_{\mathrm{id}_n}, \preceq_{\mathrm{id}_n}, s_{\mathrm{id}_n}, t_{\mathrm{id}_n}).$$

The composition $P \rhd \mathrm{id}_n$ is defined iff $t_P = s_{\mathrm{id}_n}$ target interface of P is equal to the source interface of id_n, and by the definition of id_n we know that

$$s_{\mathrm{id}_n} = t_{\mathrm{id}_n}. \tag{4.11}$$

Then,

$$s_{P \rhd \mathrm{id}_n} = s_P \text{ and } t_{P \rhd \mathrm{id}_n} = t_{\mathrm{id}_n} \overset{4.11}{=} s_{\mathrm{id}_n} = t_P$$

represent the source and target interface of the composition $P \rhd \mathrm{id}_n$. It is clear that the interfaces of P remains unchanged in sequential composition with id_n. Now, we check the set of events produced by $E_{P \rhd \mathrm{id}_n}$

$$E_{P \rhd \mathrm{id}_n} = (E_P \sqcup E_{\mathrm{id}_n})_{/t_P(i) = s_{\mathrm{id}_n}(i)}$$

By the definition of id_n, we know s_{id_n} denotes E_{id_n}. Therefore, we can write $t_P(i) = s_{\mathrm{id}_n}(i)$ equal to E_{id_n}

$$E_{P \rhd \mathrm{id}_n} = (E_P \sqcup E_{\mathrm{id}_n})_{/E_{\mathrm{id}_n}} = E_P$$

Similarly, the order $\preceq_{P \rhd \mathrm{id}_n}$

$$\preceq_{P \rhd \mathrm{id}_n} = \preceq_P \cup \preceq_{\mathrm{id}_n} \cup \{E_P \setminus t_P \times E_{\mathrm{id}_n} \setminus s_{\mathrm{id}_n}\}.$$

Since $s_{\mathrm{id}_n} = E_{\mathrm{id}_n}$, $E_{\mathrm{id}_n} \setminus s_{\mathrm{id}_n}$ equals to 0,

$$\preceq_{P \rhd \mathrm{id}_n} = \preceq_P \cup \preceq_{\mathrm{id}_n} \cup \{E_P \setminus t_P \times 0\}$$

since 0 represents empty set of events, there does not exist any order pairs

(x, y) in $(E_P \setminus t_P \times 0)$ such that $x \in P \setminus t_P$ and $y \in 0$. Therefore $\{P \setminus t_P \times 0\} = 0$. Similarly, since we know $E_{\mathrm{id}_n} \subseteq E_P$, the order relation naturally follows $\preceq_{\mathrm{id}_n} \subseteq \preceq_P$,

$$\preceq_{\mathrm{id}_n} \subseteq \preceq_P \Longrightarrow \preceq_P \cup \preceq_{\mathrm{id}_n} = \preceq_P .$$

using these results, we arrive at

$$\begin{aligned}\preceq_{P \rhd \mathrm{id}_n} &= \preceq_P \cup \preceq_{\mathrm{id}_n} \cup \{E_P \setminus t_P \times 0\} \\ &= \preceq_P \cup \preceq_{\mathrm{id}_n} \cup 0 \\ &= \preceq_P \cup \preceq_{\mathrm{id}_n} \\ &= \preceq_P .\end{aligned}$$

This witness the equality in all three aspect of iposets: interfaces, set of events and order relation, and thus $P \rhd \mathrm{id}_n = P$.

► Proof of Equation (4.10):

$$P \otimes \mathrm{id}_0 = P$$

Suppose P be an iposet

$$P = (E_P, \preceq_P, s_P, t_P).$$

By the definition of id_0,

$$\mathrm{id}_0 = (E_{\mathrm{id}_0}, \preceq_{\mathrm{id}_0}, s_{\mathrm{id}_0}, t_{\mathrm{id}_0}) = (\epsilon, \epsilon, \epsilon, \epsilon).$$

We proceed with

$$s_{P \otimes \mathrm{id}_0} = s_P \otimes s_{\mathrm{id}_0} = s_P \cup \epsilon = s_P \text{ and } t_{P \otimes \mathrm{id}_0} = s_P \otimes s_{\mathrm{id}_0} = t_P \cup \epsilon = t_P$$

the source and target interface of $P \otimes \mathrm{id}_0$ by following the definition of parallel product $\otimes$. It is clear that interfaces of P remains unchanged to parallel product with id_0.

Now, we check the set of events produced by $E_{P \otimes \mathrm{id}_0}$

$$E_{P \otimes \mathrm{id}_0} = (E_P \sqcup E_{\mathrm{id}_0}) = (E_P \sqcup 0) = E_P$$

Similarly the order relation $\preceq_{P \otimes \mathrm{id}_0}$

$$\preceq_{P \otimes \mathrm{id}_0} = \preceq_P \cup \preceq_{\mathrm{id}_0} = \preceq_P \cup \epsilon = \preceq_P .$$

This proves the equality in all three aspects of iposets: the interfaces, set of events and order relations, and thus $P \otimes \mathrm{id}_0 = P$. □

The double monoid structure on individual iposet can be lifted to the powerset level of iposets languages P and Q such that $P, Q \in 2^{\mathcal{P}}$ as usual by defining complex $\rhd$ and $\otimes$ product,

$$P \rhd Q = \{p \rhd q \mid p \in P \wedge q \in Q \text{ and } p \rhd q \text{ is defined}\} \tag{4.12}$$

$$P \otimes Q = \{p \otimes q \mid p \in P \wedge q \in Q\} \tag{4.13}$$

Proposition 4. *$(2^{\mathcal{P}}, \rhd, \otimes, 1_{\rhd}, 1_{\otimes})$ forms a double monoid over $2^{\mathcal{P}}$.*

Here, $1_{\otimes} = \{id_0\}$ denotes set containing only the empty iposet i.e. set of empty ϵ string, and $1_{\rhd} = \{id_n\}$ for $n > 0$ denotes set containing only identity iposets such that $(id, [n], id) : n \to n$. The double monoid of the iposets lanauges is a composite

$$(2^{\mathcal{P}}, \rhd, \otimes, 1_{\rhd}, 1_{\otimes})$$

of two ordered monoids

$$(2^{\mathcal{P}}, \rhd, 1_{\rhd}) \quad \text{and} \quad (2^{\mathcal{P}}, \otimes, 1_{\otimes})$$

structure such that, for $P \in 2^{\mathcal{P}}$

$$P \rhd 1_{\rhd} = P$$
$$P \otimes 1_{\otimes} = P.$$

Proposition 5. *The structure $(2^{\mathcal{P}}, \cup, \rhd, \otimes, 0, 1_{\rhd}, 1_{\otimes})$ forms a bisemiring such that following equations holds, for $P, Q, R \in 2^{\mathcal{P}}$*

$$P \rhd 0 = 0 \tag{4.14}$$
$$P \otimes 0 = 0 \tag{4.15}$$
$$P \rhd 1_{\rhd} = P = 1_{\rhd} \rhd P \tag{4.16}$$
$$P \otimes 1_{\otimes} = P = 1_{\otimes} \otimes P \tag{4.17}$$
$$(P \rhd Q) \rhd R = P \rhd (Q \rhd R) \tag{4.18}$$
$$(P \otimes Q) \otimes R = P \otimes (Q \otimes R) \tag{4.19}$$
$$P \cup 0 = P \tag{4.20}$$
$$P \rhd (Q \cup R) = P \rhd Q \cup R \rhd Q \tag{4.21}$$
$$(P \cup Q) \rhd R = P \rhd R \cup Q \rhd R \tag{4.22}$$
$$P \otimes (Q \cup R) = P \otimes Q \cup P \otimes R \tag{4.23}$$
$$(P \cup Q) \otimes R = P \otimes R \cup Q \otimes R \tag{4.24}$$

Here, constant 0 denotes set of $\emptyset$ iposets and operation $\cup$ denotes choice.

Proof.

► Proof of Equation (4.14)

$$P \rhd 0 = 0$$

Let x be an iposet such that $x \in P \rhd 0$. By using the $\rhd$ Definition 4.12, there exists iposets $p \in P$ and $q \in 0$ such that

$$x = p \rhd q \text{ is defined.}$$

We know 0 is the set of $\emptyset$ iposets, which means q must be an $\emptyset$ iposet, then

$$x = p \rhd q = p \rhd \emptyset = \emptyset$$

This implies $x \in 0$. Similarly, $x \in 0$ can be found in $P \rhd 0$. This two way inclusion witness $P \rhd 0 = 0$.

► Proof of Equation (4.15)

$$P \otimes 0 = 0$$

Let x be an iposet such that $x \in P \otimes 0$. By using the $\otimes$ Definition 4.13, there exists iposets $p \in P$ and $q \in 0$ such that

$$x = p \otimes q \text{ is defined.}$$

We know 0 is the set of $\emptyset$ iposets, which means q must be an $\emptyset$ iposet then

$$x = p \otimes q = p \otimes \emptyset = \emptyset$$

This implies $x \in 0$. Similarly $x \in 0$ can be found in $P \otimes 0$. This two way inclusion witness $P \otimes 0 = 0$.

► Proof of Equation (4.16)

$$P \rhd 1_{\rhd} = P = 1_{\rhd} \rhd P$$

Let x be an iposet such that $x \in P \rhd 1_{\rhd}$. By using the $\rhd$ Definition 4.12, there exists iposets $p \in P$ and $q \in 1_{\rhd}$ such that

$$x = p \rhd q \text{ is defined.}$$

We know $1_{\rhd}$ is the set of identity iposets, which means q must be an identity iposet id_n for $n > 0$ such that

$$x = p \rhd q \overset{4.9}{=} p$$

This implies $x \in P$. Similarly $x \in P$ can be found in $P \rhd 1_{\rhd}$. This two way inclusion witness $P \rhd 1_{\rhd} = P$. Similarly, we can prove $P = 1_{\rhd} \rhd P$, and conclude that $P \rhd 1_{\rhd} = P = 1_{\rhd} \rhd P$.

► Proof of Equation (4.17)

$$P \otimes 1_{\otimes} = P = 1_{\otimes} \otimes P$$

Let x be an iposet such that $x \in P \otimes 1_{\otimes}$. By using the $\otimes$ Definition 4.13, there exists iposets $p \in P$ and $q \in 1_{\otimes}$ such that

$$x = p \otimes q \text{ is defined.}$$

We know $1_\otimes$ is the set of ϵ empty iposets, which means q must be an ϵ empty iposet then

$$x = p \otimes q \overset{4.10}{=} p$$

This implies $x \in P$. Similarly $x \in P$ can be found in $P \otimes 1_\otimes$. This two way inclusion witness $P \otimes 1_\otimes = P$. Similarly, we can prove $P = 1_\otimes \otimes P$, and conclude that $P \otimes 1_\otimes = P = 1_\otimes \otimes P$.

▶ Proof of Equation (4.18)

$$(P \rhd Q) \rhd R = P \rhd (Q \rhd R)$$

Let x be an iposet such that $x \in (P \rhd Q) \rhd R$. By using the $\rhd$ Definition 4.12, there exists iposets $p \in P$, $q \in Q$ and $r \in R$ such that

$$x = (p \rhd q) \rhd r \text{ is defined}$$

By $\rhd$ associative property 4.1 of iposets, we get $(p \rhd q) \rhd r = p \rhd (q \rhd r)$ then

$$x = p \rhd (q \rhd r)$$

This implies that there exists iposets $p \in P$, $q \in Q$ and $r \in R$ such that $x = p \rhd (q \rhd r)$ is defined, which means $x \in P \rhd (Q \rhd R)$ by the Definition 4.12. Similarly, we can show $x \in P \rhd (Q \rhd R)$ implies $x \in (P \rhd Q) \rhd R$. This two way inclusion concludes $(P \rhd Q) \rhd R = P \rhd (Q \rhd R)$.

▶ Proof of Equation (4.19)

$$(P \otimes Q) \otimes R = P \otimes (Q \otimes R)$$

Let x be an iposet such that $x \in (P \otimes Q) \otimes R$. By the $\otimes$ Definition 4.13, there exists iposets $p \in P$, $q \in Q$ and $r \in R$ such that

$$x = (p \otimes q) \otimes r \text{ is defined.}$$

By using $\otimes$ associative property 4.2 of iposets, we get $(p \otimes q) \otimes r = p \otimes (q \otimes r)$ then

$$x = p \otimes (q \otimes r)$$

This implies there exists $p \in P$, $q \in Q$ and $r \in R$ such that $x = p \otimes (q \otimes r)$ is defined, which means $x \in P \otimes (Q \otimes R)$ by the Definition 4.13. Similarly, we can show $x \in P \otimes (Q \otimes R)$ implies $x \in (P \otimes Q) \otimes R$. This two way inclusion concludes $(P \otimes Q) \otimes R = P \otimes (Q \otimes R)$.

▶ Proof of Equation (4.20)

$$P \cup 0 = P$$

Let x be an iposet such that $x \in P \cup 0$. The $x \in P \cup 0$ implies either $x \in P$ or $x \in 0$. However, 0 denotes the set of $\emptyset$ iposets which implies $x \in P$. Similarly, it can be shown that $x \in P$ implies $x \in P \cup 0$. This two way inclusion concludes $P \cup 0 = P$.

► Proof of Equation (4.20)

$$P \rhd (Q \cup R) = P \rhd Q \cup R \rhd Q$$

Let x be an iposet such that $x \in P \rhd (Q \cup R)$. By using the $\rhd$ Definition 4.12, there exists iposets $p \in P$, $q \in Q$ and $r \in R$ such that

$$\text{either } x = (p \rhd q) \text{ or } x = (p \rhd r) \text{ is defined.}$$

This implies that there exists $p \in P$, $q \in Q$ and $r \in R$ such that $x = (p \rhd q) \cup (p \rhd r)$ is defined, which means $x \in P \rhd Q \cup R \rhd Q$. Similarly, it can be shown that $x \in P \rhd Q \cup R \rhd Q$ implies $x \in P \rhd (Q \cup R)$. This two way inclusion concludes $P \rhd (Q \cup R) = P \rhd Q \cup R \rhd Q$.

► Proof of Equation (4.21)

$$(P \cup Q) \rhd R = P \rhd R \cup Q \rhd R$$

Let x be an iposet such that $x \in (P \cup Q) \rhd R$. By using the $\rhd$ Definition 4.12, there exists iposets $p \in P$, $q \in Q$ and $r \in R$ such that

$$\text{either } x = (p \rhd r) \text{ or } x = (q \rhd r) \text{ is defined.}$$

This implies that there exists $p \in P$, $q \in Q$ and $r \in R$ such that $x = (p \rhd r) \cup (q \rhd r)$ is defined, which means $x \in P \rhd R \cup Q \rhd R$ by the Definition 4.12. Similarly, it can be shown that $x \in P \rhd R \cup Q \rhd R$ implies $x \in (P \cup Q) \rhd R$. This two way inclusion concludes $(P \cup Q) \rhd R = P \rhd R \cup Q \rhd R$.

► Proof of Equation (4.22)

$$P \otimes (Q \cup R) = P \otimes Q \cup P \otimes R$$

Let x be an iposet such that $x \in P \otimes (Q \cup R)$. By using the $\otimes$ Definition 4.13, there exists iposets $p \in P$, $q \in Q$ and $r \in R$ such that

$$\text{either } x = (p \otimes q) \text{ or } x = (p \otimes r).$$

This implies that there exists $p \in P$, $q \in Q$ and $r \in R$ such that $x = (p \otimes q) \cup (p \otimes r)$, which means $x \in P \otimes Q \cup P \otimes R$ by the Definition 4.13. Similarly, we can show that $x \in P \otimes Q \cup P \otimes R$ implies $x \in P \otimes (Q \cup R)$. This two way inclusion concludes $P \otimes (Q \cup R) = P \otimes Q \cup P \otimes R$.

► Proof of Equation (4.22)

$$(P \cup Q) \otimes R = P \otimes R \cup Q \otimes R$$

Let x be an iposet such that $x \in (P \cup Q) \otimes R$. By using the $\otimes$ Definition 4.13, there exists iposets $p \in P$, $q \in Q$ and $r \in R$ such that

$$\text{either } x = (p \otimes r) \text{ or } x = (q \otimes r).$$

This implies that there exists $p \in P$, $q \in Q$ and $r \in R$ such that $x = (p \otimes r) \cup (q \otimes r)$ is defined, which means $x \in P \otimes R \cup Q \otimes R$ by the Definition 4.13. Similarly, we can show that $x \in P \otimes R \cup Q \otimes R$ implies $x \in (P \cup Q) \otimes R$. This two way inclusion concludes $(P \cup Q) \otimes R = P \otimes R \cup Q \otimes R$. $\square$

4.3 The structured theory of iposets

In this section, we expose a hierarchy of iposet languages generated by a finite number of series and parallel compositions of singleton iposets.

We can find four *types of singleton iposets* once the labelling of underlying posets are uniquely determined. We use notation $\mathcal{S}$ to denote the class of singleton iposets as follows

$$\begin{aligned} S = \{ & [0] \to [1] \leftarrow [0], \\ & [1] \to [1] \leftarrow [1], \\ & [0] \to [1] \leftarrow [1], \\ & [1] \to [1] \leftarrow [0] \}. \end{aligned}$$

We are interested in the sets of iposets generated by a finite number of series $\rhd$ and parallel $\otimes$ operations over $\mathcal{S}$. Contrary to singleton pomsets, singleton iposets are factorizable into the sequential $\rhd$ factors of identity iposets.

$$[0] \to [1] \leftarrow [0] = [0] \to [1] \leftarrow [1] \rhd [1] \to [1] \leftarrow [1] \rhd [1] \to [1] \leftarrow [0]$$

The decomposition of singleton iposet into a set of singleton identity iposets can be seen as an internal (endogenous or temporal) decomposition. While decomposing into a set of singleton identity iposets, the event induced by singleton iposets remains static but its temporal property gets reconfigured. For example, assume execution of $[0] \to [1] \leftarrow [0]$ takes an unit time interval under the assumption of global clock. Then, the decomposed factors

$$[0] \to [1] \leftarrow [0] = [0] \to [1] \leftarrow [1] \rhd [1] \to [1] \leftarrow [0]$$

take twice of $[0] \to [1] \leftarrow [0]$ unit time to terminate its computation while they induce same event, a single event. It says singleton iposets can delay its computation by decomposing internally as long as it requires to respect its temporal precedence in causal structure of computation. This notion of internal decomposition of a singleton iposet might be useful to model synchronization of communicating threads in a concurrent environment.

Theorem 6 (Interval order)**.** *An ordered set is an* interval order [10] *iff it has no suborder isomorphic to* $2+2$*.*

Definition 39 (Interval representation)**.** *A poset is an interval order if it has an interval representation [10]. An* interval representation of a poset $P = (E_P, \preceq_P)$ *with* $E_P = \{x_1, \dots, x_n\}$ *consists of a set of real intervals*

$$I = \{[l_i, r_i] \mid i = 1, \dots, n, \forall i : l_i \preceq r_i\} \subseteq 2^{\mathbb{R}}$$

with the property that $x_i \preceq_P x_j$ *iff* $r_i \preceq l_j$*.*

Remark 15. A linear extension of poset $P = (E_P, \preceq_P)$ is a linear order $L = (E_P, \preceq_L)$ so that $x \preceq_L$ if $x \preceq_P y$. A linear extension of an poset P is a set of linear orders whose intersection is P, and an interval realizer of an poset P is a set of interval orders whose intersection is P.

Let $\mathcal{S}^{\otimes}$ denote the set of iposets generated from $\mathcal{S}$ by parallel product, and $(\mathcal{S}^{\otimes})^{\triangleright}$ the set of i-posets generated from $\mathcal{S}^{\otimes}$ by sequential product.

Theorem 7. *An iposet is in* $(\mathcal{S}^{\otimes})^{\triangleright}$ *iff it is an interval order.*

Proof.

If $P, Q \in \mathcal{S}^{\otimes}$ then P and Q have interval order representation by Remark 15 and Thereom 6. Similarly, $P \triangleright Q \in (\mathcal{S}^{\otimes})^{\triangleright}$ produces an iposet with partial order $\preceq_{P \triangleright Q}$

$$\preceq_{P \triangleright Q} = \preceq_P \cup \preceq_Q \cup \{E_P \setminus t_P \times E_Q \setminus s_Q\},$$

where $\{E_P \setminus t_P \times E_Q \setminus s_Q\}$ establish linear order from poset P to Q; hence $\preceq_{P \triangleright Q}$ preserves interval realizer of resulting poset $P \triangleright Q$. □

Let $\mathcal{S}^{\otimes, \triangleright}$ denote the set of iposets generated from $\mathcal{S}$ by parallel and sequential product. We now want to expose a hierarchy of generated iposets and compare them with series-parallel posets.

Definition 40 (Iposet hierarchy)**.** *Let* $\mathcal{P}$ *denotes the set of all iposets. For any* $\mathcal{Q} \subseteq \mathcal{P}$*,*

$$\mathcal{Q}^{\otimes} = \{P_1 \otimes \dots \otimes P_n \mid n \in \mathbb{N}, P_1, \dots, P_n \in \mathcal{Q}\},$$
$$\mathcal{Q}^{\triangleright} = \{P_1 \triangleright \dots \triangleright P_n \mid n \in \mathbb{N}, P_1, \dots, P_n \in \mathcal{Q}\}.$$

Then, iposet hierarchy *is given by*

$$\mathcal{C}_0 = \mathcal{D}_0 = \mathcal{S}$$
$$\mathcal{C}_{2n+1} = \mathcal{C}_{2n}^{\otimes} \qquad \mathcal{D}_{2n+1} = \mathcal{D}_{2n}^{\triangleright}$$
$$\mathcal{C}_{2n+2} = \mathcal{C}_{2n+1}^{\triangleright} \qquad \mathcal{D}_{2n+2} = \mathcal{D}_{2n+1}^{\otimes}$$

Lemma 2. *For all n we have,*

1. $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ *and* $\mathcal{C}_n \subseteq \mathcal{D}_{n+1}$.
2. $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ *and* $\mathcal{D}_n \subseteq \mathcal{C}_{n+1}$.
3. $\mathcal{C}_n \cup \mathcal{D}_n \subseteq \mathcal{C}_{n+1} \cap \mathcal{D}_{n+1}$.

