Representations of reflection groups

Lectures by R. Carter

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Notes by M. Yotov and I. Slungaard

Edited by K. Ranestad

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Lecture 1

Representation of finite groups

Let G be a finite group, and k be a field. The mapping $\rho: G \to GL_n(k)$ of G into the group of all nonsingular $n \times n$-matrices over k is called a representation of G if it is a group homomorphism:

$$x \in G \mapsto \rho(x) \in GL_n(k)$$
$$\forall x, y \in G \quad \rho(xy) = \rho(x)\rho(y).$$

A vector space V (over k) is called a *G-module*, if there is an action of G on V:

$$V \times G \to V$$

$(v, x) \mapsto vx$ with the properties

1) $v \mapsto vx$ is a linear map $V \to V$ for all $x \in G$
2) $v(xy) = (vx)y \ \forall x, y \in G$
3) $v1 = v \ \forall v \in V$ (the so-called "nonsingularity of the action")

There is a natural bijection between representations of G and G-modules. Indeed, let V be a G-module, $\dim_k V = n$; let $\{e_1, \ldots, e_n\}$ be a basis of V (over k). We have

$$e_i x = \sum_{i=1}^{n} \rho_{ij}(x) e_j \qquad i = 1, \ldots, n$$

Consider the mapping $\rho: G \mapsto GL(V)$, $x \mapsto \rho(x) = (\rho_{ij}(x))$. It is easy to check that ρ is a representation of G. Obviously, any representation $\rho: G \to GL(V)$ provides V with a structure of a G-module.

Let $\rho: G \to GL(V)$ be a representation and $\dim V = n$; any representation of the form

$$\rho'(x) = T\rho(x)T^{-1} \qquad \text{for } T \in GL(n, K) \ \forall x \in G$$

is called an *equivalent representation*.

Fact. There is 1–1 correspondence between G-modules (up to isomorphism) and classes of equivalent representations.

G-submodules

First basic theorem (Maschke): Let k be a field ($\mathrm{char}\, k = 0$ or $\mathrm{char}(k)$ does not divide $|G|$). Let V be a finite dimensional G-module over k, and W a G-submodule. Then there exists a submodule W' of V such that

$$V = W \oplus W'$$

(This theorem can fail if $\mathrm{char}(k)$ divides $|G|$.) □

A G-module V is called *irreducible* if its only submodules are V and $\{0\}$.

Corollary. Let V be a finite dimensional G-module over k, where $\mathrm{char}(k)$ does not divide $|G|$. Then there exist irreducible G-submodules $V_1, \ldots, V_k$ of V:

$$V = V_1 \oplus \cdots \oplus V_k.$$

□

We'll consider irreducible modules only. More precisely, we'll consider finite dimensional irreducible G-modules over $\mathbf{C}$.

Let $x, y \in G$; x is called *conjugate* to y if for some $g \in G$ we have $y = g^{-1}xg$. Conjugacy is an equivalence relation on G. So

$$G = \coprod \{\text{Conjugacy classes of } G\}.$$

Second basic theorem. The number of inequivalent irreducible representations of G is equal to the number of the conjugacy classes in the group. □

Character of a representation. Let $g : G \to GL(V)$ be a representation. The mapping $\chi : G \to \mathbf{C}$ given by $\chi(x) = \mathrm{trace}\, \rho(x)$, $x \in G$, is called the character of ρ.

- Characters are class functions, i.e.

  $\chi(x) = \chi(y)$, whenever x and y are conjugate.

- Equivalent representation have the same character:

  $\rho'(x) = T\rho(x)T^{-1} \Rightarrow \chi'(x) = \chi(x).$ □

Let $G = K_1 \coprod K_2 \coprod \ldots \coprod K_r$ be the decomposition of G into a union of its conjugacy classes (by definition $K_1 = \{1\}$).

The "second basic theorem" implies that there are exactly $\mathbf{r}$ inequivalent irreducible representations $\rho^1, \ldots, \rho^r$. Let $\chi^1, \ldots, \chi^r$ be the corresponding characters.

Example. The *unit representation* $x \mapsto (1) \in GL(\mathbf{C})$ is irreducible, and $\chi(x) = 1 \quad \forall x \in G$; $\chi^1 := \chi$ is called the *unit character* of G.

Since characters are class functions $\chi^i(x)$ depends only on the conjugacy class of x, so we may define

$\chi^i_j := \chi^i(x)$ for any $x \in K_j$.

The matrix (χ^i_j) is called the *character table* of G:

$$\begin{array}{c|c} & \begin{array}{cccc} K_1 & K_2 & \cdots & K_r \end{array} \\ \begin{array}{c} \chi^1 \\ \chi^2 \\ \vdots \\ \chi^r \end{array} & \begin{pmatrix} d_1 = 1 & 1 & \cdots & 1 \\ d_2 & & & \\ \vdots & & \chi^i_j & \\ d_r & & & \end{pmatrix} \end{array}$$

We have: $d_i = \chi^i(1) = \operatorname{trace} \rho^i(1) = \operatorname{trace} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in GL_{d_i}(\mathbf{C})$. The dimension d_i is called the *degree* of the representation ρ^i.

Example. Let $G = C_n$ be the cyclic group with n-elements. G is abelian so the conjugacy classes are the single elements of G:

$$G = \langle a \rangle \Rightarrow K_i = \{a^{i-1}\}, \qquad i = 1, \ldots, n \ .$$

Let $\omega = e^{2\pi i/n}$ be a primitive nth root of unity. Define $\rho^i : G \to GL_1(\mathbf{C}) = \mathbf{C}^*$ to be the representation

$$\rho^i(a) = \omega^i, \qquad i = 0, \ldots, n-1 \ .$$

It is obvious that $\rho^0, \rho^1, \ldots, \rho^{n-1}$ are not equivalent, and are irreducible. Hence they are all the irreducible representations of G. So we have the character table of $G = C_n$:

$$\begin{array}{c|c} & \begin{array}{ccccc} \{1\} & \{a\} & \{a^2\} & \cdots & \{a^{n-1}\} \end{array} \\ \begin{array}{c} \chi^1 \\ \chi^2 \\ \vdots \\ \chi^n \end{array} & \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{n-2} & \cdots & \omega \end{pmatrix} \end{array}$$

□

Let ϕ and ψ be class functions on G. Define the scalar product

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{x \in G} \phi(x) \, \overline{\psi(x)} \ .$$

Third basic theorem. For the characters $\{\chi^i\}$ of G we have $\langle \chi^i, \chi^j \rangle = \delta_{ij}$ (the Kronecker symbol). □

Some consequences. Let V be a G-module. Maschke's theorem says that $V = V_1 \oplus \cdots \oplus V_k$, where $V_1, \ldots, V_k$ are irreducible G-submodules of V. Let χ_i be the character of V_i, $i = 1 - n$. Hence the character χ of V is the sum of χ_i:

$$\chi = \chi_1 + \cdots + \chi_k \ .$$

Hence each character of G has the form

$$\chi = m_1\chi^1 + \cdots + m_r\chi^r, \qquad m_i \in \mathbf{Z},\ m_i \geq 0,\ i = 1,\ldots,r,$$

where $\chi^1,\ldots,\chi^r$ are all the irreducible characters of G. By the "third basic theorem" we get $m_i = \langle\chi,\chi^i\rangle$, $i = 1,\ldots,r$; thus $\langle\chi,\chi\rangle = m_1^2 + \cdots + m_r^2$, and

$$\boxed{\langle\chi,\chi\rangle = 1 \Leftrightarrow \chi \text{ is irreducible}}$$

□

Corollary (orthogonality relations). a) Row relations

$$\sum_{s=1}^{r} |K_s|\chi_s^i\overline{\chi_s^j} = \delta_{ij}|G|$$

b) Column relations

$$\sum_{i=1}^{r} \chi_j^i\overline{\chi_k^i} = \delta_{jk}\frac{|G|}{|K_j|}$$

Corollary. $\sum_{i=1}^{r} d_i^2 = |G|$.