Proof.

► $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ and $\mathcal{C}_n \subseteq \mathcal{D}_{n+1}$. Since the $n+1$ iterations are build from the n iterations, we have

$$\mathcal{C}_n \subseteq \mathcal{C}_{n+1} \text{ and } \mathcal{D}_n \subseteq \mathcal{D}_{n+1}$$

by definition. Similarly,

$$\mathcal{C}_0 \subseteq \mathcal{C}_0^{\triangleright} = (\mathcal{D}_0^{\triangleright} = \mathcal{D}_1).$$

Since $\mathcal{C}_n$ is made from $\mathcal{C}_0$ by the same alternative applications of the two operators $\otimes$ and $\triangleright$ as $\mathcal{D}_{n+1}$ is made from $\mathcal{D}_1$, we obtain the stated inclusion.

► $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ and $\mathcal{D}_n \subseteq \mathcal{C}_{n+1}$. We have

$$\mathcal{C}_n \subseteq \mathcal{C}_{n+1} \text{ and } \mathcal{D}_n \subseteq \mathcal{D}_{n+1}$$

Analogously, by definition

$$\mathcal{D}_0 \subseteq \mathcal{D}_0^{\otimes} = (\mathcal{C}_0^{\otimes} = \mathcal{C}_1).$$

Since $\mathcal{D}_n$ is made from $\mathcal{D}_0$ by the same alternative applications of the two operators $\triangleright$ and $\otimes$ as $\mathcal{C}_{n+1}$ is made from $\mathcal{C}_1$, we obtain the stated inclusion.

► $\mathcal{C}_n \cup \mathcal{D}_n \subseteq \mathcal{C}_{n+1} \cap \mathcal{D}_{n+1}$

This is a generalization of the inclusion [1,2] above in Lemma 2. Since $\mathcal{C}_n \cup \mathcal{D}_n \subseteq \mathcal{C}_{n+1}$ and $\mathcal{C}_n \cup \mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ by definition, their intersection includes $\mathcal{C}_n \cup \mathcal{D}_n$.

□

Lemma 3. *For all n we have,*

1. $\mathcal{D}_{2n+1} \otimes \mathcal{D}_{2n} \subseteq \mathcal{D}_{2n+2}$, *and* $\mathcal{D}_{2n} \otimes \mathcal{D}_{2n+1} \subseteq \mathcal{D}_{2n+2}$.
2. $\mathcal{C}_{2n+1} \triangleright \mathcal{C}_{2n} \subseteq \mathcal{C}_{2n+2}$, *and* $\mathcal{C}_{2n} \triangleright \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n+2}$.

Proof.

► $\mathcal{D}_{2n+1} \otimes D_{2n} \subseteq \mathcal{D}_{2n+2}$ follows from

1. $\mathcal{D}_{2n+2} = \mathcal{D}_{2n+1}^{\otimes}$ by Definition 40.
2. $\mathcal{D}_{2n+1} \supseteq \mathcal{D}_{2n}$ by Lemma 2.
3. Hence, $\mathcal{D}_{2n+1}^{\otimes} \supseteq \mathcal{D}_{2n+1} \otimes \mathcal{D}_{2n}$, and therefore $\mathcal{D}_{2n+1} \otimes D_{2n} \subseteq \mathcal{D}_{2n+2}$.

Similarly, $\mathcal{D}_{2n} \otimes \mathcal{D}_{2n+1} \subseteq \mathcal{D}_{2n+2}$ follows.

► $\mathcal{C}_{2n+1} \rhd \mathcal{C}_{2n} \subseteq \mathcal{C}_{2n+2}$ follows from

1. $\mathcal{C}_{2n+2} = \mathcal{C}^{\rhd}_{2n+1}$ by the Definition 40.
2. $\mathcal{C}_{2n+1} \supseteq \mathcal{C}_{2n}$ by the Lemma 2.
3. $\mathcal{C}^{\rhd}_{2n+1} \supseteq \mathcal{C}_{2n+1} \otimes \mathcal{C}_{2n}$, and hence $\mathcal{C}_{2n+1} \rhd \mathcal{C}_{2n} \subseteq \mathcal{C}_{2n+2}$.

Similarly, $\mathcal{C}_{2n} \rhd \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n+2}$ follows.

□

Lemma 4. *For all $n \geq 1$ we have,*

1. *$\mathcal{D}_{2n} \rhd \mathcal{D}_{2n-1} \subseteq \mathcal{D}_{2n+1}$, and $\mathcal{D}_{2n-1} \rhd \mathcal{D}_{2n} \subseteq \mathcal{D}_{2n+1}$.*
2. *$\mathcal{C}_{2n} \otimes \mathcal{C}_{2n-1} \subseteq \mathcal{C}_{2n+1}$, and $\mathcal{C}_{2n-1} \otimes \mathcal{C}_{2n} \subseteq \mathcal{C}_{2n+1}$.*

Proof.

► $\mathcal{D}_{2n-1} \rhd \mathcal{D}_{2n} \subseteq \mathcal{D}_{2n+1}$ follows from

1. $\mathcal{D}_{2n+1} = \mathcal{D}^{\rhd}_{2n}$ by Definition 40.
2. $\mathcal{D}_{2n} \supseteq \mathcal{D}_{2n-1}$ by Lemma 2.
3. $\mathcal{D}^{\rhd}_{2n} \supseteq \mathcal{D}_{2n-1} \rhd \mathcal{D}_{2n}$, and hence $\mathcal{D}_{2n+1} \supseteq D_{2n-1} \subseteq \mathcal{D}_{2n}$.

Similarly, $\mathcal{D}_{2n} \rhd \mathcal{D}_{2n-1} \subseteq \mathcal{D}_{2n+1}$ follows.

► $\mathcal{C}_{2n-1} \otimes \mathcal{C}_{2n} \subseteq \mathcal{C}_{2n+1}$ follows from

1. $\mathcal{C}_{2n+1} = \mathcal{C}^{\otimes}_{2n}$ by Definition 40.
2. $\mathcal{C}_{2n} \supseteq \mathcal{C}_{2n-1}$ by Lemma 2.
3. $\mathcal{C}^{\otimes}_{2n} \supseteq \mathcal{C}_{2n-1} \otimes \mathcal{C}_{2n}$, and hence $\mathcal{C}_{2n-1} \otimes \mathcal{C}_{2n} \subseteq \mathcal{C}_{2n+1}$.

Similarly, $\mathcal{C}_{2n} \otimes \mathcal{C}_{2n-1} \subseteq \mathcal{C}_{2n+1}$ follows.

□

Definition 41 (SP-poset hierarchy)**.** *Let S_0 be an singleton iposet such that*

$$\mathcal{S}_0 = \{[0] \to [1] \leftarrow [0]\}$$

corresponding to the singleton poset. Then, SP-poset hierarchy *is given by*

$$\begin{gathered} \mathcal{T}_0 = \mathcal{U}_0 = \mathcal{S}_0 \\ \mathcal{T}_{2n+1} = \mathcal{T}^{\otimes}_{2n} \qquad \mathcal{U}_{2n+1} = \mathcal{U}^{\rhd}_{2n} \\ \mathcal{T}_{2n+2} = \mathcal{T}^{\rhd}_{2n+1} \qquad \mathcal{U}_{2n+2} = \mathcal{U}^{\otimes}_{2n+1} \end{gathered}$$

Corollary 8. *For all n, $\mathcal{T}_n \cup \mathcal{U}_n = \mathcal{T}_{n+1} \cap \mathcal{U}_{n+1}$.*

Corollary 9. *For all n we have,*

1. *$\mathcal{U}_{2n+1} \otimes \mathcal{U}_{2n} \subseteq \mathcal{U}_{2n+2}$, and $\mathcal{U}_{2n} \otimes \mathcal{U}_{2n+1} \subseteq \mathcal{U}_{2n+2}$.*

2. $\mathcal{T}_{2n+1} \rhd \mathcal{T}_{2n} \subseteq \mathcal{T}_{2n+2}$, *and* $\mathcal{T}_{2n} \rhd \mathcal{T}_{2n+1} \subseteq \mathcal{T}_{2n+2}$.

Corollary 10. *For all $n \geq 1$ we have,*

1. $\mathcal{U}_{2n} \rhd U_{2n-1} \subseteq \mathcal{U}_{2n+1}$.
2. $\mathcal{T}_{2n} \otimes T_{2n-1} \subseteq \mathcal{T}_{2n+1}$.

Remark 16. From hierarchy of SP-posets and iposets, it is clear that

$$\mathcal{S}_0 \subsetneq \mathcal{S} \implies \mathcal{T}_n \subsetneq \mathcal{C}_n \text{ and } \mathcal{U}_n \subsetneq \mathcal{D}_n \text{ For all } n.$$

The expressiveness of iposets compared to SP posets can be illustrated by following example as well.

$$Q = \begin{pmatrix} \begin{tikzcd}[row sep=small] 1\cdot \arrow[r] \arrow[drr] & 3\cdot \arrow[r] & \cdot 5 \\ 2\cdot \arrow[r] \arrow[urr] & 4\cdot \arrow[r] & \cdot 6 \end{tikzcd} \end{pmatrix} = \begin{pmatrix} 1\cdot \longrightarrow \cdot 3 \\ 2\cdot \longrightarrow \cdot 4 \end{pmatrix} \overset{(3\overset{\circ}{,}4)}{\rhd} \begin{pmatrix} 3\cdot \longrightarrow \cdot 5 \\ 4\cdot \longrightarrow \cdot 6 \end{pmatrix}$$

Here, $(3\overset{\circ}{,}4)$ denote agreeing interfaces between the iposets. We showed poset Q is easily expressible in iposets hierarchy given in Definition 40, while it not expressible in sp-posets hierarchy given in Definition 10 . Q contains an N-pomset structure based on $1 \preceq 5, 2 \preceq 5$ and $2 \preceq 4$ causal structure of events in Q. Similar argument applies for all the posets listed in the proof section of Proposition 12, except P itself in the proposition.

4.3.1 The non-collapsing hierarchy, conjectured

Conjecture 1. *Let P_1, P_2, Q_1, Q_2 be iposets such that $P_1 \otimes P_2 = Q_1 \rhd Q_2$, then one of the following is true:*

1. $P_1 = \emptyset$ *or* $P_2 = \emptyset$
2. $Q_1 = id_n$ *or* $Q_2 = id_n$ *for some* $n \in \mathbb{N}$

Proof. Let P_1, P_2, Q_1, Q_2 be iposets.

Assume $x \in Q_1$ and $y \in Q_2$ such that $x \preceq_{Q_1 \rhd Q_2} y$, then $x \preceq_{P_1 \otimes P_2} y$ by either $x \preceq_{P_1} y$ or $x \preceq_{P_2} y$. Consider the case $x \preceq_{P_1} y$ and assume $z \in P_2$ then we find neither $x \preceq_{P_1 \otimes P_2} z$ nor $y \preceq_{P_1 \otimes P_2} z$ which implies neither $z \in Q_1$ nor $z \in Q_2$ (otherwise z will be connected to either one of x and y) unless either $Q_1 \in \mathrm{id}_n$ such that $z \in t_{Q_1}$ and $z = y$ or $Q_2 \in \mathrm{id}_n$ such that $z \in s_{Q_2}$ and $x = z$.

Now, lets consider the case $Q_1, Q_2 \notin \mathrm{id}_n$ and stick to neither $z \in Q_1$ nor $z \in Q_2$.

However, $P_1 \otimes P_2 = Q_1 \rhd Q_2$ that implies $P_2 \subseteq Q_1 \rhd Q_2$, and we find

$$z \in P_2 \implies z \in Q_1 \rhd Q_2$$

which leads to contradiction since neither $z \in Q_1$ nor $z \in Q_2$. Therefore, $P_2 = \emptyset$. Similarly, by choosing $x \preceq_{P_2} y$ and assuming $z \in P_1$, we arrive at $P_1 = \emptyset$.

□

Conjecture 2. *$P_n \in \mathcal{C}_{2n} \setminus \mathcal{C}_{2n-1}$ for all $n \geq 1$.*

Proof. We proceeds with induction on base case $n = 1$; P_1 is not a multiset. Hence $P_1 \notin \mathcal{C}_1$. But $Q \in \mathcal{C}_0 \subseteq \mathcal{C}_1$, hence $P_1 = Q \rhd Q \in \mathcal{C}_2 = \mathcal{C}_1^{\rhd}$.

Now let $n \geq 1$ and assume $P_n \in \mathcal{C}_{2n}$ and $P_n \notin \mathcal{C}_{2n-1}$. Using Conjecture 1, we can show that $P_n \otimes P_n \in \mathcal{C}_{2n+1} \setminus \mathcal{C}_{2n}$:

Obviously $P_n \otimes P_n \in \mathcal{C}_{2n+1} = \mathcal{C}_{2n}^{\otimes}$. Assume $P_n \otimes P_n \in \mathcal{C}_{2n} = \mathcal{C}_{2n-1}^{\rhd}$, then $P_n \otimes P_n = Q^1 \rhd Q^2 \rhd \dots$ for $Q^i \in \mathcal{C}_{2n-1}$.

Note that P_n is a *strongly connected component (SCC)* since the event from Q reaches all events from the P_{n-1} components. This makes $P_n \otimes P_n$ union of two SCC, call these SCC_1 and SCC_2. Assume that each Q^i is non-trivial, i.e., for Q^1 there exists at least one event that is not part of the t-interface, call this e^1. Assume $e^1 \in SCC_1$ and thus is not part of SCC_2. By the definition of concatenation $\rhd$ then e^1 will have a dependency to all events in SCC_2, which cannot be true, unless all the events of SCC_2 are part of the s-interface of $Q^2 \rhd \dots$. Since SCC_2 contains a 2+2 it means that there are at least two events which are not minimal in the partial order, and thus cannot be part of the s-interface. This finishes the contradiction, and thus $P_n \rhd P_n \notin \mathcal{C}_{2n}$.

Now to $P_{n+1} = Q \rhd (P_n \otimes P_n)$. Trivially, $P_{n+1} \in \mathcal{C}_{2n+2} = \mathcal{C}_{2n+1}^{\rhd}$. Assume $P_{n+1} \in \mathcal{C}_{2n+1} = \mathcal{C}_{2n}^{\otimes}$. P_{n+1} is not a parallel product, hence $P_{n+1} \in \mathcal{C}_{2n} = \mathcal{C}_{2n-1}^{\rhd}$. Now $P_{n+1} = Q \rhd (P_n \otimes P_n)$ is the only non-trivial $\rhd$-decomposition of P_{n+1}, thus $P_n \in \mathcal{C}_{2n-1}$, a contradiction. We have shown that $P_{n+1} \notin \mathcal{C}_{2n+1}$. □

Corollary 11. *$\mathcal{C}_{2n-1} \subsetneq \mathcal{C}_{2n}$ for all $n \geq 1$, hence the $\mathcal{C}_n$ hierarchy does not collapse.*

4.3.2 The incomplete hierarchy, conjectured

Proposition 12. *Let $P = \begin{pmatrix} 1\cdot & \longrightarrow & \cdot 4 \\ 2\cdot & \longrightarrow & \cdot 5 \\ 3\cdot & \longrightarrow & \cdot 6 \end{pmatrix}$. Then for all $n \geq 0$, $P \notin C_n$.*

Proof. We fix the decomposition $P = P_1 \rhd P_2$ non-trivial iff P_1 and P_2 are neither null nor identity iposets. Similarly, the decomposition $P = P_1 \rhd P_2$ is trivial iff $P_1 \in \mathrm{id}_n$ or $P_2 \in \mathrm{id}_n$ for $n \geq 1$. Here, $\mathrm{id}_n \in \mathrm{Id}$ such that

$$\mathrm{Id} = \{(E_{\mathrm{id}_n}, \preceq_{\mathrm{id}_n}, s_{\mathrm{id}_n}, 0), (E_{\mathrm{id}_n}, \preceq_{\mathrm{id}_n}, 0, t_{\mathrm{id}_n}), (E_{\mathrm{id}_n}, \preceq_{\mathrm{id}_n}, s_{\mathrm{id}_n}, t_{\mathrm{id}_n})\}$$

Given the iposet $P = (E_P, \preceq_P, 0, 0) = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] \arrow[dr] & \cdot 4 \\ 2\cdot \arrow[r] \arrow[dr] & \cdot 5 \\ 3\cdot \arrow[r] & \cdot 6 \end{tikzcd}\right)$ is composed of set of $E_1 = \begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix}$ minimal and $E_2 = \begin{pmatrix} \cdot 4 \\ \cdot 5 \\ \cdot 6 \end{pmatrix}$ maximal events. It is clear that E_1 and E_2 denotes two least subsets of events of P *i.e.*, if P has factors Q and R then $E_1 \subseteq E_Q$ and $E_2 \subseteq E_R$ such that $E_P = E_{Q \rhd R}$.

We proceed with the possible factors $P_1 = (E_{P_1}, \preceq_{P_1}, 0, 0)$ and $P_2 = (E_{P_2}, \preceq_{P_2}, 0, 0)$ such that $E_1 = E_{P_1}$ and $E_2 = E_{P_2}$, and arrived at $P_1 \rhd P_2 \neq P$ due to the fact that $3 \prec_{P_1 \rhd P_2} 4$ and $3 \not\prec_P 4$. This can be illustrated by the following diagram,

$$\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix} \rhd \begin{pmatrix} \cdot 4 \\ \cdot 5 \\ \cdot 6 \end{pmatrix} = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] \arrow[dr] \arrow[ddr] & \cdot 4 \\ 2\cdot \arrow[ur] \arrow[r] \arrow[dr] & \cdot 5 \\ 3\cdot \arrow[uur] \arrow[ur] \arrow[r] & \cdot 6 \end{tikzcd}\right) \neq P$$

Following P_1 and P_2 as the least possible factors of P, we list all possible decomposition $P = Q \rhd R$ such that $E_{P_1} \subseteq E_Q$ and $E_{P_2} \subseteq E_R$, where $Q = (E_Q, \preceq_Q, 0, t_Q)$ and $R = (E_R, \preceq_R, s_R, 0)$ such that $t_Q = s_R$.

We first check for trivial case $P = Q \rhd R$ such that $E_{P_1} \subseteq E_Q$ and $E_{P_2} \subseteq E_R$, where $Q \in \mathrm{Id}$ or $R \in \mathrm{Id}$,

1. It covers the case $P = Q \rhd R$ such that $Q \in \mathrm{Id}$
   $$\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(1,2,3)}{\rhd} \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] \arrow[dr] & \cdot 4 \\ 2\cdot \arrow[r] \arrow[dr] & \cdot 5 \\ 3\cdot \arrow[r] & \cdot 6 \end{tikzcd}\right) = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] \arrow[dr] & \cdot 4 \\ 2\cdot \arrow[r] \arrow[dr] & \cdot 5 \\ 3\cdot \arrow[r] & \cdot 6 \end{tikzcd}\right)$$
   where $^{(1,2,3)}$ denotes the agreeing interfaces
2. It covers the case $P = Q \rhd R$ such that $R \in \mathrm{Id}$
   $$\left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] \arrow[dr] & \cdot 4 \\ 2\cdot \arrow[r] \arrow[dr] & \cdot 5 \\ 3\cdot \arrow[r] & \cdot 6 \end{tikzcd}\right) \overset{(4,5,6)}{\rhd} \begin{pmatrix} 4\cdot \\ 5\cdot \\ 6\cdot \end{pmatrix} = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] \arrow[dr] & \cdot 4 \\ 2\cdot \arrow[r] \arrow[dr] & \cdot 5 \\ 3\cdot \arrow[r] & \cdot 6 \end{tikzcd}\right)$$

Now, we check for non-trivial case $P = Q \rhd R$ such that $E_{P_1} \subseteq E_Q$ and $E_{P_2} \subseteq E_R$ where Q and R are neither null nor identity iposets,

1. We first proceed with fixed Q such that $E_{P_1} \subseteq E_Q$ and find all possible R such that $E_{P_2} \subseteq E_R$; the composition should respect $E_{Q \rhd R} = E_P$ *i.e.*, the $Q \rhd R$ should produce exactly same set of events in P.

   (a) Lets fix $Q = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] & \cdot 4 \\ 2\cdot & \\ 3\cdot & \end{tikzcd}\right)$ such that $E_{P_1} \subseteq E_Q$ then, possible R such that $E_{P_2} \subseteq E_R$ covers following case

   i. $$\left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] & \cdot 4 \\ 2\cdot & \\ 3\cdot & \end{tikzcd}\right) \overset{(4)}{\rhd} \begin{pmatrix} 4\cdot \\ 5\cdot \\ 6\cdot \end{pmatrix} = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] \arrow[dr] \arrow[ddr] & \cdot 4 \\ 2\cdot \arrow[r] \arrow[dr] & \cdot 5 \\ 3\cdot \arrow[ur] \arrow[r] & \cdot 6 \end{tikzcd}\right) \neq P$$

   ii. $$\left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] & \cdot 4 \\ 2\cdot & \\ 3\cdot & \end{tikzcd}\right) \overset{(4,2)}{\rhd} \left(\begin{tikzcd}[row sep=tiny] 4\cdot & \\ 2\cdot \arrow[r] & \cdot 5 \\ 6\cdot & \end{tikzcd}\right) = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[r] \arrow[dr] \arrow[ddr] & \cdot 4 \\ 2\cdot \arrow[r] & \cdot 5 \\ 3\cdot \arrow[uur] \arrow[ur] \arrow[r] & \cdot 6 \end{tikzcd}\right) \neq P$$

iii. $\begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(4,3)}{\rhd} \begin{pmatrix} 4\cdot \\ 5\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

iv. $\begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(4,2,3)}{\rhd} \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

Here, we exclude the case $R = \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 6\cdot \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ to avoid $E_{Q \rhd R} \neq E_P$.

(b) Lets fix $Q = \begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix}$ such that $E_{P_1} \subseteq E_Q$ then, possible R such that $E_{P_2} \subseteq E_R$ covers following case

i. $\begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix} \overset{(5)}{\rhd} \begin{pmatrix} 4\cdot \\ 5\cdot \\ 6\cdot \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

ii. $\begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix} \overset{(1,5)}{\rhd} \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 6\cdot \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

iii. $\begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix} \overset{(5,3)}{\rhd} \begin{pmatrix} 4\cdot \\ 5\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

iv. $\begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix} \overset{(1,5,3)}{\rhd} \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

Here, we exclude the case $R = \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ to avoid $E_{Q \rhd R} \neq E_P$.