Proof. Put $j = k = 1$ in b) above, and use $|K_1| = 1$. □

Example. $G = D_6$ – the dihedral group with six elements.

$$D_6 = \langle a, b \mid a^3 = 1, b^2 = 1, bab = a^2\rangle = \{1, a, a^2, ab, a^2b, b\}$$

The conjugacy classes of D_6 are: $\{1\}, \{a, a^2\}, \{b, ab, a^2b\}$

The 1-dimensional representations of D_6 are given by the formulas:

$$a \mapsto (\alpha), b \mapsto (\beta), \alpha^3 = 1, \beta^2 = 1, \beta\alpha\beta = \alpha^2 \Rightarrow \alpha = 1, \beta = \pm 1.$$

$(\alpha,\beta) = (1,1)$ corresponds to the unit representation so we have ρ^1 and ρ^2 – two of the three irreducible representations of D_6. We can find the character table of D_6 by using the (column) orthogonality relations: $\chi^1 = (1,1,1)$, $\chi^2 = (1,1,-1)$, $d_1 = d_2 = 1$. $|G| = 6 = 1^2 + 1^2 + d_3^2 \Rightarrow d_3 = 2$. We have

$$\begin{array}{cc} & \begin{array}{ccc} (1) & (a,a^2) & (b,ab,a^2b) \end{array} \\ \begin{array}{c} \chi^1 \\ \chi^2 \\ \chi^3 \end{array} & \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{array}\right) \end{array}$$

We'll see below the irreducible representation to which χ^3 corresponds.

Induced representations

Let G be a finite group, and $H \subset G$ – a subgroup. Each G-module V is an H-module in an obvious way. We are going to construct a G-module provided that an H-module is given. Let $|G : H| = m = \frac{|G|}{|H|}$. Then there are m right cosets of H in G:

$$G = Hg_1\coprod Hg_2\coprod\ldots\coprod Hg_m, \quad g_1,\ldots,g_m \text{ – right coset representatives.}$$

Let $\sigma : H \to GL(W)$ be a representation of H of degree r. We define a representation $\rho : G \to GL(W^{\oplus m})$ (of degree mr) in the following way:

1. $\forall x \in G$, $\rho(x)$ is a matrix with m^2 blocks of $r \times r$ matrices
2. Given j, we have $g_j x \in G$ so $\exists! \ell : g_j x \in H g_\ell$. Thus $g_j x g_\ell^{-1} \in H$ and $\sigma(g_j x g_\ell^{-1}) \in GL(W)$.
3. By definition, the j-th row of $\rho(X)$ consists of zero-matrix blocks outside the $(j\ell)$th block, which is $\sigma(g_j x g_\ell^{-1})$.

$$\rho(x) = \begin{array}{r} \\ \\ j \rightarrow \\ \\ \end{array} \left(\begin{array}{ccccc} & & \vdots & & \\ & & \vdots & & \\ \hline 0_{r,r} & \cdots & \sigma(g_j x g_\ell^{-1}) & \cdots & 0_{r,r} \\ \hline & & \vdots & & \end{array} \right)$$

It is not difficult to show that ρ is in fact a representation of G – the induced representation of G.

Example. $G = D_6$ as above, and $H = \langle a \rangle \subset G$. Let $\sigma : H \to \mathbf{C}^*$ be a 1-dimensional representation of H defined by

$$\sigma(a) = (\omega) \in GL(\mathbf{C}), \qquad \omega = e^{2\pi i/3}.$$

$|G : H| = 2 \Rightarrow G = H1 \coprod Hb$ ($g_1 = 1$, $g_2 = b$).

Let $\rho : G \to GL(\mathbf{C}^2)$ be the induced representation of G. $1 \cdot a \cdot 1^{-1} = a \in H$ and $bab^{-1} = a^2 \in H$ so

$$\rho(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

$1 \cdot b \cdot b^{-1} = 1 \in H$ and $b \cdot b \cdot 1^{-1} = 1 \in H$ so

$$\rho(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, $\rho(a)$ and $\rho(b)$ determine the representation ρ.