(c) Lets fix $Q = \begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \longrightarrow 6\cdot \end{pmatrix}$ such that $E_{P_1} \subseteq E_Q$ then, possible R such that $E_{P_2} \subseteq E_R$ covers following case

i. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \longrightarrow 6\cdot \end{pmatrix} \overset{(6)}{\rhd} \begin{pmatrix} 4\cdot \\ 5\cdot \\ 6\cdot \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

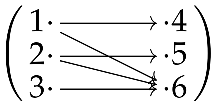
ii. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \longrightarrow 6\cdot \end{pmatrix} \overset{(2,6)}{\rhd} \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

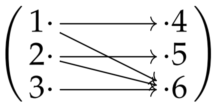
iii. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \longrightarrow 6\cdot \end{pmatrix} \overset{(1,6)}{\rhd} \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 6\cdot \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

iv. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \longrightarrow 6\cdot \end{pmatrix} \overset{(1,2,6)}{\rhd} \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

Here, we exclude the case $R = \begin{pmatrix} 4\cdot & \\ 5\cdot & \\ 3\cdot \longrightarrow & \cdot 6 \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ to avoid $E_{Q \rhd R} \neq E_P$.

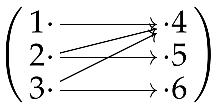
(d) Lets fix $Q = \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix}$ such that $E_{P_1} \subseteq E_Q$ then, possible R such that $E_{P_2} \subseteq E_R$ covers following case

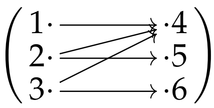
i. $\begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix} \overset{(4,5)}{\rhd} \begin{pmatrix} \cdot 4 \\ \cdot 5 \\ \cdot 6 \end{pmatrix} =$  $\neq P$

ii. $\begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix} \overset{(4,5,3)}{\rhd} \begin{pmatrix} 4\cdot & \\ 5\cdot & \\ 3\cdot \longrightarrow & \cdot 6 \end{pmatrix} =$  $\neq P$

Here, we exclude the case $\left\{ \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 6\cdot \end{pmatrix}, \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix}, \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix} \right\} \in R$ such that $E_{P_2} \subseteq E_R$ to avoid $E_{Q \rhd R} \neq E_P$.

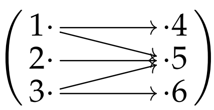
(e) Lets fix $Q = \begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix}$ such that $E_{P_1} \subseteq E_Q$ then, possible R such that $E_{P_2} \subseteq E_R$ covers following case

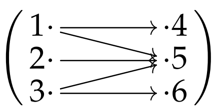
i. $\begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} \overset{(5,6)}{\rhd} \begin{pmatrix} \cdot 4 \\ \cdot 5 \\ \cdot 6 \end{pmatrix} =$  $\neq P$

ii. $\begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} \overset{(1,5,6)}{\rhd} \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 6\cdot \end{pmatrix} =$  $\neq P$

Here, we exclude the case $\left\{ \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix}, \begin{pmatrix} 4\cdot & \\ 5\cdot & \\ 3\cdot \longrightarrow & \cdot 6 \end{pmatrix}, \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} \right\} \in R$ such that $E_{P_2} \subseteq E_R$ for similar reason like the cases above.

(f) Lets fix $Q = \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix}$ such that $E_{P_1} \subseteq E_Q$ then, possible R such that $E_{P_2} \subseteq E_R$ covers following case

i. $\begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} \overset{(4,6)}{\rhd} \begin{pmatrix} \cdot 4 \\ \cdot 5 \\ \cdot 6 \end{pmatrix} =$  $\neq P$

ii. $\begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} \overset{(4,2,6)}{\rhd} \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix} =$  $\neq P$

Here, we exclude the case $\left\{ \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 6\cdot \end{pmatrix}, \begin{pmatrix} 4\cdot & \\ 5\cdot & \\ 3\cdot \longrightarrow & \cdot 6 \end{pmatrix}, \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} \right\} \in R$ such that $E_{P_2} \subseteq E_R$ to avoid $E_{Q \rhd R} \neq E_P$.

2. We now proceed with fixed R such that $E_{P_2} \subseteq E_R$ and find for all the possible Q such that $E_{P_1} \subseteq E_Q$; such that the composition should respect $E_{Q \rhd R} = E_P$ *i.e.*, the $Q \rhd R$ should produce exactly same set of events in P.

   (a) Lets fix $R = \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 6\cdot \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ then, possible Q such that $E_{P_1} \subseteq E_Q$ covers following case

      i. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(1)}{\rhd} \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 6\cdot \end{pmatrix} = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[rr] & & \cdot 4 \\ 2\cdot \arrow[rr] \arrow[urr] \arrow[drr] & & \cdot 5 \\ 3\cdot \arrow[rr] \arrow[urr] \arrow[uurr] & & \cdot 6 \end{tikzcd}\right) \neq P$

      ii. Case (b).(ii)

      iii. Case (c).(iii)

   Here, we exclude the case $Q = \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \end{pmatrix}$ such that $E_{P_1} \subseteq E_Q$ to avoid $E_{Q \rhd R} \neq E_P$.

   (b) Lets fix $R = \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ then, possible Q such that $E_{P_1} \subseteq E_Q$ covers following case

      i. Case (a).(ii)

      ii. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(2)}{\rhd} \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix} = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[rr] \arrow[drr] \arrow[ddrr] & & \cdot 4 \\ 2\cdot \arrow[rr] & & \cdot 5 \\ 3\cdot \arrow[rr] \arrow[urr] \arrow[uurr] & & \cdot 6 \end{tikzcd}\right) \neq P$

      iii. Case (c).(ii)

   Here, we exclude the case $Q = \begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix}$ such that $E_{P_1} \subseteq E_Q$ to avoid $E_{Q \rhd R} \neq E_P$.

   (c) Lets fix $R = \begin{pmatrix} 4\cdot \\ 5\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ then, possible Q such that $E_{P_1} \subseteq E_Q$ covers following case

      i. Case (a).(iii)

      ii. Case (b).(ii)

      iii. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(3)}{\rhd} \begin{pmatrix} 4\cdot \\ 5\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} = \left(\begin{tikzcd}[row sep=tiny] 1\cdot \arrow[rr] \arrow[drr] \arrow[ddrr] & & \cdot 4 \\ 2\cdot \arrow[rr] \arrow[urr] \arrow[drr] & & \cdot 5 \\ 3\cdot \arrow[rr] & & \cdot 6 \end{tikzcd}\right) \neq P$

   Here, we exclude the case $Q = \begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix}$ such that $E_{P_1} \subseteq E_Q$ to avoid $E_{Q \rhd R} \neq E_P$.

(d) Lets fix $R = \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ then, possible Q such that $E_{P_1} \subseteq E_Q$ covers following case

i. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(1,2)}{\rhd} \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \longrightarrow \cdot 5 \\ 6\cdot \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

ii. Case (c).(iv)

Here, we exclude the case $\left\{ \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \end{pmatrix}, \begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix}, \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix} \right\} \in$ Q such that $E_{P_1} \subseteq E_Q$ to avoid $E_{Q \rhd R} \neq E_P$.

(e) Lets fix $R = \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ then, possible Q such that $E_{P_1} \subseteq E_Q$ covers following case

i. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(2,3)}{\rhd} \begin{pmatrix} 4\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

ii. Case (a).(iv)

Here, we exclude the case $\left\{ \begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \end{pmatrix}, \begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix}, \begin{pmatrix} 1\cdot \\ 2\cdot \longrightarrow \cdot 5 \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} \right\} \in$ Q such that $E_{P_2} \subseteq E_R$ for similar reason like the cases above.

(f) Lets fix $R = \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix}$ such that $E_{P_2} \subseteq E_R$ then, possible Q such that $E_{P_1} \subseteq E_Q$ covers following case

i. $\begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \end{pmatrix} \overset{(1,3)}{\rhd} \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 5\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} = \begin{pmatrix} 1\cdot & \cdot 4 \\ 2\cdot & \cdot 5 \\ 3\cdot & \cdot 6 \end{pmatrix} \neq P$

ii. Case (b).(iv)

Here, we exclude the case $\left\{ \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \end{pmatrix}, \begin{pmatrix} 1\cdot \\ 2\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix}, \begin{pmatrix} 1\cdot \longrightarrow \cdot 4 \\ 2\cdot \\ 3\cdot \longrightarrow \cdot 6 \end{pmatrix} \right\} \in$ Q such that $E_{P_2} \subseteq E_R$ to avoid $E_{Q \rhd R} \neq E_P$.

These all the non-trivial cases $P \neq Q \rhd R$ such that $E_{P_1} \subseteq E_Q$ and $E_{P_2} \subseteq E_R$ witness $P \notin C_n$ for $n \geq 0$. $\square$

4.4 The theory of iposets under subsumption

In this section, we present algebraic results of iposets under subsumption order. The subsumption order is an important concept in standard pomset theory (see for example [13, 17]) which captures the difference between "implementation" and "specification" of concurrent systems in the sense that the specification preserve more concurrent behaviour than the implementation.

Definition 42. *The* subsumption order $P \leqslant Q$ on posets

$$P = (E_P, \preceq_P) \text{ and } Q = (E_Q, \preceq_Q)$$

is defined [13] if there exists a bijection $h : E_Q \to E_P$ such that

$$x \preceq_Q y \implies h(x) \preceq_P h(y) \text{ for all } x, y \in Q.$$

Subsumption says that the posets are higher in the subsumption order if they have less sequential dependencies among their set of events. The downward-closed sets of poset contains all possible linearization of its set of events. This guarantees that the weak exchange law holds on posets and downward-closed sets of posets. Below, we precisely define the order subsumption and isomorphism on iposets.

Definition 43 (Subsumption on iposets)**.** *The subsumption order $P \leqslant Q$ on iposets*

$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } Q = E_Q, \preceq_Q, s_Q, t_Q$$

is defined if there exists bijection $h : E_Q \to E_P$ such that

$$x \preceq_Q y \implies h(x) \preceq_P h(y) \text{ for all } x, y \in Q$$

with the source and target interface bijections

$$h : s_Q[n_Q] \to s_P[n_P] \text{ and } h : t_Q[m_Q] \to t_P[m_P].$$

Definition 44 (Isomorphism on iposets)**.** *The isomorphism $P = Q$ on iposets*

$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } Q = E_Q, \preceq_Q, s_Q, t_Q$$

is defined if there exists bijection $h : E_Q \to E_P$ such that

$$x \preceq_Q y \text{ iff } h(x) \preceq_P h(y) \text{ for all } x, y \in Q$$

along with

$$h : s_Q[n_Q] \to s_P[n_P] \text{ and } h : t_Q[m_Q] \to t_P[m_P]$$

the source and target interfaces bijections.

The Lemma 5 defines subsumption order for empty iposets.

Lemma 5 (Empty iposet)**.** *Let P be an iposet such that $P \leqslant id_0$ or $id_0 \leqslant P$ and id_0 denotes an ϵ empty iposet. Then $P = id_0$.*

Proof.

Let P and id_0 be iposets

$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } \mathrm{id}_0 = (E_{\mathrm{id}_0}, \preceq_{\mathrm{id}_0}, s_{\mathrm{id}_0}, t_{\mathrm{id}_0}).$$

Consider the case $P \leqslant \mathrm{id}_0$ which witness the bijections $h : E_{\mathrm{id}_0} \to E_P$ such that

$$x \preceq_{\mathrm{id}_0} y \implies h(x) \preceq_P h(y) \text{ for all } x, y \in \mathrm{id}_0$$

along with

$$h : s_{\mathrm{id}_0} \to s_P \text{ and } h : t_{\mathrm{id}_0} \to t_P$$

the source and target interface bijections. The id_0 is an empty iposet, *i.e.,* $E_{id_0} = \epsilon$, then we have $\mathrm{id}_0 = (E_{\mathrm{id}_0}, \preceq_{\mathrm{id}_0}, s_{\mathrm{id}_0}, t_{\mathrm{id}_0}) = (\epsilon, \epsilon, \epsilon, \epsilon)$. Then, we get $E_P = \epsilon$ along with $s_P = \epsilon$ and $t_P = \epsilon$ since the bijections h. The $x \preceq_{\mathrm{id}_n} y \implies h(x) \preceq_P h(y)$ naturally follows since the bijection $h^{-1}(P) = \epsilon$ on underlying empty set. Finally, $P = (E_P, \preceq_P, s_P, t_P) = (\epsilon, \epsilon, \epsilon, \epsilon)$ witness $P = \mathrm{id}_0$.

Similarly, it follows that $h^{-1} : P \to \mathrm{id}_0$ is a subsumption order witnessing $\mathrm{id}_0 \leqslant P$, which naturally concludes $P = \mathrm{id}_0$. □

The Lemma 6 defines subsumption order for singleton iposets.

Lemma 6 (Singleton iposet)**.** *Let P and Q be iposets, with Q a singleton iposet such that $Q \leqslant P$ or $P \leqslant Q$, then $P = Q$.*

Proof.

Let P and Q be iposets

$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } Q = (E_Q, \preceq_Q, s_Q, t_Q)$$

Considering the case $P \leqslant Q$ which witness the bijection $h : E_Q \to E_P$ such that

$$x \preceq_Q y \implies h(x) \preceq_P h(y) \text{ for all } x, y \in Q$$

along with

$$h : s_Q \to s_P \text{ and } h : t_Q \to t_P$$

the source and target interface bijections. Since Q is a singleton iposet, the bijection $h : E_Q \to E_P$ ensures that E_P contains single events. The $x \preceq_Q y \implies h(x) \preceq_P h(y)$ naturally follows since the bijection on the underlying single event. Finally, the bijections h on the interfaces ensures P and Q are isomorphic singleton iposets, hence $P = Q$.

Similarly, consider the case $Q \leqslant P$. Since Q is singleton iposets, if $x, y \in P$ such that $x \preceq_P y$ then $x = y$, which leads to

$$h^{-1}(x) = h^{-1}(y) \text{ and thus } h^{-1}(x) \preceq_Q h^{-1}(y)$$

along with

$$h^{-1}(s_Q) = s_P \text{ and } h^{-1}(t_Q) = t_P$$

It follows that $h^{-1} : Q \to P$ establish a subsumption order witnessing $Q \leqslant P$, hence $P = Q$. $\square$

Remark 17. *The Conjecture 1 relies on the fact that each iposet P is either a singleton or there is a unique i such that P is either $\rhd_i$ or $\otimes_i$ -reducible as Defined 4.3. Moreover, P has, up to associativity, a unique maximal decomposition into a $\rhd_i$ or $\otimes_i$ -product of i-reducible n-iposets. Then, we know that P is an isomorphic class of iposets.*

The Lemma 7 defines subsumption order for $\rhd$ decomposition of iposets.

Lemma 7 ($\rhd$ Factorization)**.** *Let P, Q_0 and Q_1 be iposets such that $P \leqslant Q_0 \rhd Q_1$. Then there exist iposets P_0 and P_1 such that*

$$P = P_0 \rhd P_1, P_0 \leqslant Q_0 \text{ and } P_1 \leqslant Q_1$$

Proof.

Let P, Q_0 and Q_1 be iposets

$$P = (E_P, \preceq_P, s_P, t_P), Q_0 = (E_{Q_0}, \preceq_{Q_0}, s_{Q_0}, t_{Q_0}), \text{ and } Q_1 = (E_{Q_1}, \preceq_{Q_1}, s_{Q_1}, t_{Q_1})$$

Consider the case $P \leqslant Q_0 \rhd Q_1$ which witness the bijection $h : E_{Q_0 \rhd Q_1} \to E_P$

$$\begin{aligned} E_P &= h(E_{Q_0 \rhd Q_1}) \\ &= h((E_{Q_0} \cup E_{Q_1})_{/t_{Q_0}(i) = s_{Q_1}(i)}) \end{aligned}$$

such that

$$x \preceq_{Q_0 \rhd Q_1} y \implies h(x) \preceq_P h(y) \text{ for all } x, y \in Q_0 \rhd Q_1$$

along with

$$h : s_{Q_0 \rhd Q_1} \to s_P \text{ and } h : t_{Q_0 \rhd Q_1} \to t_P$$

the source and target interfaces bijections. Now, we choose the pairwise disjoint iposets

$$P_0 = (E_{P_0}, \preceq_{P_0}, s_{P_0}, t_{P_0}) \text{ and } P_1 = (E_{P_1}, \preceq_{P_1}, s_{P_1}, t_{P_1})$$

such that

$E_{P_0} = E_P \cap h(E_{Q_0})$ denotes the restriction of E_{Q_0} to the E_P such that

$$x \preceq_{Q_0} y \implies h(x) \preceq_{P_0} h(y)$$

for all $x, y \in Q_0$ along with the source $s_{P_0} = s_P \cap h(s_{Q_0})$ and target $t_{P_0} = E_P \cap h(t_{Q_0})$ interfaces.

In short,

$$P_0 = (E_P \cap h(E_{Q_0}), x \preceq_{Q_0} y \implies h(x) \preceq_{P_0} h(y), s_P \cap h(s_{Q_0}), E_P \cap h(t_{Q_0}))$$

Similarly,

$$P_1 = (E_P \cap h(E_{Q_1}), x \preceq_{Q_1} y \implies h(x) \preceq_{P_1} h(y), E_P \cap h(s_{Q_1}), t_P \cap h(t_{Q_1})).$$

To established $P_0 \leqslant Q_0$, we proceed with the bijection on set of events $h : E_{Q_0} \to E_{P_0}$

$$E_{P_0} = E_P \cap h(E_{Q_0}) = h(E_{Q_0 \triangleright Q_1}) \cap h(E_{Q_0}) = h(E_{Q_0 \triangleright Q_1} \cap E_{Q_0}) = h(E_{Q_0})$$

along with the source and target $h : s_{Q_0} \to s_{P_0}$ and $h : t_{Q_0} \to t_{P_0}$ interface bijections

$$s_{P_0} = s_P \cap h(s_{Q_0}) = h(s_{Q_0 \triangleright Q_1}) \cap h(s_{Q_0}) = h(s_{Q_0}) \cap h(s_{Q_0}) = h(s_{Q_0})$$

$$t_{P_0} = E_P \cap h(t_{Q_0}) = h(E_{Q_0 \triangleright Q_1}) \cap h(t_{Q_0}) = h(E_{Q_0 \triangleright Q_1} \cap t_{Q_0}) = h(t_{Q_0})$$

such that $x \preceq_{Q_0} y \implies h(x) \preceq_{P_0} h(y)$ for all $x, y \in Q_0$. Let $x, y \in Q_0$ such that $x \preceq_{Q_0} y$. Then, we know $x \preceq_{Q_0 \triangleright Q_1} y$ and $x \preceq_P y$. However $x, y \in Q_0$ thus $x \preceq_{P_0} y$. Similarly, we can establish $P_1 \leqslant Q_1$.

To see $P = P_0 \triangleright P_1$, we proceed with sets of events

$$\begin{aligned}
E_{P_0 \triangleright P_1} &= (E_{P_0} \cup E_{P_1})_{/t_{P_0}(i)=s_{P_1}(i)} \\
&= (E_{P_0} \cup E_{P_1})_{/t_{P_0}(i)=s_{P_1}(i)} \\
&= ((E_P \cap h(E_{Q_0})) \cup (E_P \cap h(E_{Q_1})))_{/t_{P_0}(i)=s_{P_1}(i)}
\end{aligned}$$

▶After simplifying,

$$= (E_P \cap h(E_{Q_0} \cup E_{Q_1}))_{/t_{P_0}(i)=s_{P_1}(i)}$$

▶ Since $P_0 \leqslant Q_0$ and $P_1 \leqslant Q_1$, $t_{P_0}(i) = s_{P_1}(i)$ and $h(t_{Q_0}(i)) = h(s_{Q_1}(i))$ denotes same set of events. Then, $h(t_{Q_0}(i)) = h(s_{Q_1}(i))$ can be written as $h(t_{Q_0}(i) = s_{Q_1}(i))$ and after substitution we get

$$\begin{aligned}
&= E_P \cap h((E_{Q_0} \cup E_{Q_1})_{/t_{Q_0}(i)=s_{Q_1}(i))}) \\
&= E_P \cap h(E_{Q_0 \triangleright Q_1}) \\
&= E_P \cap E_P = E_p
\end{aligned}$$

along with source and target interfaces bijections

$$s_{P_0 \triangleright P_1} = s_{P_0} = s_P \cap h(s_{Q_0}) = s_P \cap h(s_{Q_0 \triangleright Q_1}) = s_P \cap s_P = s_P$$

$$t_{P_0 \triangleright P_1} = t_{P_1} = t_P \cap h(t_{Q_1}) = t_P \cap h(t_{Q_0 \triangleright Q_1}) = t_P \cap t_P = t_P$$

and order isomophism such that $x \preceq_P y$ iff $x \preceq_{P_0 \rhd P_1} y$ for all $x, y \in P$. To claim order isomorphism, let $x, y \in P_0 \rhd P_1$ such that $x \preceq_{P_0 \rhd P_1} y$, then

1. if $x, y \in Q_0$ then $x \preceq_{Q_0} y$, and $x \preceq_{Q_0 \rhd Q_1} y$. Thus, $x \preceq_P y$.
2. if $x, y \in Q_1$ then $x \preceq_{Q_1} y$, and $x \preceq_{Q_0 \rhd Q_1} y$. Thus, $x \preceq_P y$.
3. if $x \in Q_0$ and $y \in Q_1$, then there are three case to consider
   (a) if $x \in Q_0$ and $y \in Q_1$ such that $y \in s_{Q_1}$, then case (1).
   (b) if $x \in Q_0$ and $y \in Q_1$ such that $x \in t_{Q_0}$, then case (2).
   (c) if $x \in Q_0$ and $y \in Q_1$ such that $x \notin t_{Q_0}$ and $y \notin s_{Q_1}$, then $x \preceq_{Q_0 \rhd Q_1} y$. Thus, $x \preceq_P y$.
4. if $y \in Q_0$ and $x \in Q_1$ such that $x \preceq_{Q_0 \rhd Q_1} y$ implies $x = y$ by antisymmetry, and thus $x = y$ explains the fusion such that $y \in t_{Q_0}$ and $x \in s_{Q_1}$.

Similarly, let $x, y \in P$ such that $x \preceq_P y$, then

1. if $x, y \in Q_0$ then $x \preceq_{P_0} y$, and thus $x \preceq_{P_0 \rhd P_1} y$.
2. if $x, y \in Q_1$ then $x \preceq_{P_1} y$, and thus $x \preceq_{P_0 \rhd P_1} y$.
3. if $x \in Q_0$ and $y \in Q_1$ then there are three case to consider
   (a) if $x \in Q_0$ and $y \in Q_1$ such that $y \in s_{Q_1}$, then case (1).
   (b) if $x \in Q_0$ and $y \in Q_1$ such that $x \in t_{Q_0}$, then case (2).
   (c) if $x \in Q_0$ and $y \in Q_1$ such that $x \notin t_{Q_0}$ and $y \notin s_{Q_1}$, then $x \in P_0$ and $y \in P_1$. Thus, $x \preceq_{P_0 \rhd P_1} y$.
4. if $y \in Q_0$ and $x \in Q_1$ then $y \in P_0$ and $x \in P_1$, thus $x \preceq_{P_0 \rhd P_1} y$ implies $x = y$ by antisymmetry. This explains $P_0 \rhd P_1$ the fusion such that $y \in t_{P_0}$ and $x \in s_{P_1}$.

This proves the $P = P_0 \rhd P_1$. □

The Lemma 8 defines subsumption order for $\otimes$ decomposition of iposets.

Lemma 8 ($\otimes$ Factorization). *Let P_0, P_1 and Q be iposets such that $P_0 \otimes P_1 \leqslant Q$. Then there exist iposets Q_0 and Q_1 such that*

$$Q = Q_0 \otimes Q_1, P_0 \leqslant Q_0 \text{ and } P_1 \leqslant Q_1$$

Proof.

Let P_0, P_1 and Q be iposets

$$P_0 = (E_{P_0}, \preceq_{P_0}, s_{P_0}, t_{P_0}), P_1 = (E_{P_1}, \preceq_{P_1}, s_{P_1}, t_{P_1}), \text{ and } Q = (E_Q, \preceq_Q, s_Q, t_Q)$$

Consider the case $P_0 \otimes P_1 \leqslant Q$ which witness the bijection $h : E_Q \to E_{P_0 \otimes P_1}$

$$E_Q = h^{-1}(E_{P_0 \otimes P_1}) = h^{-1}(E_{P_0} \cup E_{P_1})$$

such that

$$x \preceq_Q y \implies h(x) \preceq_{P_0 \otimes P_1} h(y) \text{ for all } x, y \in Q$$

along with

$$h : s_Q \to s_{P_0 \otimes P_1} \text{ and } h : t_Q \to t_{P_0 \otimes P_1}$$

the source and target interfaces bijections. Now, we choose the pairwise disjoint iposets

$$Q_0 = (E_{Q_0}, \preceq_{Q_0}, s_{Q_0}, t_{Q_0}) \text{ and } Q_1 = (E_{Q_1}, \preceq_{Q_1}, s_{Q_1}, t_{Q_1})$$

such that

$$Q_0 = (E_Q \cap h^{-1}(E_{P_0}), x \preceq_{Q_0} y \implies h(x) \preceq_{P_0} h(y), s_Q \cap h^{-1}(s_{P_0}), E_Q \cap h^{-1}(t_{P_0}))$$

$$Q_1 = (E_Q \cap h^{-1}(E_{P_1}), x \preceq_{Q_1} y \implies h(x) \preceq_{P_1} h(y), E_Q \cap h^{-1}(s_{P_1}), t_Q \cap h^{-1}(t_{P_1})).$$

To established $P_0 \leqslant Q_0$, we proceed with the bijection $h : E_{Q_0} \to E_{P_0}$

$$\begin{aligned}
E_{Q_0} &= E_Q \cap h^{-1}(E_{P_0}) \\
&= h^{-1}(E_{P_0 \otimes P_1}) \cap h^{-1}(E_{P_0}) \\
&= h^{-1}(E_{P_0} \cup E_{P_1}) \cap h^{-1}(E_{P_0}) \\
&= h^{-1}((E_{P_0} \cup E_{P_1}) \cap E_{P_0}) \\
&= h^{-1}(E_{P_0})
\end{aligned}$$

along with the source and target $h : s_{Q_0} \to s_{P_0}$ and $h : t_{Q_0} \to t_{P_0}$ interface bijections

$$\begin{aligned}
s_{Q_0} = s_Q \cap h^{-1}(s_{P_0}) = h^{-1}(s_{P_0 \otimes P_1}) \cap h^{-1}(s_{P_0}) &= h^{-1}(s_{P_0} \cup s_{P_1}) \cap h^{-1}(s_{P_0}) \\
&= h^{-1}((s_{P_0} \cup s_{P_1}) \cap s_{P_0}) = h^{-1}(s_{P_0})
\end{aligned}$$

$$\begin{aligned}
t_{Q_0} = E_Q \cap h^{-1}(t_{P_0}) = h^{-1}(E_{P_0 \otimes P_1}) \cap h^{-1}(t_{P_0}) &= h^{-1}(E_{P_0} \cup E_{P_1}) \cap h^{-1}(t_{P_0}) \\
&= h^{-1}((E_{P_0} \cup E_{P_1}) \cap t_{P_0}) = h^{-1}(t_{P_0})
\end{aligned}$$

such that $x \preceq_{Q_0} y \implies h(x) \preceq_{P_0} h(y)$ for all $x, y \in Q_0$. Let $x, y \in Q_0$ be such that $x \preceq_{Q_0} y$. We then know $x \preceq_Q y$, and $x \preceq_{P_0 \otimes P_1} y$. However $x, y \in Q_0$, thus $x \preceq_{P_0} y$. Similarly, we can establish $P_1 \leqslant Q_1$.

To see $Q = Q_0 \otimes Q_1$, we proceed with sets of events

$$\begin{aligned}
E_{Q_0 \otimes Q_1} &= E_{Q_0} \cup E_{Q_1} \\
&= (E_Q \cap h^{-1}(E_{P_0})) \cup (E_Q \cap h^{-1}(E_{P_1}))
\end{aligned}$$

$$\begin{aligned}
&\blacktriangleright\text{After simplifying,}\\
&= E_Q \cap h^{-1}(E_{P_0} \cup E_{P_1})\\
&= E_Q \cap h^{-1}(E_{P_0 \otimes P_1})\\
&= E_Q \cap E_Q = E_Q
\end{aligned}$$

along with source and target interfaces

$$\begin{aligned}
s_{Q_0 \otimes Q_1} &= s_{Q_0} \cup s_{Q_1}\\
&= (s_Q \cap h^{-1}(s_{P_0})) \cup (E_Q \cap h^{-1}(s_{P_1}))\\
&\blacktriangleright\text{Since } s_Q \subseteq E_Q \text{ and after simplifying,}\\
&= s_Q \cap h^{-1}(s_{P_0} \cup s_{P_1})\\
&= s_Q \cup h^{-1}(s_{P_0 \otimes P_1})\\
&= s_Q \cap s_Q = s_Q\\
t_{P_0 \otimes P_1} &= t_{P_0} \cup t_{P_1}\\
&= (E_Q \cap h^{-1}(t_{P_0})) \cup (t_Q \cap h^{-1}(t_{P_1}))\\
&\blacktriangleright\text{Since } t_Q \subseteq E_Q \text{ and after simplifying,}\\
&= t_Q \cap h^{-1}(t_{P_0} \cup t_{P_1})\\
&= t_Q \cap h^{-1}(t_{P_0 \otimes P_1})\\
&= t_Q \cap t_Q = t_Q
\end{aligned}$$

and order isomorphism such that $x \preceq_Q y$ iff $x \preceq_{Q_0 \otimes Q_1} y$ for all $x, y \in Q$. To prove order isomorphism, let $x, y \in Q_0 \otimes Q_1$ such that $x \preceq_{Q_0 \otimes Q_1} y$ then

1. if $x, y \in P_0$ then $x \preceq_{P_0} y$, and $x \preceq_{P_0 \otimes P_1} y$. Thus, $x \preceq_Q y$.
2. if $x, y \in P_1$ then $x \preceq_{P_1} y$, and $x \preceq_{P_0 \otimes P_1} y$. Thus, $x \preceq_Q y$.

Similarly, let $x, y \in Q$ such that $x \preceq_Q y$, then

1. if $x, y \in P_0$ then $x \preceq_{Q_0} y$, and thus $x \preceq_{Q_0 \otimes Q_1} y$.
2. if $x, y \in P_1$ then $x \preceq_{Q_1} y$, and thus $x \preceq_{Q_0 \otimes Q_1} y$.

This proves $Q = Q_0 \otimes Q_1$. □

Definition 45. *The iposets P is said to be prime when it can not be uniquely decomposable into either $\rhd_i$ or $\otimes_i$ -product of i-reducible n non-empty non-identity iposets.*

The Lemma 9 defines uniqueness of iposets under $\rhd$ and $\otimes$ decomposition.

Lemma 9 (Uniqueness)**.** *Let P be an iposet, then exactly one of the following case holds for P*

1. *P is an empty iposet, or*
2. *P is an singleton iposet, or*

3. *There exists non-empty non-identity iposets P_0 and P_1 such that $P = P_0 \rhd P_1$.* *or,*
4. *There exists non-empty non-identity iposets P_0' and P_1' such that $P = P_0' \otimes P_1'$.* *or,*
5. *P is prime iposet, i.e., $P \notin C_n$ for all $n \geq 0$.*

Proof.

Let P be an iposet such that

$$P = (E_P, \preceq_P, s_P, t_P).$$

Then, one of following case holds for P

1. if $E_P = E_{\mathrm{id}_0}$ then P is an empty iposet by lemma 5. Thus, P neither be an identity nor there exists non-empty non-identity iposets P_0, P_1, P_0' and P_1' such that either $P = P_0 \rhd P_1$ or $P = P_0' \otimes P_1'$.
2. if $E_P = E_Q$ such that Q is a singleton iposets then P is an singleton iposets by lemma 6. Thus, P neither be an empty iposet nor there exists non-empty non-identity iposets P_0, P_1, P_0' and P_1' such that either $P = P_0 \rhd P_1$ or $P = P_0' \otimes P_1'$.
3. To prove the last two cases; there exists non-empty non-identity iposets P_0, P_1, P_0' and P_1' such that either $P = P_0 \rhd P_1$ or $P = P_0' \otimes P_1'$, we postulate a contradiction $P_0 \rhd P_1 = P_0' \otimes P_1'$ such as

$$P_0 = (E_{P_0}, \preceq_{P_0}, s_{P_0}, t_{P_0}), P_1 = (E_{P_1}, \preceq_{P_1}, s_{P_1}, t_{P_1}).$$

$$P_0' = (E_{P_0'}, \preceq_{P_0'}, s_{P_0'}, t_{P_0'}), P_1' = (E_{P_1'}, \preceq_{P_1'}, s_{P_1'}, t_{P_1'}).$$

Then, we know $E_{P_0 \rhd P_1}$ and $E_{P_0' \otimes P_1'}$ denotes same set of events

$$E_{P_0 \rhd P_1} = E_{P_0' \otimes P_1'} \tag{4.25}$$

such that $x \preceq_{P_0 \rhd P_1} y$ iff $x \preceq_{P_0' \otimes P_1'} y$ for all $x, y \in P_0 \rhd P_1$. Suppose $x \in P_0$ and $y \in P_1$ such that $x \preceq_{P_0 \rhd P_1} y$ then $x \preceq_{P_0' \otimes P_1'} y$ by either $x \preceq_{P_0'} y$ or $x \preceq_{P_1'} y$. Let take the case $x \preceq_{P_0'} y$ and assume $z \in P_1'$ then $x \not\preceq_{P_0' \otimes P_1'} z$ and $y \not\preceq_{P_0' \otimes P_1'} z$, and this imply neither $z \in P_0$ nor $z \in P_1$ (otherwise z will be connected to either one of x and y). Since $E_{P_1'} \subseteq E_{P_0 \rhd P_1}$ by Equation (4.25), we find

$$z \in P_1' \implies z \in E_{P_0 \rhd P_1}$$

which leads to contradiction since neither $z \in P_0$ nor $z \in P_1$. This proves either $P = P_0 \rhd P_1$ or $P = P_0' \otimes P_1'$.

4. if none of the case (mentioned above) hold, then P is prime iposet defined by $P \notin C_n$ for all $n \geq 0$, i.e., P is not defined by iposets hierarchy given in Definition 40.

$\square$

The Lemma 10 defines subsumption order for $\rhd$ and $\otimes$ composition of iposets under subsumption.

Lemma 10 (Iposets composition). *Let P_0, P_1, Q_0 and Q_1 be iposets such that $P_0 \leqslant Q_0$ and $P_1 \leqslant Q_1$. Then,*

$$P_0 \rhd P_1 \leqslant Q_0 \rhd Q_1 \text{ and } P_0 \otimes P_1 \leqslant Q_0 \otimes Q_1.$$

Proof.

Let P_0, P_1, Q_0 and Q_1 be iposets such that

$$P_0 = (E_{P_0}, \preceq_{P_0}, s_{P_0}, t_{P_0}) \text{ and } P_1 = (E_{P_1}, \preceq_{P_1}, s_{P_1}, t_{P_1})$$

$$Q_0 = (E_{Q_0}, \preceq_{Q_0}, s_{Q_0}, t_{Q_0}) \text{ and } Q_1 = (E_{Q_1}, \preceq_{Q_1}, s_{Q_1}, t_{Q_1})$$

Consider the case $P_0 \leqslant Q_0$ which witness the bijection $h : E_{Q_0} \to E_{P_0}$

$$E_{Q_0} = h^{-1}(P_0)$$

such that $x \preceq_{Q_0} y \implies h(x) \preceq_{P_0} h(y)$ for all $x, y \in Q_0$ along with the

$$h : s_{Q_0} \to s_{P_0} \text{ and } h : t_{Q_0} \to t_{P_0}$$

source and target interfaces bijections. The case $P_1 \leqslant Q_1$ follows accordingly.

To establish $P_0 \rhd P_1 \leqslant Q_0 \rhd Q_1$, we proceed with bijection $h : E_{Q_0 \rhd Q_1} \to E_{P_0 \rhd P_1}$

$$\begin{aligned}
E_{Q_0 \rhd Q_1} &= (E_{Q_0} \cup E_{Q_1})_{/t_{Q_0}} \qquad \blacktriangleright \text{ since } Q_0 \rhd Q_1 \text{ is defined by } t_{Q_0} = s_{Q_1} \\
&= (E_{Q_0} \cup E_{Q_1})_{/t_{Q_0}} \\
&= (h^{-1}(E_{P_0}) \cup h^{-1}(E_{P_1}))_{/h^{-1}(t_{P_0})} \\
&= h^{-1}((E_{P_0} \cup E_{P_1})_{/t_{P_0}}) \\
&= h^{-1}(E_{P_0 \rhd P_1})
\end{aligned}$$

along with the source and target interfaces bijections

$$s_{Q_0 \rhd Q_1} = s_{Q_0} = h^{-1}(s_{P_0}) = h^{-1}(s_{P_0 \rhd P_1})$$

$$t_{Q_0 \rhd Q_1} = t_{Q_1} = h^{-1}(t_{P_1}) = h^{-1}(t_{P_0 \rhd P_1})$$

such that $x \preceq_{Q_0 \rhd Q_1} y \implies h(x) \preceq_{P_0 \rhd P_1} h(y)$ for all $x, y \in Q_0 \rhd Q_1$. To establish order inclusion, we assume $x, y \in Q_0 \rhd Q_1$ such that $x \preceq_{Q_0 \rhd Q_1} y$ then

1. if $x, y \in Q_0$ then $x \preceq_{P_0} y$, and thus $x \preceq_{P_0 \rhd P_1} y$.
2. if $x, y \in Q_1$ then $x \preceq_{P_1} y$, and thus $x \preceq_{P_0 \rhd P_1} y$.
3. if $x \in Q_0$ and $y \in Q_1$ then we have three case
   (a) if $x \in Q_0$ and $y \in Q_1$ such that $y \in s_{Q_1}$ then case 1.
   (b) if $x \in Q_0$ and $y \in Q_1$ such that $x \in t_{Q_0}$ then case 2.
   (c) if $x \in Q_0$ and $y \in Q_1$ such that $x \notin t_{Q_0}$ and $y \notin s_{Q_1}$ then $x \in P_0$ and $y \in P_1$, and thus $x \preceq_{P_0 \rhd P_1} y$.
4. if $y \in Q_0$ and $x \in Q_1$ then $y \in P_0$ and $x \in P_1$, thus $x \preceq_{P_0 \rhd P_1} y$ implies $x = y$ by antisymmetry. This explains the fusion such that $y \in t_{P_0}$ and $x \in s_{P_1}$.

This proves $P_0 \rhd P_1 \leqslant Q_0 \rhd Q_1$. To establish $P_0 \otimes P_1 \leqslant Q_0 \otimes Q_1$, we proceed with bijection $h : E_{Q_0 \otimes Q_1} \to E_{P_0 \otimes P_1}$

$$\begin{aligned} E_{Q_0 \otimes Q_1} &= E_{Q_0} \cup E_{Q_1} \\ &= h^{-1}(E_{P_0}) \cup h^{-1}(E_{P_1}) \\ &= h^{-1}(E_{P_0} \cup E_{P_1}) \\ &= h^{-1}(E_{P_0 \otimes P_1}) \end{aligned}$$

along with the source and target interfaces bijections

$$s_{Q_0 \otimes Q_1} = s_{Q_0} \cup s_{Q_1} = h^{-1}(s_{P_0}) \cup h^{-1}(s_{P_1}) = h^{-1}(s_{P_0} \cup s_{P_1}) = h^{-1}(s_{P_0 \otimes P_1})$$

$$t_{Q_0 \otimes Q_1} = t_{Q_0} \cup t_{Q_1} = h^{-1}(t_{P_0}) \cup h^{-1}(t_{P_1}) = h^{-1}(t_{P_0} \cup t_{P_1}) = h^{-1}(t_{P_0 \otimes P_1})$$

such that $x \preceq_{Q_0 \otimes Q_1} y \implies h(x) \preceq_{P_0 \otimes P_1} h(y)$ for all $x, y \in Q_0 \otimes Q_1$. To show order inclusion, let $x, y \in Q_0 \otimes Q_1$ such that $x \preceq_{Q_0 \otimes Q_1} y$ then we have following case to consider

1. if $x, y \in Q_0$ then $x \preceq_{P_0} y$, and thus $x \preceq_{P_0 \otimes P_1} y$.
2. if $x, y \in Q_1$ then $x \preceq_{P_1} y$, and thus $x \preceq_{P_0 \otimes P_1} y$.

We ignore the case $x \in Q_0$ and $y \in Q_1$ that implies $x \npreceq_{Q_0 \otimes Q_1} y$.

These proves $P_0 \otimes P_1 \leqslant Q_0 \otimes Q_1$. □

Remark 18 (Extended subsumption)**.** *The subsumption order $P \leqslant Q$ on iposets*

$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } Q = E_Q, \preceq_Q, s_Q, t_Q$$

is defined iff there exists bijection $h : E_Q \to E_P$ such that

$$x \preceq_Q y \implies h(x) \preceq_P h(y) \textit{ for all } x, y \in Q$$

along with

$$x \in s_P \implies h^{-1}(x) \in s_Q \textit{ and } x \in t_P \implies h^{-1}(x) \in t_Q \textit{ , forall } x \in (s_P \cup t_P)$$

the source and target interface morphisms.

We have three different class of singletone iposets

$$\begin{aligned} S_1 =& \{[0] \to [1] \leftarrow [0]\} \\ s_2 =& \{[0] \to [1] \leftarrow [1], [1] \to [1] \leftarrow [0]\} \\ s_3 =& \{[1] \to [1] \leftarrow [1]\} \end{aligned}$$

and according to the extended subsumption definition, we get following subsumption order between them

$$s_1 \leqslant s_2 \leqslant s_3$$

The $\rhd$ composition between $s_1 := [0] \to [1] \leftarrow [0]$ singleton iposets is always serial composition $\rhd_{series}$, and the $\rhd$ composition between $s_3 := [1] \to [1] \leftarrow [1]$ single iposets is always gluing composition $\rhd_{gluing}$ iff defined. However, the $\rhd$ composition between $s_2 := [0] \to [1] \leftarrow [1], [1] \to [1] \leftarrow [0]$ depends on the target and source interfaces of sequential component. For instance,

$$[0] \to [1] \leftarrow [1] \rhd [1] \to [1] \leftarrow [0] \textit{ wil be } \rhd_{gluing} \textit{ composition iff defined}$$

$$[1] \to [1] \leftarrow [0] \rhd [0] \to [1] \leftarrow [1] \textit{ wil be } \rhd_{series} \textit{ composition.}$$

Similarly, the $\rhd$ composition between class of s_1 and s_2 is $s_1 \rhd_{series} s_2$ and $s_2 \rhd_{series} s_1$ iff defined. Likewise, s_2 and s_3 is $s_2 \rhd_{gluing} s_3$ and $s_3 \rhd_{gluing} s_2$ iff defined.