Let χ be the character of ρ. We have $\chi(1) = 2$ (the degree of p), $\chi(a) = \operatorname{tr} \rho(a) = \omega + \omega^2 = -1$, and $\chi(b) = \operatorname{tr} \rho(b) = 0$. Since $1, a$, and b represent the conjugacy classes of D_6, we have that χ is given by $(2, -1, 0)$ and this is precisely the character χ^2 from the previous example. Therefore ρ is irreducible – the third irreducible representation of D_6. This representation $\rho = \rho^3$ is a complex reflection representation of D_6. □

The character of the induced representation

$G = Hg_1 \coprod \ldots \coprod Hg_r$, let $\sigma : H \to GL(V)$ be representation with character ϕ. Let χ be the character of the induced representation of G (χ is called an induced character of G). Then obviously

$$\chi(x) = \sum_{j : g_j x g_j^{-1} \in H} \phi(g_j x g_j^{-1}).$$

Here are a couple of other formulae for the induced character. Note that if $g = hg_i$, $h \in H$, then $gxg^{-1} \in H \Leftrightarrow g_j x g_j^{-1} \in H$.

First formula (very useful in theory)

$$\chi(x) = \frac{1}{|H|} \cdot \sum_{\substack{g \in G \\ gxg^{-1} \in H}} \phi(gxg^{-1})$$

□

Second formula (very useful in practice)

$$\chi(x) = \frac{|G:H|}{|K_j|} \sum_{g \in K_j \cap H} \phi(g) \; ,$$

where $x \in K_j$ (a conjugacy class of G). □

Lecture 2

The Symmetric groups

Let S_n be the symmetric group on n letters. If $\sigma \in S_n$, $r \in \{1, \ldots, n\}$, then $r\sigma$ denotes the image of r under σ. Multiplication in S_n is performed from left to the right. Every permutation $\sigma \in S_n$ can be written as a product of disjoint cycles:

$$\sigma = (i_1, \ldots, i_{k_1})(j_1, \ldots, j_{k_2}) \ldots (\ell_1, \ldots, \ell_{k_r}).$$

In this case we say that σ has the cycle type $(k_1 k_2 \ldots k_r)$.

Proposition. Two permutations are conjugate (in S_n) iff they have the same cycle type.

□

Partitions. A partition of n is a set of numbers $\{\lambda_1, \ldots, \lambda_r\}$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0, \quad \text{and} \quad \lambda_1 + \cdots + \lambda_r = n.$$

The mapping $p : \mathbf{N} \to \mathbf{N}$

$$p(n) = \#\{\text{partitions of } n\}$$

is called the partition function. Obviously, the number of the conjugacy classes of S_n is exactly $p(n)$.

Example. The values of $p(n)$ for small n:

n	1	2	3	4	5	6	7	...
$p(n)$	1	2	3	5	7	11	15	...

□

If λ is a partition of n, then we write "$\lambda \vdash n$".

If $\lambda = (\lambda_1, \ldots, \lambda_r)$, we write $\lambda_{r+1} = \lambda_{r+2} = \cdots = 0$.

Our purpose is to write down the character table of S_n.

Young subgroups of S_n. Let $\lambda \vdash n$; let $\{1,\dots,n\} = I_1 \coprod I_2 \coprod \dots \coprod I_r$ and $\#\{I_j\} = \lambda_j$, $j = 1,\dots,r$. Consider the subgroup of S_n defined as follows:

$$S_\lambda := \{\sigma \in S_n \mid i \in I_k \Rightarrow i\sigma \in I_k, k = 1,\dots,r\}.$$

Obviously, $S_\lambda \cong S_{\lambda_1} \times \dots \times S_{\lambda_r}$, and is called a Young subgroup of S_n of type λ. As a subgroup of S_n, S_λ depends on the sets $I_1,\dots,I_k$. But all Young subgroups of type λ in S_n are conjugate.

For $\lambda \vdash n$, and S_λ a Young subgroup of S_n of type λ, let 1_{S_λ} be the unit character of S_λ, and let $\xi^\lambda = (1_{S_\lambda})^{S_n}$ be the corresponding induced character of S_n. Since any two Young subgroups of type λ are conjugate, ξ^λ does not depend on the particular choice of S_λ.

We'll see further that the induced characters are enough for writing down the character table of S_n.

Examples We will show now how to find the character tables of S_1, S_2 and S_3 (which are already known) by using the induced characters.