Based on the fact above, the factorization lemma for iposets can be redefined as follows

1. *iff $s_1 \leqslant s_2 \wedge s'_1 \leqslant s'_2$ then $s_1 \rhd_{series} s'_1 \leqslant s_2 \rhd_{series} s'_2$ and $s_1 \otimes s'_1 \leqslant s_2 \otimes s'_2$*
2. *iff $s_1 \leqslant s_2 \wedge s'_2 \leqslant s''_2$ then $s_1 \rhd_{series} s'_2 \leqslant s_2 \rhd_{series} s''_2$ and $s_1 \otimes s'_2 \leqslant s_2 \otimes s''_2$*
3. *iff $s_2 \leqslant s'_2 \wedge s_1 \leqslant s''_2$ then $s_2 \rhd_{series} s_1 \leqslant s'_2 \rhd_{series} s''_2$ and $s_2 \otimes s_1 \leqslant s'_2 \otimes s''_2$*
4. *iff $s_2 \leqslant s_3 \wedge s'_2 \leqslant s'_3$ then $s_2 \rhd_{gluing} s'_2 \leqslant s_3 \rhd_{gluing} s'_3$ and $s_2 \otimes s'_2 \leqslant s_3 \otimes s'_3$*
5. *iff $s_2 \leqslant s_3 \wedge s'_3 \leqslant s''_3$ then $s_2 \rhd_{gluing} s'_3 \leqslant s_3 \rhd_{gluing} s''_3$ and $s_2 \otimes s'_3 \leqslant s_3 \otimes s''_3$*
6. *iff $s_3 \leqslant s'_3 \wedge s_2 \leqslant s''_3$ then $s_3 \rhd_{gluing} s_2 \leqslant s'_3 \rhd_{gluing} s''_3$ and $s_3 \otimes s_2 \leqslant s'_3 \otimes s''_3$*

The factorization lemma for the subsumption order between the s_1 classes of iposets (class of SP posets)

$$\textit{iff } s_1 \leqslant s'_2 \wedge s''_1 \leqslant s'''_1 \textit{ then } s_1 \rhd_{series} s''_1 \leqslant s'_1 \rhd_{series} s'''_1 \textit{ and } s_1 \otimes s''_1 \leqslant s'_1 \otimes s'''_1$$

Between s_3 classes of iposets,

8. *iff* $s_3 \leqslant s'_3 \wedge s''_3 \leqslant s'''_3$ *then* $s_3 \rhd_{gluing} s''_3 \leqslant s'_3 \rhd_{gluing} s'''_3$ *and* $s_3 \otimes s''_3 \leqslant s'_3 \otimes s'''_3$

And between s_2 classes of iposets,

iff $s_2 \leqslant s'_2 \wedge s''_2 \leqslant s'''_2$ *then* $s_2 \rhd_{series} s''_2 \leqslant s'_2 \rhd_{series} s'''_2$ *and* $s_2 \otimes s''_2 \leqslant s'_2 \otimes s'''_2$

iff $s_2 \leqslant s'_2 \wedge s''_2 \leqslant s'''_2$ *then* $s_2 \rhd_{gluing} s''_2 \leqslant s'_2 \rhd_{gluing} s'''_2$ *and* $s_2 \otimes s''_2 \leqslant s'_2 \otimes s'''_2$.

We now prove Lemma 11 for the proof of successive Lemma 12 and the Lemma 17.

Lemma 11. *Let P, Q, U, V be iposets such that $P \rhd Q \leqslant U \rhd V$. There exists an iposets R such that either $P \leqslant U \rhd R$ and $R \rhd Q \leqslant V$ or $P \rhd R \leqslant U$ and $Q \leqslant R \rhd V$.*

Proof.

Let P, Q, U and V be iposets

$$P = (E_P, \preceq_P, s_P, t_P)\ , Q = (E_Q, \preceq_Q, s_Q, t_Q),$$

$$U = (E_U, \preceq_U, s_U, t_U) \text{ and } V = (E_V, \preceq_V, s_V, t_V)$$

Consider the case $P \rhd Q \leqslant U \rhd V$ which witness the bijection $h : E_{U \rhd V} \to E_{P \rhd Q}$

$$E_{U \rhd V} = h^{-1}(E_{P \rhd Q})$$

such that

$$x \preceq_{U \rhd V} y \implies h(x) \preceq_{P \rhd Q} h(y) \text{ for all } x, y \in U \rhd V$$

along with the source and target interface bijections

$$h : s_{U \rhd V} \to s_{P \rhd Q} \text{ and } h : t_{U \rhd V} \to t_{P \rhd Q}.$$

Now, by lemma 7 we can find $U' \leqslant U$ and $V' \leqslant V$ such that $P \rhd Q = U' \rhd V'$ which witness the bijection

$h : E_{P \rhd Q} \to E_{U' \rhd V'}$ such that $x \preceq_{P \rhd Q} y$ iff $h(x) \preceq_{U' \rhd V'} h(y)$ for all $x, y \in P \rhd Q$

along with $h : s_{P \rhd Q} \to s_{U' \rhd V'}$ and $h : t_{P \rhd Q} \to t_{U' \rhd V'}$.

Consider a contradiction such that $P \nsubseteq U'$ and $U' \nsubseteq P$, then there exists events $x \in P \setminus U'$ and $y \in U' \setminus P$. Since $x \notin U'$, it follows that $x \in V'$ and by the same reasoning $y \in Q$. Then, we get $x \preceq_{P \rhd Q} y$ and $y \preceq_{U' \rhd V'} x$. Since $P \rhd Q$ and $U' \rhd V'$ is isomorphic by definition, we find $x = y$ by antisymmetry. This is a contradiction, since $x \in P$ and $y \notin P$. Thus, either $P \subseteq U'$ or $U' \subseteq P$.

Let take case $P \subseteq U'$, while $U' \subseteq P$ will follow the similar procedure. Let assume an iposet $R = U' \setminus (P \setminus id_{t_P})$, then $R = (E_R, \preceq_R, s_R, t_R)$ is defined by

$$R = (h^{-1}(E_{U'}) \setminus (E_P \setminus t_P), \preceq_{U' \setminus (P \setminus id_{t_P})}, h^{-1}(E_{U'}) \cap t_P, h^{-1}(t_{U'})).$$

Now we claim that $U' = P \triangleright R$. To show $U' = P \triangleright R$, we procced with sets of events

$$\begin{aligned} E_{P \triangleright R} &= (E_P \setminus t_P) \cup E_R \quad \blacktriangleright\text{since } t_P = s_R \text{ by the definition of } R \\ &= (E_P \setminus t_P) \cup (h^{-1}(E_{U'}) \setminus (E_P \setminus t_P)) \\ &= h^{-1}(E_{U'}) \end{aligned}$$

along with source and target

$$s_{P \triangleright R} = s_P = s_{P \triangleright Q} = h^{-1}(s_{U' \triangleright V'}) = h^{-1}(s_{U'})$$

$$t_{P \triangleright R} = t_R = h^{-1}(t_{U'})$$

interface bijections such that $x \preceq_{U'} y$ iff $h(x) \preceq_{P \triangleright R} h(y)$ for all $x, y \in U'$. The order isomorphism follows by

1. Suppose, $x, y \in U'$ such that $x \preceq_{U'} y$, then
   (a) if $x, y \in P$ such that $x \preceq_P y$, then $x \preceq_{P \triangleright R} y$.
   (b) if $x, y \in R$ such that $x \preceq_R y$, then $x \preceq_{P \triangleright R} y$.
   (c) if $x \in P$ and $y \in R$, then there are three case to consider,
      i. if $x \in P$ and $y \in s_R$, then case (a).
      ii. if $x \in t_P$ and $y \in R$, then case (b).
      iii. if $x \in P$ and $y \in R$ such that $y \notin s_R$ and $x \notin t_P$, then $x \preceq_{P \triangleright R} y$.
   (d) if $y \in P$ and $x \in R$, then $x \preceq_{P \triangleright R}$ y implies $y = x$ by antisymmetry. This implies $x \in U'$ such that x denotes fusion of $y \in t_P$ and $x \in s_Q$.
2. Similarly, suppose $x, y \in U'$ such that $x \preceq_{P \triangleright R} y$, then
   (a) if $x, y \in P$ such that $x \preceq_P y$ then $x \preceq_{P \triangleright Q} y$, and $x \preceq_{U' \triangleright V'} y$. Since $x, y \in U'$, thus $x \preceq_{U'} y$.
   (b) if $x, y \in R$ such that $x \preceq_R y$. Since $E_R \subseteq h^{-1}(E_{U'}) \setminus (E_P \setminus t_P)$, thus $x \preceq_{U'} y$.
   (c) if $x \in P$ and $y \in R$, then there are three case to consider
      i. if $x \in P$ and $y \in s_R$, then case (a).
      ii. if $x \in t_P$ and $y \in R$, then case (b).

iii. if $x \in P$ and $y \in R$ such that $y \notin s_R$ and $x \notin t_P$, then $y \in Q$ therefore $x \preceq_{P \rhd Q} y$. Since $P \rhd Q = U' \rhd V'$, we have $x \preceq_{U' \rhd V'} y$. Then $x, y \in U'$, thus $x \preceq_{U'} y$.

(d) if $y \in P$ and $x \in R$ then $x \preceq_{P \rhd R}$ y implies $y = x$ (by antisymmetry), and thus $x \in U'$ such that x denotes fusion of $y \in t_P$ and $x \in s_Q$.

We choose R to find that $U' = P \rhd R$. Since $U' \leqslant U$, thus $P \rhd R \leqslant U$. Similarly, we arrive at $P \leqslant U \rhd R$ by following the optional case $U' \subseteq P$.

We now claim $R \rhd V' \leqslant Q$ by following similar procedure as above. Consider a contradiction $Q \nsubseteq V'$ and $V' \nsubseteq Q$, then we eventually arrive at either $V' \subseteq Q$ or $Q \subseteq V'$. Lets take the case $V' \subseteq Q$ and assume iposets $R = Q \setminus (V' \setminus id_{s_{V'}})$ such that

$$R = (h(E_Q) \setminus (E_{V'} \setminus s_{V'}), \preceq_{Q \setminus (V' \setminus id_{s_{V'}})}, h(s_Q), h(E_Q) \cap s_{V'}).$$

To show $Q = R \rhd V'$, we now proceed with sets of events

$$\begin{aligned} E_{R \rhd V'} &= E_R \cup (E_{V'} \setminus s_{V'}) \quad \blacktriangleright \text{ since } t_R = s_{V'} \text{ by definition of } R \\ &= (h(E_Q) \setminus (E_{V'} \setminus s_{V'})) \cup (E_{V'} \setminus s_{V'}) \\ &= h(E_Q) \end{aligned}$$

along with the source

$$s_{R \rhd V'} = s_R = h(s_Q)$$

and target interface

$$t_{R \rhd V'} = t_{V'} = t_{U' \rhd V'} = h(t_{P \rhd Q}) = h(t_Q)$$

bijections such that $x \preceq_Q y$ iff $h(x) \preceq_{R \rhd V'} h(y)$ for all $x, y \in Q$, denotes order isomorphism. The order isomorphism follows by

1. Suppose $x, y \in Q$ such that $x \preceq_Q y$. There are three case to consider

   (a) if $x, y \in R$ then $x \preceq_R y$. Thus, $x \preceq_{R \rhd V'} y$.

   (b) if $x, y \in V'$ then $x \preceq_{V'} y$. Thus, $x \preceq_{R \rhd V'} y$.

   (c) if $x \in R$ and $y \in V'$, then there are three case to consider

      i. if $x \in R$ and $y \in t_R$, then case (a).

      ii. if $x \in s_{V'}$ and $y \in V'$, then case (b).

      iii. if $x \in R$ and $y \in V'$ such that $y \notin t_R$ and $x \notin s_{V'}$, then $x \preceq_{R \rhd V'} y$.

   (d) if $y \in R$ and $x \in V'$, then $y \preceq_{R \rhd V'} x$ implies $y = x$ by antisymmetry. This implies $x \in Q$ such that x denotes fusion of $y \in t_R$ and $x \in s_{V'}$.

2. Similarly, suppose $x, y \in Q$ such that $x \preceq_{R \rhd V'} y$. There are three cases to consider

(a) if $x, y \in R$, then $x \preceq_{U'} y$ and $x \preceq_{U' \rhd V'} y$. Thereafter $x \preceq_{P \rhd Q} y$, and thus it follows $x \preceq_Q y$.

(b) if $x, y \in V'$ then $x \preceq_{U' \rhd V'} y$. Thereafter $x \preceq_{P \rhd Q} y$, and thus it follows $x \preceq_Q y$.

(c) if $x \in R$ and $y \in V'$, then there are three case to consider

i. if $x \in R$ and $y \in s_{V'}$, then case (a).

ii. if $x \in t_R$ and $y \in V'$, then case (b).

iii. if $x \in R$ and $y \in V'$ such that $x \notin t_R$ and $y \notin s_{V'}$, then $x \in U'$, thus $x \preceq_{U' \rhd V'} y$. Thereafter $x \preceq_{P \rhd Q} y$, and thus it follows $x \preceq_Q y$.

(d) if $y \in R$ and $x \in V'$, then $y \preceq_{R \rhd V'} x$ implies $y = x$ (by antisymmetry) and thus $x \in Q$ such that x denotes fusion of $y \in t_R$ and $x \in s_{V'}$.

We choose R to find that $Q = R \rhd V'$. Since $V' \leqslant V$, thus $Q \leqslant R \rhd V$. Similarly, we arrive at $R \rhd Q \leqslant V$ by following the optional case $Q \subseteq V'$. □

We state Lemma 12 for the proof of successive Lemma 13 for unique $\rhd$ decomposition of iposets.

Lemma 12. *Let P, Q, U, V be iposets such that $P \rhd Q = U \rhd V$. There exists an iposets R such that either $P = U \rhd R$ and $R \rhd Q = V$ or $P \rhd R = U$ and $Q = R \rhd V$.*

Proof. Trivial by lemma 11 with h^{-1} inverse morphisms in order relation. □

The Lemma 13 defines uniqueness $\rhd$ decomposition of iposets.

Lemma 13 (Unique $\rhd$ Factorization). *Let P be an iposets such that $U_1 \rhd U_2 \dots U_n$ and $V_1 \rhd V_2 \dots V_m$ denotes the Sequential factorization of P for some $n, m \in \mathbb{N}$, then*

$$U_1 \rhd U_2 \dots U_n = V_1 \rhd V_2 \dots V_m$$

Proof.

We follow similar method [18, lemma 3.8] in our case as well.
Given iposets $U_1, U_2 \dots, U_n$ and $V_1, V_2 \dots, V_m$ such that

$$U_1 \rhd U_2 \dots U_n = V_1 \rhd V_2 \dots V_m.$$

We proceed to prove $U_n = V_m$ for $1 \leq (n = m) \leq \mathbb{N}$ by induction based on lemma 12 such as

1. Trivial by Lemma 9 if $n = m = 0$.
2. Trivial by Lemma 9 if $n = m = 1$.

3. Let assume claim holds for n', m' such that $n' < n = m' < m$ and $(n = m) > 1$. Then by Lemma 12, consider an iposet R such that either

$$U_1 \rhd R = V_1 \text{ and } U_2 \rhd \dots U_n = R \rhd V_2 \dots V_m$$

or

$$V_1 \rhd R = U_1 \text{ and } V_2 \rhd \dots V_m = R \rhd U_2 \dots U_n.$$

If we assume $R = \mathrm{id}_n$, then either part becomes $U_1 = V_1$ and $U_2 \rhd \dots U_n = V_2 \rhd \dots V_m$, where $U_2 \rhd \dots U_n = V_2 \rhd \dots V_m$ again holds by following induction. Similar argument can be made for later case as well.

□

We prove Lemma 14 for the proof of successive Lemma 15.

Lemma 14. *Let P, Q, U, V be iposets such that $P \otimes Q \leqslant U \otimes V$ there exists an iposets R such that either $P \leqslant U \otimes R$ and $R \otimes Q \leqslant V$ or $P \otimes R \leqslant U$ and $Q \leqslant R \otimes V$.*

Proof.

Let P, Q, U and V be iposets

$$P = (E_P, \preceq_P, s_P, t_P)\ , Q = (E_Q, \preceq_Q, s_Q, t_Q),$$

$$U = (E_U, \preceq_U, s_U, t_U) \text{ and } V = (E_V, \preceq_V, s_V, t_V)$$

Consider the case $P \otimes Q \leqslant U \otimes V$ which witness the bijection $h : E_{U\otimes V} \to E_{P\otimes Q}$

$$E_{U\otimes V} = h^{-1}(E_{P\otimes Q})$$

such that

$$x \preceq_{U\otimes V} y \implies h(x) \preceq_{P\otimes Q} h(y) \text{ for all } x, y \in U \otimes V$$

along with the source and target interfaces bijections

$$h : s_{U\otimes V} \to s_{P\otimes Q} \text{ and } h : t_{U\otimes V} \to t_{P\otimes Q}.$$

Now, by Lemma 8 we can find $U' \leqslant U$ and $V' \leqslant V$ such that $P \otimes Q = U' \otimes V'$ which witness the bijection

$$h : E_{P\otimes Q} \to E_{U'\otimes V'} \text{ such that } x \preceq_{P\otimes Q} y \text{ iff } h(x) \preceq_{U'\otimes V'} h(y) \text{ for all } x, y \in P \otimes Q$$

$$\text{along with } h : s_{P\otimes Q} \to s_{U'\otimes V'} \text{ and } h : t_{P\otimes Q} \to t_{U'\otimes V'}$$

Assume a contradiction such that $P \nsubseteq U'$ and $U' \nsubseteq P$, then there exists events $x \in P \setminus U'$ and $y \in U' \setminus P$. Since $x \notin U'$, it follows $x \in V'$, and by the same reasoning $y \in Q$. But, then $x \otimes y \in P \otimes Q$ and $y \otimes x \in U' \otimes V'$. Since $P \otimes Q$ and $U' \otimes V'$ is isomorphic by definition , we find $x \otimes y = y \otimes x$. This is a contradiction, since $x \otimes y \neq y \otimes x$ once the iposets are labelled. Thus, either $P \subseteq U'$ or $U' \subseteq P$ is true.

Let take case $P \subseteq U'$, while $U' \subseteq P$ will follow the similar procedure. Lets assume iposets $R = U' \setminus P$ such that

$$R = (h^{-1}(E_{U'}) \setminus E_P, \preceq_{U' \setminus P}, h^{-1}(s_{U'}) \setminus s_P, h^{-1}(t_{U'}) \setminus t_P).$$

We now claim $U' = P \otimes R$. To show $U' = P \otimes R$, we proceed with the sets of events

$$\begin{aligned} E_{P \otimes R} =& E_P \cup E_R \\ =& \, E_P \cup (h^{-1}(E_{U'}) \setminus E_P) \\ =& \, h^{-1}(E_{U'}) \end{aligned}$$

along with source

$$s_{P \otimes R} = s_P \cup s_R = s_P \cup (h^{-1}(s_{U'}) \setminus s_P) = h^{-1}(s_{U'})$$

and target

$$t_{P \otimes R} = t_P \cup t_R = t_P \cup (h^{-1}(t_{U'}) \setminus t_P) = h^{-1}(t_{U'})$$

interface bijections such that $x \preceq_{U'} y$ iff $h(x) \preceq_{P \otimes R} h(y)$ for all $x, y \in U'$ denotes order isomorphism. The order isomorphism follows by

1. Suppose, $x, y \in U'$ such that $x \preceq_{U'} y$, then
   (a) if $x, y \in P$ such that $x \preceq_P y$, then $x \preceq_{P \otimes R} y$.
   (b) if $x, y \in R$ such that $x \preceq_R y$, then $x \preceq_{P \otimes R} y$.

   We list an additional case if $x, y \in U'$ such that $x \not\preceq_{U'} y$ then, $x \not\preceq_{P \otimes R} y$ by $x \in P$ and $y \in R$.

2. Similarly, suppose $x, y \in U'$ such that $x \preceq_{P \otimes R} y$, then
   (a) if $x, y \in P$ such that $x \preceq_P y$ then $x \preceq_{P \otimes Q} y$, and $x \preceq_{U' \otimes V'} y$. Since $x, y \in U'$, thus $x \preceq_{U'} y$.
   (b) if $x, y \in R$ such that $x \preceq_R y$. Since $E_R \subseteq (h^{-1}(E_{U'}) \setminus E_P)$, thus $x \preceq_{U'} y$.

   We list an additional case $x \in P$ and $y \in R$ such that $x \not\preceq_{P \otimes R} y$ (since $R = (U' \setminus P)$, the disjointness of P and R) then, by 2(a) and 2(b) $x \not\preceq_{U'} y$.

We choose R to find that $U' = P \otimes R$. Since $U' \leqslant U$, thus $P \otimes R \leqslant U$. Similarly, we arrive at $P \leqslant U \otimes R$ by following the optional case $U' \subseteq P$.

We now claim $R \otimes V' \leqslant Q$ by following similar procedure as above. Consider a contradiction $Q \nsubseteq V'$ and $V' \nsubseteq Q$, then we eventually arrive at either

$V' \subseteq Q$ or $Q \subseteq V'$. Lets take the case $V' \subseteq Q$ and assume iposets $R = Q \setminus V'$ such that

$$R = (h(E_Q) \setminus E_{V'}, \preceq_{Q \setminus V'}, h(s_Q) \setminus s_{V'}, h(t_Q) \setminus t_{V'}).$$

To show $Q = R \otimes V'$, we now proceed with sets of events

$$\begin{aligned} E_{R \otimes V'} &= E_R \cup E_{V'}) \\ &= (h(E_Q) \setminus E_{V'}) \cup E_{V'} \\ &= h(E_Q) \end{aligned}$$

along with the source

$$s_{R \otimes V'} = s_R \cup s_{V'} = (h(s_Q) \setminus s_{V'}) \cup s_{V'} = h(s_Q)$$

and target interface

$$t_{R \rhd V'} = t_R \cup t_{V'} = (h(t_Q) \setminus t_{V'}) \cup t_{V'} = h(t_Q)$$

bijections such that $x \preceq_Q y$ iff $h(x) \preceq_{R \otimes V'} h(y)$ for all $x, y \in Q$ denotes order isomorphism. The order isomorphism follows by

1. Suppose $x, y \in Q$ such that $x \preceq_Q y$, then
   (a) if $x, y \in R$ then $x \preceq_R y$. Thus, $x \preceq_{R \otimes V'} y$.
   (b) if $x, y \in V'$ then $x \preceq_{V'} y$. Thus, $x \preceq_{R \otimes V'} y$.

   We list an additional case if $x, y \in Q$ such that $x \not\preceq_Q y$ then, $x \not\preceq_{R \otimes V'} y$ by $x \in R$ and $y \in V'$.

2. Similarly, suppose $x, y \in Q$ such that $x \preceq_{R \otimes V'} y$, then
   (a) if $x, y \in R$, then $x \preceq_{U'} y$ and $x \preceq_{U' \otimes V'} y$. Thereafter $x \preceq_{P \otimes Q} y$, and thus it follows $x \preceq_Q y$.
   (b) if $x, y \in V'$ then $x \preceq_{U' \otimes V'} y$. Thereafter $x \preceq_{P \otimes Q} y$, and thus it follows $x \preceq_Q y$.

   We list an additional case $x \in R$ and $y \in V'$ such that $x \not\preceq_{R \otimes V'} y$ (since $R = (Q \setminus V')$, the disjointness of R and V') then, by 2(a) and 2(b) $x \not\preceq_Q y$.

We choose R to find that $Q = R \otimes V'$. Since $V' \leqslant V$, thus $Q \leqslant R \otimes V$. Similarly, we arrive at $R \otimes Q \leqslant V$ by following the optional case $Q \subseteq V'$. □

We state Lemma 15 for the proof of successive Lemma 16.

Lemma 15. *Let P, Q, U, V be iposets such that $P \otimes Q = U \otimes V$ there exists an iposets R such that either $P = U \otimes R$ and $R \otimes Q = V$ or $P \otimes R = U$ and $Q = R \otimes V$.*

Proof. Trivial by lemma 14 with h^{-1} inverse morphisms in order relation. □

The Lemma 16 defines uniqueness of $\otimes$ decomposition of iposets.

Lemma 16 (Unique $\otimes$ Factorization). *Let P be an iposet such that $U_1 \otimes U_2 \dots U_n$ and $V_1 \otimes V_2 \dots V_m$ denotes the parallel factorization of P for some $n, m \in \mathbb{N}$, then*

$$U_1 \otimes U_2 \dots U_n = V_1 \otimes V_2 \dots V_m$$

Proof.

Given iposets $U_1, U_2 \dots, U_n$ and $V_1, V_2 \dots, V_m$ such that

$$U_1 \otimes U_2 \dots U_n = V_1 \otimes V_2 \dots V_m.$$

We proceed to prove $U_n = V_m$ for $1 \leq (n = m) \leq \mathbb{N}$ by induction based on Lemma 15 such as

1. Trivial by lemma 9 if $n = m = 0$.
2. Trivial by lemma 9 if $n = m = 1$.
3. Let assume claim holds for n', m' such that $n' < n = m' < m$ and $(n = m) > 1$. Then by lemma 15, consider an iposet R such that either

   $$U_1 \otimes R = V_1 \text{ and } U_2 \otimes \dots U_n = R \otimes V_2 \dots V_m$$

   or

   $$V_1 \otimes R = U_1 \text{ and } V_2 \otimes \dots V_m = R \otimes U_2 \dots U_n$$

   If we assume $R = \mathrm{id}_0$ (an empty iposet), then either part becomes $U_1 = V_1$ and $U_2 \otimes \dots U_n = V_2 \otimes \dots V_m$, where $U_2 \otimes \dots U_n = V_2 \otimes \dots V_m$ again holds by following induction. Similarly, claim for later case can be established.

□

We now prove Levi's Lemma 17 for iposets.

Lemma 17 (Levi). *Let P and Q be iposets, and let $W_0, W_1, ..., W_{n-1}$ with $n > 0$ be non-empty iposets such that $P \rhd Q \leqslant W_0 \rhd W_1 \rhd ... \rhd W_{n-1}$. Then, there exists an $m < n$ and iposets U, V such that*

$$U \rhd V = W_m, P \leqslant W_0 \rhd W_1 \rhd ... \rhd W_{m-1} \rhd U \text{ and } V \leqslant V \rhd W_{m+1} \rhd W_{m+2} \rhd ... \rhd W_{n-1}.$$

Proof.

We follow proof similar to the literature [17, lemma 3.4].
We proceed with induction on n,

1. The base case when $n = 1$, then $m = 0$. We get,

   $$P \rhd Q \leqslant W_0 \quad \text{and} \quad U \rhd V \leqslant W_0$$

we choose $P = U$ and $Q = V$ to satisfy the claim.

2. We assume (1) holds for induction step $n-1$. Then, we can write

$$P \triangleright Q \leqslant W_0 \triangleright W_1 \triangleright ... \triangleright W_{n-1}$$

rewritting

$$P \triangleright Q \leqslant W_0 \triangleright (W_1 \triangleright ... \triangleright W_{n-1})$$

Then, by Lemma 11, we have following two cases

(a) Consider an iposet R such that

$$P \leqslant W_0 \triangleright R \text{ and } R \triangleright Q \leqslant W_1 \triangleright ... \triangleright W_{n-1}.$$

By induction, $1 \leq m < n$ and iposets U, V such that $U \triangleright V \leqslant W_m$ and $R \leqslant W_1 \triangleright ... \triangleright W_{m-1} \triangleright U$ and $Q \leqslant V \triangleright W_{m+1} \triangleright ... \triangleright W_n$. Since $P \leqslant W_0 \triangleright R$, it follows $P \leqslant W_0 \triangleright W_1 \triangleright ... \triangleright W_{m-1} \triangleright U$.

(b) Consider an iposet R such that $P \triangleright R \leqslant W_0$ and $Q \leqslant R \triangleright (W_1 \triangleright ... \triangleright W_{n-1})$.Again we choose $m = 0$, then $P = U$ and $Q = V$ follows the claims.

□

We now prove soft version of interpolation Lemma 18 for parallel factors of iposets.

Lemma 18. *Let P, Q, U, V be iposets such that $P \otimes Q \leqslant U \otimes V$. Then there exist iposets U_0, U_1, V_0, V_1 such that*

$$U_0 \otimes U_1 \leqslant U, V_0 \otimes V_1 \leqslant V, P \leqslant U_0 \otimes V_0, \textit{ and } Q \leqslant U_1 \otimes V_1.$$

Proof.

Let P, Q, U, V be iposets

$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } Q = (E_Q, \preceq_Q, s_Q, t_Q)$$

$$U = (E_U, \preceq_U, s_U, t_U) \text{ and } V = (E_V, \preceq_V, s_V, t_V)$$

Consider the case $P \otimes Q \leqslant U \otimes V$ which establish the bijection

$$h : E_{U \otimes V} \to E_{P \otimes Q} \tag{4.26}$$

such that

$$x \preceq_{U \otimes V} y \implies h(x) \preceq_{P \otimes Q} h(y) \text{ for all } x, y \in (U \otimes V)$$

along with the source and target interfaces bijections

$$h : s_{U \otimes V} \to s_{P \otimes Q} = s_P \cup s_Q \text{ and } h : t_{U \otimes V} \to t_{P \otimes Q} = t_P \cup t_Q.$$

Now, we choose the pairwise disjoint iposets

$$U_0 = (E_{U_0}, \preceq_{U_0}, s_{U_0}, t_{U_0}) \text{ and } U_1 = (E_{U_1}, \preceq_{U_1}, s_{U_1}, t_{U_1})$$

$$V_0 = (E_{V_0}, \preceq_{V_0}, s_{V_0}, t_{V_0}) \text{ and } V_1 = (E_{V_1}, \preceq_{V_1}, s_{V_1}, t_{V_1})$$

such that

$E_{U_0} = E_U \cap h^{-1}(E_P)$ denotes the restriction of E_P to the E_U such that

$\preceq_U \cap h^{-1}(\preceq_P)$ along with $s_{U_0} = s_U \cap h^{-1}(s_P)$ and $t_{U_0} = E_U \cap h^{-1}(t_P)$.

In short

$$\blacktriangleright\ U_0 = (E_U \cap h^{-1}(E_P), \preceq_U \cap h^{-1}(\preceq_P), s_U \cap h^{-1}(s_P), E_U \cap h^{-1}(t_P)).$$

Similarly,

$$U_1 = (E_U \cap h^{-1}(E_Q), \preceq_U \cap h^{-1}(\preceq_Q), E_U \cap h^{-1}(s_Q), t_U \cap h^{-1}(t_Q))$$

$$V_0 = (E_V \cap h^{-1}(E_P), \preceq_V \cap h^{-1}(\preceq_P), s_V \cap h^{-1}(s_P), E_V \cap h^{-1}(t_P))$$
$$V_1 = (E_V \cap h^{-1}(E_Q), \preceq_V \cap h^{-1}(\preceq_Q), E_V \cap h^{-1}(s_Q), t_V \cap h^{-1}(t_Q)).$$

To establish $P \leqslant U_0 \otimes V_0$, we proceed with the bijection $h : E_{U_0 \otimes V_0} \to E_P$

$$\begin{aligned}
E_{U_0 \otimes V_0} &= E_{U_0} \cup E_{V_0} \\
&= (E_U \cap h^{-1}(E_P)) \cup (E_V \cap h^{-1}(E_P)) \\
&\blacktriangleright \text{ After simplifying,} \\
&= (E_U \cup E_V) \cap h^{-1}(E_P) \\
&= E_{U \otimes V} \cap h^{-1}(E_P) \\
&= h^{-1}(E_{P \otimes Q} \cap E_P) \\
&= h^{-1}((E_P \cup E_Q) \cap E_P) \\
&= h^{-1}(E_P)
\end{aligned}$$

along with the source $h : s_{U_0 \otimes V_0} \to s_P$ and target $h : t_{U_0 \otimes V_0} \to t_P$

$$\begin{aligned}
s_{U_0 \otimes V_0} &= s_{U_0} \cup s_{V_0} \\
&= (s_U \cap h^{-1}(s_p)) \cup (s_V \cap h^{-1}(s_P)) \\
&\blacktriangleright \text{ After simplifying,} \\
&= (s_U \cup s_V) \cap h^{-1}(s_P) \\
&= s_{U \otimes V} \cap h^{-1}(s_P) \\
&= h^{-1}(s_{P \otimes Q}) \cap h^{-1}(s_P) \\
&= h^{-1}((s_P \cup s_Q) \cap s_P) \\
&= h^{-1}(s_P)
\end{aligned}$$

$$
\begin{aligned}
t_{U_0 \otimes V_0} =& t_{U_0} \cup t_{V_0} \\
&= (E_U \cap h^{-1}(t_p)) \cup (E_V \cap h^{-1}(t_P)) \\
&\blacktriangleright \text{After simplifying,} \\
&= (E_U \cup E_V) \cap h^{-1}(t_P) \\
&= E_{U \otimes V} \cap h^{-1}(t_P) \\
&= h^{-1}(E_{P \otimes Q}) \cap h^{-1}(t_P) \\
&= h^{-1}((E_P \cup E_Q) \cap t_P) \\
&\blacktriangleright \text{Since, } t_P \subseteq E_P \\
&= h^{-1}(t_P)
\end{aligned}
$$

interface bijections and order subsumptions such that $x \preceq_{U_0 \otimes V_0} y \implies h(x) \preceq_P h(y)$. Suppose $x, y \in P$ such that $x \preceq_{U_0 \otimes V_0} y$, which is either $x \preceq_{U_0} y$ or $x \preceq_{V_0} y$. if $x \preceq_{U_0} y$ then $x \preceq_U y$ by choice of U_0. But then $x \preceq_{U \otimes V} y$, thus $x \preceq_{P \otimes Q} y$ by equation 4.26. Since $x, y \in P$, we conclude $x \preceq_P y$. Similarly, we can show $x \preceq_P y$ when $x \preceq_{V_0} y$. This proves $P \leqslant U_0 \otimes V_0$. Similarly, we can establish $Q \leqslant U_1 \otimes V_1$ as well.

To establish $U_0 \otimes U_1 \leqslant U$, we now proceed with the bijection $h : E_U \to E_{U_0 \otimes U_1}$

$$
\begin{aligned}
E_{U_0 \otimes U_1} &= E_{U_0} \cup E_{U_1} \\
&= (E_U \cap h^{-1}(E_P)) \cup (E_U \cap h^{-1}(E_Q)) \\
&= (E_U \cup E_U) \cap (E_U \cup h^{-1}(E_Q)) \cap (h^{-1}(E_P) \cup E_U) \cap (h^{-1}(E_P) \cup h^{-1}(E_Q)) \\
&\quad \blacktriangleright \text{By simplifying first 3 term, we get} \\
&= E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q)) \\
&= E_U \cap h^{-1}(E_P \cup E_Q) \\
&= E_U \cap h^{-1}(E_{P \otimes Q}) \\
&= E_U \cap (E_{U \otimes V}) \\
&= E_U \cap (E_U \cup E_V) \\
&= E_U
\end{aligned}
$$

along with the source and target $h : s_U \to s_{U_0 \otimes U_1}$ and $h : t_U \to t_{U_0 \otimes U_1}$ interface bijections

$$
\begin{aligned}
s_{U_0 \rhd U_1} &= s_{U_0} \cup s_{U_1} \\
&= (s_U \cap h^{-1}(s_P)) \cup (E_U \cap h^{-1}(s_Q)) \\
&= (s_U \cup E_U) \cap (s_U \cup h^{-1}(s_Q)) \cap (h^{-1}(s_P) \cup E_U) \cap (h^{-1}(s_P) \cup h^{-1}(s_Q)) \\
&\blacktriangleright \text{Since } s_U \subseteq E_U \text{, then simplifying} \\
&= s_U \cap (h^{-1}(s_P) \cup h^{-1}(s_Q)) \\
&= s_U \cap h^{-1}(s_{P \otimes Q}) \\
&= s_U \cap s_{U \otimes V}
\end{aligned}
$$

$$\begin{aligned}
&= s_U \cap (s_U \cup s_V) \\
&= s_U
\end{aligned}$$

$$\begin{aligned}
t_{U_0 \otimes U_1} &= t_{U_0} \cup t_{U_0} \\
&= (E_U \cap h^{-1}(t_P)) \cup (t_U \cap h^{-1}(t_Q)) \\
&= (E_U \cup t_U) \cap (E_U \cup h^{-1}(t_Q)) \cap (h^{-1}(t_P) \cup t_U) \cap (h^{-1}(t_P) \cup h^{-1}(t_Q)) \\
&\blacktriangleright \text{Since } t_U \subseteq E_U \text{, then simplifying} \\
&= t_U \cap (h^{-1}(t_P) \cup h^{-1}(t_Q)) \\
&= t_U \cap h^{-1}(t_{P \otimes Q}) \\
&= t_U \cap t_{U \otimes V} \\
&= t_U \cap (t_U \cup t_V) \\
&= t_U
\end{aligned}$$

such that $x \preceq_U y \implies h(x) \preceq_{U_0 \otimes U_1} h(y)$. Suppose $x, y \in U$ such that $x \preceq_U y$. Then, we know that $x \preceq_U y$ mean $x \preceq_{U \otimes V} y$, and thus $x \preceq_{P \otimes Q} y$. Then, we have following case to consider

1. if $x, y \in P$ then $x \preceq_{U_0} y$, and thus $x \preceq_{U_0 \otimes U_1} y$.
2. if $x, y \in Q$ then $x \preceq_{U_1} y$, and thus $x \preceq_{U_0 \otimes U_1} y$.

We ignore the case $x \in P$ and $y \in Q$ that implies $x \not\preceq_{P \otimes Q} y$. The order subsumptions above witness $U_0 \otimes U_1 \leqslant U$. Similarly, we can establish $V_0 \otimes V_1 \leqslant V$ as well. □

We generalise Lemma 19 for iposets from soft version of interpolation Lemma 18 above.

Lemma 19. *Let P, Q, U, V be iposets such that $P \otimes Q = U \otimes V$. Then there exist iposets U_0, U_1, V_0, V_1 such that*

$$U_0 \otimes U_1 = U, V_0 \otimes V_1 = V, P = U_0 \otimes V_0, \text{ and } Q = U_1 \otimes V_1.$$

Proof. Trivial by Lemma 18 with h^{-1} inverse morphisms in order relation. □

We now prove interpolation Lemma 17 for iposets.

Lemma 20 (Interpolation). *Let P, Q, U, V be iposets such that $P \rhd Q \leqslant U \otimes V$. Then there exist iposets U_0, U_1, V_0, V_1 such that*

$$U_0 \rhd U_1 \leqslant U, V_0 \rhd V_1 \leqslant V, P \leqslant U_0 \otimes V_0, \text{ and } Q \leqslant U_1 \otimes V_1.$$

Proof.

Let P, Q, U, V be iposets

$$P = (E_P, \preceq_P, s_P, t_P) \text{ and } Q = (E_Q, \preceq_Q, s_Q, t_Q)$$

$$U = (E_U, \preceq_U, s_U, t_U) \text{ and } V = (E_V, \preceq_V, s_V, t_V).$$

Consider the case $P \rhd Q \leqslant U \otimes V$ which witness the bijection

$$h : E_{U \otimes V} \to E_{P \rhd Q} \tag{4.27}$$

such that

$$x \preceq_{U \otimes V} y \implies h(x) \preceq_{P \rhd Q} h(y) \text{ for all } x, y \in (U \otimes V)$$

along with the

$$h : s_{U \otimes V} \to s_{P \rhd Q} = s_P \text{ and } h : t_{U \otimes V} \to t_{P \rhd Q} = t_Q$$

source and target interfaces bijections. Now, we choose the pairwise disjoint iposets

$$U_0 = (E_{U_0}, \preceq_{U_0}, s_{U_0}, t_{U_0}) \text{ and } U_1 = (E_{U_1}, \preceq_{U_1}, s_{U_1}, t_{U_1})$$

$$V_0 = (E_{V_0}, \preceq_{V_0}, s_{V_0}, t_{V_0}) \text{ and } V_1 = (E_{V_1}, \preceq_{V_1}, s_{V_1}, t_{V_1})$$

such that

$$E_{U_0} = E_U \cap h^{-1}(E_P) \text{ denotes the restriction of } E_P \text{ to the } E_U \text{ such that}$$

$$\preceq_U \cap h^{-1}(\preceq_P) \text{ along with } s_{U_0} = s_U \cap h^{-1}(s_P) \text{ and } t_{U_0} = E_U \cap h^{-1}(t_P).$$

In short,

$$\blacktriangleright\ U_0 = (E_U \cap h^{-1}(E_P), \preceq_U \cap h^{-1}(\preceq_P), s_U \cap h^{-1}(s_P), E_U \cap h^{-1}(t_P)).$$

Similarly,

$$U_1 = (E_U \cap h^{-1}(E_Q), \preceq_U \cap h^{-1}(\preceq_Q), E_U \cap h^{-1}(s_Q), t_U \cap h^{-1}(t_Q))$$

$$V_0 = (E_V \cap h^{-1}(E_P), \preceq_V \cap h^{-1}(\preceq_P), s_V \cap h^{-1}(s_P), E_V \cap h^{-1}(t_P))$$

$$V_1 = (E_V \cap h^{-1}(E_Q), \preceq_V \cap h^{-1}(\preceq_Q), E_V \cap h^{-1}(s_Q), t_V \cap h^{-1}(t_Q)).$$

To established $P \leqslant U_0 \otimes V_0$, we proceed with the bijection $h : E_{U_0 \otimes V_0} \to E_P$

$$\begin{aligned} E_{U_0 \otimes V_0} &= E_{U_0} \cup E_{V_0} \\ &= (E_U \cap h^{-1}(E_P)) \cup (E_V \cap h^{-1}(E_P)) \\ &\blacktriangleright \text{After simplifying,} \\ &= (E_U \cup E_V) \cap h^{-1}(E_P) \\ &= E_{U \otimes V} \cap h^{-1}(E_P) \end{aligned}$$

$$
\begin{aligned}
&= h^{-1}(E_{P \rhd Q} \cap E_P) \\
&= h^{-1}(E_P)
\end{aligned}
$$

along with the source $h : s_{U_0 \otimes V_0} \to s_P$ and target $h : t_{U_0 \otimes V_0} \to t_P$

$$
\begin{aligned}
s_{U_0 \otimes V_0} &= s_{U_0} \cup s_{V_0} \\
&= (s_U \cap h^{-1}(s_p)) \cup (s_V \cap h^{-1}(s_P)) \\
&\blacktriangleright \text{ After simplifying,} \\
&= (s_U \cup s_V) \cap h^{-1}(s_P) \\
&= s_{U \otimes V} \cap h^{-1}(s_P) \\
&= h^{-1}(s_{P \rhd Q}) \cap h^{-1}(s_P) \\
&= h^{-1}(s_{P \rhd Q} \cap s_P) \\
&= h^{-1}(s_P \cap s_P) \\
&= h^{-1}(s_P)
\end{aligned}
$$

$$
\begin{aligned}
t_{U_0 \otimes V_0} &= t_{U_0} \cup t_{V_0} \\
&= (E_U \cap h^{-1}(t_p)) \cup (E_V \cap h^{-1}(t_P)) \\
&\blacktriangleright \text{ After simplifying,} \\
&= (E_U \cup E_V) \cap h^{-1}(t_P) \\
&= E_{U \otimes V} \cap h^{-1}(t_P) \\
&= h^{-1}(E_{P \rhd Q}) \cap h^{-1}(t_P) \\
&= h^{-1}(E_{P \rhd Q} \cap t_P) \\
&= h^{-1}(t_P)
\end{aligned}
$$

interface bijections and order subsumptions such that $x \preceq_{U_0 \otimes V_0} y \implies h(x) \preceq_P h(y)$. Suppose $x, y \in P$ such that $x \preceq_{U_0 \otimes V_0} y$, which is either $x \preceq_{U_0} y$ or $x \preceq_{V_0} y$. If $x \preceq_{U_0} y$ then $x \preceq_U y$ by choice of U_0. But then $x \preceq_{U \otimes V} y$, thus $x \preceq_{P \rhd Q} y$ by Equation (4.27). Since $x, y \in P$, we conclude $x \preceq_P y$. Similarly, we can show $x \preceq_P y$ when $x \preceq_{V_0} y$. This witness $P \leqslant U_0 \otimes V_0$. Similarly, we can establish $Q \leqslant U_1 \otimes V_1$ as well.