$S_1 = \langle e \rangle$: $\Rightarrow \xi^1 = \chi^1$

$S_2 \cong \mathbf{Z}_2$: The conjugacy classes are

$$\begin{array}{ccccc} S_2 = & \langle 1 \rangle & \coprod & \langle -1 \rangle, & 1 \text{ has cycle type (1,1):} \quad (1)(2) = e \\ & \| & & \| & \\ & K_{11} & & K_2 & -1 \text{ has a cycle type (2):} \quad (12) = -1 \end{array}$$

ξ^2 is the unit character of S_2 ($\Rightarrow$ irreducible), while ξ^{11} is a character of a representation of degree 2, and $\xi^{11}((12)) = 0$, so the induced characters are: $\begin{array}{c|cc} & K_{11} & K_2 \\ \hline \xi^2 & 1 & 1 \\ \xi^{11} & 2 & 0 \end{array}$ We know that $\langle \xi^{11}, \xi^2 \rangle$ is the multiplicity of ξ^2 as a component of ξ^{11};

$$\langle \xi^{11}, \xi^2 \rangle = \frac{1 \cdot 2 + 1 \cdot 0}{|S_2|} = 1.$$

Hence, $\xi^{11} - \xi^2$ is a character of S_2. But it has degree 1 and so is irreducible. Therefore the character table of S_2 is the following:

$$\begin{array}{c|cc} & K_{11} & K_2 \\ \hline \xi^2 & 1 & 1 \\ \xi^{11} - \xi^2 & 1 & -1 \end{array} \qquad \xi^2 = \chi^1, \xi^{11} - \xi^2 = \chi^2$$

(Compare with the previous example for the character tables of C_n, $n = 2$).

S_3 The conjugacy classes are

$$K_{111} = \{e\}, K_{21} = \{(12)(3), (13)(2), (23)(1)\} \quad \text{and} \quad K_3 = \{(123), (132)\}.$$

For the Young subgroups we can take

$$\langle e \rangle, \mathbf{Z}_2 \times \mathbf{Z}_1 \subset S_3 \quad \text{and} \quad \mathbf{Z}_3 \cong A_3 \subset S_3.$$

It is easy to see that the table for ξ^3, ξ^{21}, and ξ^{111} is as follows

$$\begin{array}{c} \\ \xi^3 \\ \xi^{21} \\ \xi^{111} \end{array} \begin{array}{c} \begin{array}{ccc} K_{111} & K_{21} & K_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 6 & 0 & 0 \end{pmatrix} \end{array}.$$

As before, we see that $\langle \xi^3, \xi^{21} \rangle = 1$ so $\xi^{21} - \xi^3$ is a character of S_3. Moreover $\langle \xi^{21} - \xi^3, \xi^{21} - \xi^3 \rangle = 1$, so it is irreducible. Further, $\langle \xi^{111}, \xi^3 \rangle = 1$ hence $\xi^{111} - \xi^3$ is a character of S_3, also $\langle \xi^{111}, \xi^{21} - \xi^3 \rangle = 2$ hence $\xi^{111} - \xi^3 - 2(\xi^{21} - \xi^3)$ is a character too. $\xi^{111} - 2\xi^{21} + \xi^3$ is of degree 1, so it is irreducible, and the character table of S_3 is as follows

$$\begin{array}{c} \\ \xi^3 \\ \xi^{21} - \xi^3 \\ \xi^{111} - 2\xi^{21} + \xi^3 \end{array} \begin{array}{c} \begin{array}{ccc} K_{111} & K_{21} & K_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} \end{array}$$

If we compare with the example about $D_6 (\cong S_3)$ we'll see that $\xi^3 = \chi^1$, $\xi^{21} - \xi^3 = \chi^3$, $\xi^{111} - 2\xi^{21} + \xi^3 = \chi^2$. □

In general, ξ^n is the unit character of S_n. S_n has a second (distinguished) character of degree 1 – "the sign character" ε:

$$\begin{aligned} &\varepsilon(\sigma) = 1 \quad \text{for even } \sigma \in S_n \\ &\varepsilon(\tau) = -1 \quad \text{for odd } \tau \in S_n. \end{aligned}$$

(In fact, ε is a representation, the sign representation, of degree 1; hence it coincides with its character). By using the sign representation we define a new series of characters of S_n: for any $\lambda \vdash n$

$$\eta^\lambda := (\varepsilon_{S_\lambda})^{S_n}.$$

There is a close relation between ξ^λ and η^μ, $\lambda \vdash n$, $\mu \vdash n$.