To establish $U_0 \rhd U_1 \leqslant U$, we now proceed with the bijection $h : E_U \to E_{U_0 \rhd U_1}$

$$
\begin{aligned}
E_{U_0 \rhd U_1} &= (E_{U_0} \cup E_{U_1})_{/t_{U_0}} \blacktriangleright \text{ since } t_{U_0} = s_{U_1} \\
&= ((E_U \cap h^{-1}(E_P)) \cup (E_U \cap h^{-1}(E_Q)))_{/t_{U_0}} \\
&= ((E_U \cup E_U) \cap (E_U \cup h^{-1}(E_Q)) \cap \\
&\qquad (h^{-1}(E_P) \cup E_U) \cap (h^{-1}(E_P) \cup h^{-1}(E_Q)))_{/t_{U_0}} \\
&\blacktriangleright \text{ By simplifying first 3 term, we get} \\
&= (E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q)))_{/t_{U_0}}
\end{aligned}
$$

$$
\begin{aligned}
&= (E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q)))_{/E_U \cap h^{-1}(t_P)}\\
&= (E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q))) \cap \overline{E_U \cap h^{-1}(t_P)}\\
&= (E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q))) \cap (\overline{E_U} \cup \overline{h^{-1}(t_P)})\\
&= ((E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q))) \cap \overline{E_U}) \cup\\
&\qquad\qquad ((E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q))) \cap \overline{h^{-1}(t_P)})
\end{aligned}
$$

▶ By simplifying first term, we get

$$= \emptyset \cup ((E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q))) \cap \overline{h^{-1}(t_P)})$$

▶ While simplifying rest, we get

$$= ((E_U \cap (h^{-1}(E_P) \cup h^{-1}(E_Q))))_{/h^{-1}(t_P)}$$

▶ Again simplifying, we get

$$
\begin{aligned}
&= E_U \cap h^{-1}((E_P \cup E_Q)_{/t_P})\\
&= E_U \cap h^{-1}(E_{P \rhd Q})\\
&= E_U \cap (E_U \cup E_V)\\
&= E_U.
\end{aligned}
$$

along with the source $h : s_U \to s_{U_0 \rhd U_1}$ and target $h : t_U \to t_{U_0 \rhd U_1}$ interface bijections are given

$$
\begin{aligned}
s_{U_0 \rhd U_1} = s_{U_0} = s_U \cap h^{-1}(s_P) = s_U \cap h^{-1}(s_{P \rhd Q})&\\
= s_U \cap s_{U \otimes V} = s_U \cap (s_U \cup s_V) = s_U&
\end{aligned}
$$

and

$$
\begin{aligned}
t_{U_0 \rhd U_1} = t_{U_1} = t_U \cap h^{-1}(t_Q) = t_U \cap h^{-1}(t_{P \rhd Q})&\\
= t_U \cap t_{U \otimes V} = t_U \cap (t_U \cup t_V) = t_U&
\end{aligned}
$$

such that $x \preceq_U y \implies h(x) \preceq_{U_0 \rhd U_1} h(y)$. For order inclusion, suppose $x, y \in U$ such that $x \preceq_U y$. Then, $x \preceq_U y$ implies $x \preceq_{U \otimes V} y$, and thus $x \preceq_{P \rhd Q} y$. Now, we have three cases to consider

1. if $x, y \in P$ then $x \preceq_{U_0} y$, and thus $x \preceq_{U_0 \rhd U_1} y$.
2. if $x, y \in Q$ then $x \preceq_{U_1} y$, and thus $x \preceq_{U_0 \rhd U_1} y$.
3. if $x \in P$ and $y \in Q$, then there are three case to consider
   (a) if $x \in P$ and $y \in Q$ such that $y \in s_Q$, then case 1.
   (b) if $x \in P$ and $y \in Q$ such that $x \in t_P$, then case 2.
   (c) if $x \in P$ and $y \in Q$ such that $x \notin t_P$ and $x \notin s_Q$, then $x \in U_0$ and $y \in U_1$, and thus $x \preceq_{U_0 \rhd U_1} y$.
4. if $y \in P$ and $x \in Q$, then $y \in U_0$ and $x \in U_1$. But, $x \preceq_{U_0 \rhd U_1} y$ implies $x = y$ by antisymmetry. Thus, $x \in U_0 \rhd U_1$ explains the fusion such that $y \in t_{U_0}$ and $x \in s_{U_1}$.

The order subsumptions above proves $U_0 \rhd U_1 \leqslant U$. Similarly, we get $V_0 \rhd V_1 \leqslant V$ as well. $\square$

Lemma 21. *Let P, Q, U, V be iposets such that $P \rhd Q = U \otimes V$. Then there exist iposets U_0, U_1, V_0, V_1 such that*

$$U_0 \rhd U_1 = U, V_0 \rhd V_1 = V, P = U_0 \otimes V_0, \text{ and } Q = U_1 \otimes V_1.$$

Proof. Trivial by the proof of Lemma 20. $\square$

Remark 19. *We have proved most of the property of order structure of SP posets under subsumption [13] for the order structure of iposets under subsumption. These algebraic results of iposets under subsumption orders are rich enough to generalise the weak class of iposet languages.*

4.5 Summary

In this chapter, we presented our results of iposets theory. We gave an equational theory of iposets algebra in Section 4.1. We established that ordered bisemigroup of iposets forms a concurrent semigroup, and thereby satisfies exchange law given in Equation (4.3). We also established that the identities of concurrent monoid in iposet algebra do not imply exchange law in Lemma 1; replicating the original results of Concurrent Kleene algebra [14, Definition 6.8.]. We gave an equational theory of iposet languages in Section 4.2. We have shown that the double monoid structure of iposets in Proposition 3 defines language bisemirings of iposets in Proposition 5.

We defined structured theory for iposet languages and outlined their hierarchy in Section 4.3. We generalized properties of iposets hierarchy in the Lemmas 2, 3 and 4. Further, we derived properties of iposets under subsumption order in Section 4.4 and established that the order structure of iposets under subsumption respects most of the property of order structure of SP posets under subsumption.

Chapter 5

Domain and modal operators for iposets languages

In this chapter, we present axioms of domain operations for iposets languages and generalise their corresponding modal operators by following earlier exposition of literature in Section 2.3.

5.1 Domain operators

Relations are a natural model of computation such as input-output aspects of programs. These input-output structures (or labelled transition systems) are models of program execution and provide logic for programs such as dynamic logic. For example, a pair in a binary relation can be seen as relating a start state to an end state in program execution. These input-output examples of relation capture all possible end states that the program can reach from the point of its executions. A domain operation on such a relation would then return all the start states, i.e., all those states from which the program can execute. Peleg gives similar justifications for multi-relations for concurrent programs in the setting of Concurrent dynamic logic [30] and the domain of multi-relations [12] capture similar input-output relational aspect.

Relations can be extracted from labelled iposets by looking at the individual source s and target t interfaces as follows, for some iposets P, Q

$$R(P) = \{(i, j) \mid s(i) \leq t(j)\}.$$

This resulting relation will be over two different sets, or it can be made over the same set if we impose some restrictions on how the interfaces of the iposet should always look like. However, we also want that such a translation should capture the following relational compositions

$$R(P \triangleright Q) = R(P) \circ R(Q).$$

However, this cannot be true because the iposet concatenation introduces dependencies between the s interfaces of P and the t interfaces of Q which cannot be avoided no matter how we construct the iposet.

Now, if we look at languages of iposets then there is a way to translate these into relations. Consider the following two translations between relation $\mathcal{R}$, the set of relations $R \subseteq \mathbb{N} \times \mathbb{N}$, and iposets languages PL, $PL \in 2^{\mathcal{P}}$ the set of all sets of iposets,

$$\mathcal{R}(PL) = \{(i,j) \mid [i] \to P \leftarrow [j] \in PL\} \tag{5.1}$$

$$\mathcal{L}(R) = \{id : [i] \to [i+j] \leftarrow [j] : id \mid (i,j) \in R.\} \tag{5.2}$$

Theorem 13. *For any relation R and iposets $PL, QL \in 2^{\mathcal{P}}$, the maps in Equaitons (5.1) and (5.2) respect the following*

$$\mathcal{R}(PL \rhd QL) = \mathcal{R}(PL) \circ \mathcal{R}(QL), \tag{5.3}$$

$$\mathcal{L}(R \circ R') = \mathcal{L}(R) \rhd \mathcal{L}(R'). \tag{5.4}$$

Proof.

▶ Proof of Equation (5.3): the language composition is defined pointwise. For some $[i] \to P \leftarrow [j] \in PL$ and $[j'] \to P \leftarrow [k] \in QL$ are composable iff $[j] = [j']$, i.e., are the same number. Then, the resulting ipomset in the new language has the source interface $[i]$ and the target interface $[k]$. This is the same as in relational composition, where the two corresponding pairs (i,j) and (j',k) can be composed only if $j = j'$ to form the new pair (i,k).

▶ Proof of Equation (5.4): for each pair $(i,j) \in R$ and $(j',k) \in R'$, there exist the corresponding iposets $id : [i] \to [i+j] \leftarrow [j] : id \in \mathcal{L}(R)$ and $id : [j'] \to [j'+k] \leftarrow [k] : id \in \mathcal{L}(R')$. The relational composition between R and R' is defined only when $j = j'$, thus making the iposet concatenation also defined, with the resulting iposet being translated into the corresponding relation pair. □

We call any $I \subseteq 1_{\rhd}$ a subidentity. The set of all subidentities together with the concatenation and union form a Boolean algebra with $1_{\rhd}$ as the top element and $\emptyset$ as the bottom element. The concatenation $\rhd$ acts as conjunction (i.e., $I \rhd J = I \cap J$ for $I, J \subseteq 1_{\rhd}$) and union acts as disjunction when applied to the elements of $1_{\rhd}$.

Theorem 14 (Boolean algebra)**.** The set of $I \subseteq 1_{\rhd}$ identity iposets generalize Boolean elements such that $\overline{I} \subseteq 1_{\rhd} \setminus I$ denotes Boolean complement of I, and satisfies following Boolean axioms

$$I \rhd \overline{I} = \emptyset, \tag{5.5}$$

$$I \cup \overline{I} = 1_{\rhd}. \tag{5.6}$$

Proof.

▶ we first proceed for

$$I \triangleright J = I \cap J \text{ for any } I, J \subseteq 1_{\triangleright}. \tag{5.7}$$

The composition $I \triangleright J$ is defined by finding those identity iposets in I and J which leads to $I' \triangleright J' = I'$ such that $I' \in I$ and $J' \in J$. This implies $I' = J'$ and the gluing between two identity iposets is the identity iposets itself if the gluing is defined. That is exactly the $I \cap J$, the composition yields those identity iposets in I and J which are isomorphics, i.e., $I' = J'$ such that $I' \in I$ and $J' \in J$.

▶ The proof of Equation (5.5) follows from Equation (5.7), which is trivial by set theory

$$I \triangleright \overline{I} = I \cap \overline{I} = I \cap (1_{\triangleright} \setminus I) = \emptyset.$$

▶ The Equation (5.6) is trivial by set theory

$$I \cup \overline{I} = I \cup (1_{\triangleright} \setminus I) = 1_{\triangleright}.$$

□

Domain definition over iposets

We follow the intuitions and results from [7, 9], and define the *domain operation* applied to an iposet as

$$dom([n] \rightarrow P \leftarrow [m]) \stackrel{\text{def}}{=} [n] \rightarrow [n] \leftarrow [n]$$

which is an element of $1_{\triangleright}$. For a set of iposets, we define domain by pointwise application

$$dom(A) \stackrel{\text{def}}{=} \{dom(P) \mid P \in A\}$$

which is a subidentity (i.e., $dom(A) \subseteq 1_{\triangleright}$). The definition of *range* is similar, except, that it returns the target interface of the iposet

$$ran([n] \rightarrow P \leftarrow [m]) \stackrel{\text{def}}{=} [m] \rightarrow [m] \leftarrow [m]$$

with the same observations as above of being a subidentity when applied to sets of iposets.

Note that now we can formulate the definedness condition for concatenation of languages of iposets using domain and range as follows:

$$L \triangleright M = \{P \triangleright Q \mid P \in L, Q \in M, \text{ iff } ran(P) = dom(Q)\}.$$

For languages of iposets we define the *antidomain* as

$$ant(A) \stackrel{\text{def}}{=} 1_{\rhd} \setminus \{dom(P) \mid P \in A\}$$

which is a subidentity.

Lemma 22. *We observe following equalities for domain of individual iposet.*

$$dom(P) \rhd P = P \tag{5.8}$$

$$dom(dom(P)) = dom(P) \tag{5.9}$$

$$dom(P \rhd Q) = dom(P) \textit{ if } ran(P) = dom(Q) \tag{5.10}$$

Proof.

▶ Proof of Equation (5.8): consider an iposet P such that $[n] \to P \leftarrow [m]$, then by following definition of domain

$$dom(P) = \{[n] \to [n] \leftarrow [n]\}$$

yields an identity iposets which contains all the events of source interface of P. This implies $dom(P) \rhd P$ composition is defined by gluing $dom(P)$ with P, which fuses all the events of identity iposet yields from $dom(P)$ to the source interface of P, and P remains unchanged by the composition $dom(P) \rhd P$.

$$dom(P) \rhd P = \mathrm{id}_n \rhd P = P.$$

▶ Proof of Equation (5.9) follows from the definition of domain. The $dom(P)$ yields an identity iposet containing all the events of the source interfaces of P, and $dom(dom(P))$ yields an identity iposets containing all the events of source interface of the identity iposet produced by $dom(P)$. Since the source interface of an identity iposet contains all the events of the iposet, the $dom(dom(P))$ and $dom(P)$ identity poset contains exactly the same events that belongs to the source interface of P, i.e., $dom(dom(P)) = dom(P)$.

▶ Proof of Equation (5.10) follows from the definition of domain

$$dom(P \rhd Q) = dom(P) \text{ since, } s_{P \rhd Q} = s_P \text{ iff } ran(P) = dom(Q)$$

□

Lemma 23. *Following equalities exist for the range of individual iposets.*

$$P \rhd ran(P) = P \tag{5.11}$$

$$ran(ran(P)) = ran(P) \tag{5.12}$$

$$ran(P \rhd Q) = ran(Q) \textit{ if } ran(P) = dom(Q) \tag{5.13}$$

$$ran(dom(P)) = dom(P) \tag{5.14}$$

$$dom(ran(P)) = ran(P). \tag{5.15}$$

Proof.

► Proof of Equation (5.11): consider an iposet P such that $[n] \to P \leftarrow [m]$. Then, by the definition of range

$$ran(P) = \{[m] \to [m] \leftarrow [m]\}$$

yields an identity iposets which contains all the events of target interfaces of P. This implies $P \rhd ran(P)$ composition is defined by gluing P with $ran(P)$, that merge all the events of identity iposets yields from $ran(P)$ to the target interface of P, and P remains unchanged by the composition $P \rhd ran(P)$

$$P \rhd ran(P) = P \rhd \mathrm{id}_m = P.$$

► Proof of Equation (5.12) follows from the definition of range. The $ran(P)$ yields an identity iposet containing all the events of the target interfaces of P, and $ran(ran(P))$ yields an identity iposet containing all the events of target interface of the identity iposet produced by $ran(P)$. Since the target interface of an identity iposet contains all the events of the iposets, $ran(ran(P))$ and $ran(P)$ identity posets contains exactly the same events that belongs to the target interface of P, i.e., $ran(ran(P)) = ran(P)$.

► Proof of Equation (5.13) follows from the definition of range,

$ran(P \rhd Q) = ran(Q)$ since composition $P \rhd Q$ is defined by $ran(P) = dom(Q)$

► Proof of Equation (5.14) and 5.15 follows from similar argument as proof of Equation (5.12) from above.

□

Theorem 15 (Domain axioms). *For some sets of iposets A, B we have:*

$$A \cup dom(A) \rhd A = dom(A) \rhd A \tag{5.16}$$

$$dom(A \rhd B) = dom(A \rhd dom(B)) \tag{5.17}$$

$$dom(A) \cup 1_{\rhd} = 1_{\rhd} \tag{5.18}$$

$$dom(\emptyset) = \emptyset \tag{5.19}$$

$$dom(A \cup B) = dom(A) \cup dom(B) \tag{5.20}$$

The domain of iposets is local wrt. concatenation and least left preserver by Equation (5.8), domain elements are subidentities by Equation (5.9), domain is strict and additive by Equation (5.20).

For some subidentity $I \subseteq 1_{\triangleright}$ we have:

$$dom(I) = I \tag{5.21}$$

Proof.

► Proof of Equation (5.16): we prove instead $A = dom(A) \triangleright A$ which then implies the axiom. By definition, $dom(A) = \{\mathrm{id}_n = \mathrm{id} : [n] \to [n] \leftarrow [n] : \mathrm{id} \mid [n] \to P \leftarrow [m] \in A\}$ contains all those identity iposets that appear as source interface of some iposet from A. An iposet from the concatenation $dom(A) \triangleright A$ is $dom(P) \triangleright Q$ for $P, Q \in A$ for which the concatenation is defined, i.e., the source interface of Q is some $[n]$ which is the same as the target interface of P and the domain becoming the identity id_n. Therefore, the identity disappears into $dom(P) \triangleright Q = \mathrm{id}_n \triangleright Q = Q$ which is in A, thus proving one direction of the equality. Proving the other direction is similar; taking any $Q \in A$ we find its domain $dom(Q) \in dom(A)$, by definition. It is obvious that $dom(Q) = \mathrm{id}_n$ can be concatenated to the left with Q and is the source interfaces of Q. Thus, we have found the iposet from $dom(A) \triangleright A$.

► Proof of Equation (5.17), we explicit both sides of the equality as follows:

$$dom(A \triangleright B) = \{dom(P \triangleright P') \mid P \in A, P' \in B, \text{ and } ran(P) = dom(P'),\} \tag{5.22}$$

which implies

1. for any $P \in A$, $P' \in B$, s.t. $ran(P) = dom(P')$ we find $dom(P \triangleright P') = dom(P)$ in the above set, and nothing else.

On the other side,

$$dom(A \triangleright dom(B)) = \{dom(P \triangleright Q) \mid P \in A, Q \in dom(B), \text{ and } ran(P) = dom(Q)\} \tag{5.23}$$

$$= \{dom(P \triangleright Q) \mid P \in A, Q = dom(P') \text{ for } P' \in B, \text{ and } ran(P) = dom(dom(P')) = dom(P'),\} \tag{5.24}$$

which implies

1. for any $P \in A$, $P' \in B$, s.t. $ran(P) = dom(P')$ we find $dom(P \triangleright dom(P')) = dom(P)$ in the above set, and nothing else.

Therefore the two sets are the same.

► It is easy to see that Equation (5.19) is true since the domain has no iposet to be applied to, thus the resulting empty set.

► Proof of Equation (5.18): since the domain is a subidentity then set operations give the stated result.

► Proof of Equation (5.20): it is enough to use properties of pointwise definition of domain over union of sets.

□

Note 2. The equations for domain from [9, Lemma 5.1] hold for languages of iposets, which one can check manually.

Remark 20. The anitdomain $ant(x)$ of a program x model the set of states from which x can not execute, compared to domain $dom(x)$ of a program x that model the set of states from which x can execute. This interprets $ant(x)$ as a Boolean complement of $dom(x)$, and they together model the input-output state spaces of program x.

Theorem 16 (Antidomain axioms)**.** *For some sets of iposets A, B we have:*

$$ant(A) \rhd A = \emptyset \tag{5.25}$$

$$ant(A \rhd B) = ant(A \rhd dom(B)) \tag{5.26}$$

$$dom(A) \cup ant(A) = 1_{\rhd} \tag{5.27}$$

$$ant(\emptyset) = 1_{\rhd} \tag{5.28}$$

$$dom(A) \rhd ant(A) = \emptyset \tag{5.29}$$

$$ant(A \cup B) = ant(A) \rhd ant(B) \tag{5.30}$$

Proof.

► Proof of Equation (5.25): we use the definition of antidomain as follows

$$ant(A) \rhd A = (1_{\rhd} \setminus dom(A)) \rhd A = (1_{\rhd} \rhd A) \setminus (dom(A) \rhd A) = A \setminus A = \emptyset.$$

► Proof of Equation (5.26): we use domain axioms as follows

$$ant(A \rhd B) = 1_{\rhd} \setminus dom(A \rhd B) \overset{5.17}{=} 1_{\rhd} \setminus dom(A \rhd dom(B)) = ant(A \rhd dom(B)).$$

► Proof of Equations 5.27 and 5.28 follow immediately from the definition.

► The proof of Equation (5.29) follows from the Equation (5.7). By definition, $ant(A)$ contains those identities that are not in $dom(A)$. Therefore, when concatenating the languages we cannot find any pair of identity iposets where the concatenation is defined, because their respective interfaces are different.

► Proof of Equation (5.30): we explicit the two sides. The left side is

$$ant(A \cup B) = 1_{\triangleright} \setminus dom(A \cup B) \overset{5.20}{=} 1_{\triangleright} \setminus (dom(A) \cup dom(B))$$

which thus contains all identities id_n that are not part of $dom(A)$ and neither of $dom(B)$. The right side is

$$ant(A) \triangleright ant(B) = (1_{\triangleright} \setminus dom(A)) \triangleright (1_{\triangleright} \setminus dom(B))$$

which thus contains iposets obtained as concatenation of an identity that is not in $dom(A)$ and another identity that is not in $dom(B)$. Since the concatenation must be defined, it means that the two identities must be the same, and thus we are left only with those identities as in the let side of the equation.

□

Theorem 17 (Boolean domain axioms)**.** *The domain and antidomain operations satisfies following axioms, for some sets of iposets A, B and C*

$$ant(A) \cup dom(A) = 1_{\triangleright} \tag{5.31}$$

$$dom(A) \triangleright (ant(A) \cup dom(B)) = dom(A) \triangleright dom(B) \tag{5.32}$$

$$dom(B) \triangleright (ant(A) \cup dom(B)) = dom(B) \tag{5.33}$$

$$\begin{aligned} ant(A) \cup (dom(B) \triangleright dom(C)) = (&ant(A) \cup dom(B)) \\ &\triangleright (ant(A) \cup dom(C)) \end{aligned} \tag{5.34}$$

Then, $1_{\triangleright}$ is a Boolean domain semiring and $dom(ant(A)) = ant(A)$.

Proof.

► The proof of Equation (5.31) follows the proof of Equation (5.27) above.

► Proof of Equation (5.32): we follow left side of equation, and by distribution of $\triangleright$ over union

$$dom(A) \triangleright (ant(A) \cup dom(B)) = (dom(A) \triangleright ant(A)) \cup (dom(A) \triangleright dom(B))$$

We have $dom(A) \triangleright ant(A) = \emptyset$ by Equation (5.29), then

$$\begin{aligned} &= \emptyset \cup (dom(A) \triangleright dom(B)) \\ &= dom(A) \triangleright dom(B)) \end{aligned}$$

we arrived at the right hand side of Equation (5.32).

► Proof of Equation (5.33): we follow the left hand side of the equation, and by distribution of $\triangleright$ with union

$$\begin{aligned} dom(B) \triangleright (ant(A) \cup dom(B)) &= (dom(B) \triangleright ant(A)) \cup (dom(B) \triangleright dom(B)) \\ &= (dom(B) \triangleright ant(A)) \cup dom(B). \end{aligned}$$

By following the right hand side of Equation (5.33)

$$\begin{aligned} dom(B) &= dom(B) \triangleright 1_{\triangleright} \\ &\blacktriangleright \text{Since } dom(A) \subseteq 1_{\triangleright}, \text{ so we can write} \\ &= dom(B) \triangleright (ant(A) \cup 1_{\triangleright}) \\ &\blacktriangleright \text{By distribution of } \triangleright \text{ with union, we get} \\ &= (dom(B) \triangleright ant(A)) \cup (dom(B) \triangleright 1_{\triangleright}) \\ &= (dom(B) \triangleright ant(A)) \cup dom(B) \end{aligned}$$

we arrive at the left hand side.

► Proof of Equation (5.34): we first proceeds with left hand side

$$\begin{aligned} (ant(A) \cup dom(B))(ant(A) \cup dom(C)) = \;& (ant(A) \triangleright ant(A)) \\ \cup (ant(A) \triangleright dom(C)) \cup (dom(B) \triangleright ant(A)) &\cup (dom(B) \triangleright dom(C)). \end{aligned}$$

Since $ant(A) \triangleright ant(A) = ant(A)$, we arrive

$$\begin{aligned} (ant(A) \cup dom(B))(ant(A) \cup dom(C)) = \;& ant(A) \cup (ant(A) \triangleright dom(C)) \\ & \cup (dom(B) \triangleright ant(A)) \cup (dom(B) \triangleright dom(C)) \end{aligned}$$

By following the left hand side of the equation 5.34

$$\begin{aligned} ant(A) \cup (dom(B) \triangleright dom(C)) = \;& (ant(A) \triangleright 1_{\triangleright}) \cup (dom(B) \triangleright dom(C)) \\ & \blacktriangleright \text{since } dom(C) \subseteq 1_{\triangleright}, \text{ we can write} \\ & = (ant(A) \triangleright (1_{\triangleright} \cup dom(C))) \cup (dom(B) \triangleright dom(C)) \\ & = ant(A) \cup (ant(A) \triangleright dom(C)) \cup (dom(B) \triangleright dom(C)) \\ & = (1_{\triangleright} \triangleright ant(A)) \cup (ant(A) \triangleright dom(C)) \cup (dom(B) \triangleright dom(C)) \\ & \blacktriangleright \text{since } dom(B) \subseteq 1_{\triangleright}, \text{ we can write} \\ & = ((dom(B) \cup 1_{\triangleright}) \triangleright ant(A)) \cup (ant(A) \triangleright dom(C)) \\ & \quad \cup (dom(B) \triangleright dom(C)) \\ & = (1_{\triangleright} \triangleright ant(A)) \cup (dom(B) \triangleright ant(A)) \\ & \quad \cup (ant(A) \triangleright dom(C)) \cup (dom(B) \triangleright dom(C)) \\ & \blacktriangleright \text{Rewritting using commutative of union, we get} \\ & = ant(A) \cup (ant(A) \triangleright dom(C)) \\ & \quad \cup (dom(B) \triangleright ant(A)) \cup (dom(B) \triangleright dom(C)), \end{aligned}$$

we arrive at the left hand side.

□

Definition 46 (Endomorphism)**.** The endomorphism $dom(A) \to dom(B)$ on domain operations for some sets of iposets A and B, is called relative pseudocomplement of $dom(B)$ with respect to $dom(A)$. The pseudocomplement $\neg dom(A)$ of $dom(A)$ is can be expressed as $dom(A) \to 0$.

Definition 47 (Galois connection)**.** The *Galois connection*, some sets of iposets A, B and C, is a biconditional endomorphism $\leftrightarrow$ on the domain operations

$$dom(A) \rhd dom(B) \leq dom(C) \leftrightarrow dom(A) \leq dom(B) \to dom(C)$$

defined by

$$dom(A) \rhd dom(B) \leq dom(C) \leftrightarrow dom(A) \leq ant(B) \wedge dom(C). \tag{5.35}$$

Corollary 18. *The Galois connection together with a closure condition*

$$dom(dom(A) \to dom(B)) = dom(A) \to dom(B)$$

The Galois connection together with a closure condition axiomatise Heyting algebra [9, Proposition 10.1] given by following axioms

$$dom(A) \to dom(A) = 1_{\rhd} \tag{5.36}$$

$$dom(A) \rhd (dom(A) \to dom(B)) = dom(A) \rhd dom(B) \tag{5.37}$$

$$dom(B) \rhd (dom(A) \to dom(B)) = dom(B) \tag{5.38}$$

$$\begin{aligned} dom(A) \to dom(B) \rhd dom(C) = (dom(A) \to dom(B)) \\ \rhd (dom(A) \to dom(C)) \end{aligned} \tag{5.39}$$

Remark 21. The opposite of a semiring swaps the order of sequential composition and runs program backwards. By duality, the opposite of all the statement about semiring hold in the opposite of the semiring. This defines $ran(x)$ range as a weak converse of $dom(x)$ domain [7, Section 5.2]. Analogously, the weak converse of an antidomain $ant(x)$ of program x should model the antirange $ar(x)$ of the program x, but anti-domain is not closed under this duality. The $ar(x)$ denotes the set of states in which program x can not terminate. This interprets $ar(x)$ as Boolean complement over $ran(x)$, and they together model the post state space of program x.

Corollary 19. *Following the Remark 21 and Theorem 15, these are routine facts about the range for some sets of iposets A, B,*

$$A \cup A \rhd ran(A) = A \rhd ran(A) \tag{5.40}$$

$$ran(A \rhd B) = ran(ran(A) \rhd B) \tag{5.41}$$

$$ran(A) \cup 1_{\rhd} = 1_{\rhd} \tag{5.42}$$

$$ran(\emptyset) = \emptyset \tag{5.43}$$

$$ran(A \cup B) = ran(A) \cup ran(B) \tag{5.44}$$

Proof.

► Proof of Equation (5.40): we know $P = P \rhd ran(P)$ for some individual iposets from Equation (5.11). The composition $A \rhd ran(A)$ at the right side of the Equation (5.40) will be defined iff there exist $P, Q \in A$ such that $P \rhd ran(Q)$ is defined. By following definition of range, $ran(Q)$ in $P \rhd ran(Q)$ implies all those sets of identity iposets which has same source interfaces as target interfaces of iposets in P. Therefore $P \rhd ran(Q)$ is defined, and yields iposets P such that $P \in A$. The events of identity iposets $ran(Q)$ disappears into the P by following definition of iposets gluing $\rhd$. This implies $A \rhd ran(A) = A$, and follows the composition $A \cup A \rhd ran(A) = A \cup A = A$ at the right hands side of the Equation (5.40).

► Proof of Equation (5.41): we proceed from left side

$$ran(A \rhd B) = \{ran(P \rhd Q) \mid P \in A, Q \in B, \text{ and } ran(P) = dom(Q)\}$$

we get $ran(P \rhd Q) = ran(Q)$ such that $Q \in B$ by Equation (5.13), which implies $ran(A \rhd B) = ran(B)$. Similarly, we take right hand side

$$ran(ran(A) \rhd B) = \{ran(P \rhd Q) \mid P \in ran(A), Q \in B, \text{ and } ran(P) = dom(Q)\}$$

since P belongs to the sets of identity iposets; $P \in ran(A)$, we get $ran(P) = P$

$$ran(ran(A) \rhd B) = \{ran(P \rhd Q) \mid P \in ran(A), Q \in B, \text{ and } P = dom(Q)\}$$

Again, $ran(P \rhd Q) = ran(Q)$ such that $Q \in B$ by Equation (5.13), which implies $ran(A \rhd B) = ran(B)$. Therefore, the both side contains same sets of identity iposets.

► The proof of the equations 5.42,5.43 and 5.44 follows similar proof like domain axioms in the theorem 15.

□

Corollary 20. *Following the Remark 21 and Theorem 15, these are some routine facts about the antirange for some sets of iposets A, B and C,*

$$ar(A) \cup ran(A) = 1_{\triangleright} \tag{5.45}$$

$$A \triangleright ar(A) = \emptyset \tag{5.46}$$

$$ar(A \triangleright B) \cup ar(ran(A) \triangleright B) = ar(ran(A) \triangleright B) \tag{5.47}$$

Proof.

▶ Proof of Equation (5.45) follows immediately from the definition of the complement of $ran(A)$, i.e., $ar(A) = 1_{\triangleright} \setminus ran(A)$.

▶ Proof of Equation (5.46): we use definition of antirange

$$A \triangleright ar(A) = A \triangleright (1_{\triangleright} \setminus ran(A)) = (A \triangleright 1_{\triangleright}) \setminus (A \triangleright ran(A)) = A \setminus A = \emptyset.$$

▶ Proof of Equation (5.47): we instead prove $ar(A \triangleright B) = ar(ran(A) \triangleright B)$.

$$ar(A \triangleright B) = 1_{\triangleright} \setminus ran(A \triangleright B) \overset{5.41}{=} 1_{\triangleright} \setminus ran(ran(A) \triangleright B) = ant(ran(A) \triangleright B),$$

which simplifies and complete proof of Equation (5.47). □

Theorem 21 (Domain and products)**.** *For some iposets P, Q we have:*

$$dom(P \otimes Q) = dom(P) \otimes dom(Q) = dom(Q \otimes P) \tag{5.48}$$

For some sets of iposets A, B we have:

$$dom(A \otimes B) = dom(A) \otimes dom(B) \tag{5.49}$$

Proof.

▶ Proof of Equation (5.48): $dom(P \otimes Q)$ produces the cumulation $[m+n]$ of source interfaces of P and Q respectively. Therefore, the domain is the identity id_{m+n} which is the same as id_{n+m} and thus is the same as the domain of $Q \otimes P$. Then, $dom(P) = \mathrm{id}_n$ and $dom(Q) = \mathrm{id}_n$ which when put in parallel their two interfaces are cumulated into the $[m+n]$ and the equation is verified.

▶ Proof of Equation (5.49): we first follow the left side

$$\begin{aligned} dom(A \otimes B) &= \{dom(P \otimes Q) \mid P \in A, Q \in B\} \\ &\overset{5.48}{=} \{dom(P) \otimes dom(Q) \mid P \in A, Q \in B\}. \end{aligned} \tag{5.50}$$

On the right side, we make the product of the respective two languages that each contain identities belonging to the respective domain. The product of languages makes products of individual iposets, i.e., of domains from each language. Thus

$$dom(A) \otimes dom(B) = \{dom(P) \mid P \in A\} \otimes \{dom(Q) \mid Q \in B\}$$
$$= \{dom(P) \otimes dom(Q) \mid P \in A, Q \in B\}.$$

□

Note 3. For some labelled iposts P, Q, we have

$$dom(P \otimes Q) \neq dom(Q \otimes P) \tag{5.51}$$

Let id_{n+m} donotes the $dom(P \otimes Q)$ produced by the disjoint union

$$s_P \cup s_Q = s : [m+n] \to (P \sqcup Q)$$

of source interfaces $s_P : [m] \to P$ and $s_Q : [n] \to Q$ of P and Q respectively. Similarly, id_{m+n} denotes the $dom(Q \otimes P)$ given by

$$s_Q \cup s_P = s : [m+n] \to (Q \sqcup P)$$

which is not the same as id_{n+m}, as the naming of interface in id_{m+n} are different from id_{n+m}. Therefore, when we put $dom(P) = \mathrm{id}_n$ and $dom(Q) = \mathrm{id}_m$ in parallel composition their interfaces are two different cummulation depending on the order we put them in parallel composition, and the Equation (5.51) is verified.

5.2 Modal operators

We define a diamond modal operation using the domain. The antidomain then gives us logical negation.

Definition 48 (Diamond modality)**.** *We define an operation* $|\cdot\rangle\cdot : 2^{\mathcal{P}} \times 2^{1_\triangleright} \to 2^{1_\triangleright}$ *taking one language of iposets and one subidentity and returning another subidentity, as follows*

$$|A\rangle I \stackrel{\text{def}}{=} dom(A \triangleright I).$$

Intuitively, the modality can be understood as follows. The subidentity I contains a set of possible interfaces (i.e., identity iposets). The concatenation $A \triangleright I$ picks only those iposets $Q \in A$ from A that finish with an interface among those in I. For all these concurrent behaviors that were selected, take their domain, thus giving all their source interfaces which represent all the possibilities that these behaviors can be preceded by other matching behaviors.

This is a set interpretation of a modal operator, i.e. the semantics is the set of those states in the Kripke model where the modal formula holds.

Remark 22. following the Definition 48 of diamond modal box above

$$|A\rangle I \stackrel{\text{def}}{=} dom(A \rhd I)$$

I models the post states space of A which it might possibly reach after execution, and $|A\rangle I$ models those pre-states of A from which executing A may lead into the post states space I, i.e., essential domain of A but only those ones which leads A to I.

It is clear that $|A\rangle I$ models pre- state spaces of A with respect to the specified post- state space I; quite similar to $dom(xy)$; only those pre- states of x from where x can execute and finds gluing with y, i.e., precisely domain restriction with respect to post condition. Since interfaces of iposets models the domain and range, $|A\rangle I$ models source interfaces of those iposets in A whose target interfaces are equal to the I. This implies I is a fixed set of identity iposets which essential models the sets of target interfaces that every iposets in A has to qualify to proceed concurrently from the pull of iposets A.

Theorem 22 (Modal axioms). *For some sets of iposets A, B and some subidentities I, I' we have:*

$$|A \cup B\rangle I = |A\rangle I \cup |B\rangle I \tag{5.52}$$

$$|A\rangle(I \cup I') = |A\rangle I \cup |A\rangle I' \tag{5.53}$$

$$|A \rhd B\rangle I = |A\rangle|B\rangle I \tag{5.54}$$

$$|A\rangle\emptyset = \emptyset \tag{5.55}$$

$$|1_{\rhd}\rangle I = I \tag{5.56}$$

$$|I\rangle I' = I \cap I' \tag{5.57}$$

Proof.

► Proof of Equation (5.52): the left side translates into $dom((A \cup B) \rhd I)$ which because of the distributivity of sum over concatenation is the same as

$$dom((A \rhd I) \cup (B \rhd I)) \stackrel{5.20}{=} dom(A \rhd I) \cup dom(B \rhd I) = |A\rangle I \cup |B\rangle I.$$

► Proof of Equation (5.53) follows from similar arguments as above apply to the left side

$$dom(A \rhd (I \cup I')) = dom((A \rhd I) \cup (A \rhd I')) \overset{5.20}{=} \\ dom(A \rhd I) \cup dom(A \rhd I') = |A\rangle I \cup |A\rangle I'..$$

► Proof of Equation (5.54): the right-hand side, using the definition

$$|A\rangle dom(B \rhd I) = dom(A \rhd dom(B \rhd I)).$$

The left-hand side also rewrites into the same domain as above

$$dom(A \rhd B \rhd I) \overset{5.17}{=} dom(A \rhd dom(B \rhd I)).$$

► For Equation (5.55)

$$|A\rangle\emptyset = dom(A \rhd \emptyset) = dom(\emptyset) \overset{5.19}{=} \emptyset.$$

► For Equation (5.56)

$$|1_{\rhd}\rangle I = dom(1_{\rhd} \rhd I) = dom(I) = I.$$

► Proof of Equation (5.57): the left side rewritten by definition as follows $dom(I \rhd I')$. The concatenation picks only those identities from I that are also present in I', then the domain just returns these. This is the same as the intersection of the two subidentities. The proof follows from the Equation (5.7), i.e., $I \rhd I' = I \cap I'$.

□

Remark 23. Following the argument at the beginning of the Section 5.2, the modal box operator can be obtained from modal diamond operator by using logical negation such as

$$\neg(|A\rangle I) \overset{\text{def}}{=} \neg(dom(A \rhd I))$$

$$|A]I \overset{\text{def}}{=} ant(A \rhd ant(I)).$$

I models those set of post- states space of A that it must reach after execution, and $|A\rangle I$ models those set of pre- states space of A from which executing A must lead into the post states spaces I, i.e., essential anti-domain of those iposets in A which does not lead to I , which are anti-domain of those iposets in A which leads to the anti-domain of I

Corollary 23. *Following Remark 23 and Thereom 22, we get following axioms of modal box operator for iposets language. For some sets of iposets A, B and $I, I' \in 1_{\rhd}$,*

$$|A \cup B]I = |A]I \cup |B]I \tag{5.58}$$

$$|A](I \cup I') = |A]I \cup |A]I' \tag{5.59}$$

$$|A \rhd B]I = |A]|B]I \tag{5.60}$$

$$|A]\varnothing = \varnothing \tag{5.61}$$

$$|1_{\rhd}]I = I \tag{5.62}$$

$$|I]I' = I \cap I' \tag{5.63}$$

Corollary 24. *Following the Remark 21 and Corollary 19, backward modal diamond operator for iposet languages can be expressed as converse of forward modal diamond operator such that*

$$(|A\rangle I)^c \stackrel{\text{def}}{=} (dom(A \rhd I))^c$$
$$\langle A|I \stackrel{\text{def}}{=} ran(I \rhd A).$$

The $\langle A|I$ yields the possible post states space of every iposets in A whose pre-states space qualifies I. For some sets of iposets A, B and $I, I' \in 1_{\rhd}$, we get

$$\langle A \cup B|I = \langle A|I \cup \langle B|I \tag{5.64}$$

$$\langle A|(I \cup I') = \langle A|I \cup \langle A|I' \tag{5.65}$$

$$\langle A \rhd B|I = \langle A|\langle B|I \tag{5.66}$$

$$\langle A|\varnothing = \varnothing \tag{5.67}$$

$$\langle 1_{\rhd}|I = I \tag{5.68}$$

$$\langle I|I' = I \cap I' \tag{5.69}$$

Corollary 25. *Following the Remark 21, the backward modal box operator for iposet languages is obtained from the backward modal diamond operator by logical negation such as*

$$\neg(\langle A|I) \stackrel{\text{def}}{=} \neg(ran(I \rhd A))$$
$$[A|I = ar(ar(I) \rhd A)$$

The $[A|I$ yields all the post- states space of every iposets in A whose pre-states space qualifies I. For some sets of iposets A, B and $I, I' \in 1_{\triangleright}$, we get

$$[A \cup B|I = [A|I \cup [B|I \tag{5.70}$$

$$[A|(I \cup I') = [A|I \cup [A|I' \tag{5.71}$$

$$[A \triangleright B|I = [A|[B|I \tag{5.72}$$

$$[A|\emptyset = \emptyset \tag{5.73}$$

$$[1_{\triangleright}|I = I \tag{5.74}$$

$$[I|I' = I \cap I' \tag{5.75}$$

5.3 Summary

In this chapter, we presented an algebraic approach to modal operators based on the axioms of domain operations for iposet languages. In Section 5.1, we defined relational semantics for iposet languages and generalised their axioms of domain operations. We have presented several axiomatisation of domain operations for iposet languages following the results of literature [7, 9]. We have shown that domain axiomatisation generalise a missing link between the algebraic model of iposet languages and their relational semantics. The axioms of domain operations capture many natural properties of the relational domain operation and provide insights into the iposet languages compositionality. They considerably argue in favour of expressiveness of iposets language, particular, for the analysis and verification of programs modelled by iposet languages.

Chapter 6

Conclusion

We have introduced a language-theoretic model of iposets for concurrency modelling in Chapter 4. The presented hierarchy for a structured theory of iposets in Definition 40 are more general than the SP posets hierarchy given in Definition 41. We have shown that posets that contain N-pomset, which are not admissible in the SP posets hierarchy, become naturally admissible in the iposets hierarchy in Remark 16. The hierarchical facts about iposets collected in Lemmas 2, 3, and 4 evidence that the structured theory of iposets is applicable to a wider spectrum of concurrency theory compared to the SP posets. Moreover, we have shown that ordered bisemigroup of iposets forms a concurrent semigroup in Proposition 2, and thereby, satisfies the exchange law given in Equation 4.3. We also proved that the identities of iposets semigroup do not imply the exchange law in Lemma 1, reproducing the result of Concurrent Kleene algebra [14, Definition 6.8.] in our iposets algebra. Further, we derived an equational theory of iposets bisemiring in Proposition 5. These results of iposets manifest promising algebraic property to adopt iposets as a language model for concurrency modelling such as Concurrent Kleene algebra and Higher dimensional automata [31].

The main contribution of this thesis are the methods and results listed in the theory of iposets under subsumption order in Section 4.4. The structured theory of iposets under subsumption order have been rigorously derived, and thereby, we have postulated that the algebraic properties of iposets under subsumption are rich enough to generalise a weak class of iposets language. The language theory under subsumption order is one of the important techniques for cutting down the search space in an automated theorem prover. The methods for writing efficient subsumption procedures pivot on unit clauses assumption. The non-unit subsumption tends to slow down the proof resolution as the complexity increases with non-unit subsumption computation [28]. However, our proofs for the iposets under subsumption order are not necessarily given based on the assumption of singleton (unit) iposets subsumption unless we are explicitly dealing with singleton iposets. They are derived by considering both singleton and non-singleton iposets subsumption clauses. Therefore, the proofs of iposets under subsumption orders are open for more complex iposet languages theory compared to the restricted class of iposets language given by the hierarchy in Definition 40.

In Chapter 5, we axiomatised domain operations for iposet languages. They generalise a missing link between the algebraic model of iposet languages and their relational semantics. Furthermore, domain axiomatisation of iposets leads to a simple algebraic approach to modal operators based on equational reasoning. In program analysis and verification tasks, modal operators are known to be suitable for automated reasoning [8]. Therefore, these preliminary results on domain definitions of iposet languages might provide a uniform hierarchical and modular reasoning framework for concurrency reasoning.

6.1 Future work

We project fundamental theory of iposets in the trajectory of a language model for Higher dimension automaton [31] and Concurrent Kleene algebra. We set our motivation for the free model of Concurrent Kleene algebra inspired by recent literature [17, 23, 24] in that direction. We also seek for an operational model of iposets language that allows a decision procedure for language equivalence inspired by recent literature [18].

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