Example. $G = S_3$. It is straightforward that

$$\begin{array}{c} \\ \eta^3 \\ \eta^{21} \\ \eta^{111} \end{array} \begin{array}{c} \begin{array}{ccc} K_{111} & K_{21} & K_3 \end{array} \\ \begin{pmatrix} 1 & -1 & 1 \\ 3 & -1 & 0 \\ 6 & 0 & 0 \end{pmatrix} \end{array}$$

And one can check that

$$\begin{cases} \langle \xi^3, \eta^{111} \rangle = 1 \\ \langle \xi^{21}, \eta^{21} \rangle = 1 \\ \langle \xi^{111}, \eta^3 \rangle = 1. \end{cases}$$

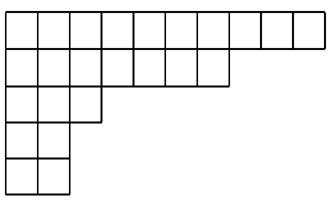
□

The Young diagram of a partition

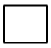
Let $\lambda \vdash n$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. The Young diagram of λ has

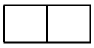
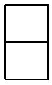
$$\begin{cases} \lambda_1 & \text{boxes in row 1} \\ \lambda_2 & \text{boxes in row 2} \\ & \\ \lambda_r & \text{boxes in row r} \end{cases}$$

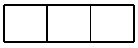
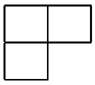
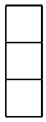
lined up on the left



$\leftrightarrow (10, 7, 3, 2, 2)$

Examples
$\lambda = (1)$ 

$\lambda = (2)$  , $\lambda = (1,1)$ 

$\lambda = (3)$  , $\lambda = (2,1)$  , $\lambda = (1,1,1)$ 

$\square$

For each partition $\lambda \vdash n$ there is a conjugate (or dual) partition λ'. The dual partition λ' is determined by λ in the following way: the Young diagram of λ' is obtained from that of λ by interchanging rows and columns.

Examples. $\lambda = (2) \Rightarrow \lambda' = (1,1)$
$\lambda = (3) \Rightarrow \lambda' = (1,1,1)$; $\lambda = (2,1) \Rightarrow \lambda' = \lambda$, $\square$

Theorem. Let $\lambda \vdash n$. Then the scalar product of ξ^λ and $\eta^{\lambda'}$ is equal to 1: $\langle \xi^\lambda, \eta^{\lambda'} \rangle = 1$. $\square$

Remark. In the proof of the last theorem one can use the following fact:

Frobenius reciprocity. Let G be a finite group. $H \subset G$ – a subgroup. Let χ be a character of G, χ_H being its restriction to H; let ϕ be a character of H, ϕ^G being the corresponding induced character of G. Then,

$$\langle \chi, \phi^G \rangle_G = \langle \chi_H, \phi \rangle_H \ .$$

$\square$

Corollary. Let $\lambda \vdash n$. Then there is just one irreducible character χ^λ of S_n which is a component of ξ^λ and $\eta^{\lambda'}$. It has multiplicity one in both of them.

Proof. Let $\chi^1, \dots, \chi^r$ be the irreducible characters of S_n.
Then $\xi^\lambda = m_1\chi^1 + \cdots + m_r\chi^r$, $\eta^{\lambda'} = n_1\chi^1 + \cdots + n_r\chi^r$ and

$$\langle \xi^\lambda, \eta^{\lambda'} \rangle = m_1 n_1 + \cdots + m_r n_r = 1.$$

(according to the theorem above). But $m_i, n_i \in \mathbf{Z}$, $m_i, n_i \geq 0$ so $\exists! m_i n_i = 1$ and all $m_j n_j = 0$ for $j \neq i$. $\square$

Theorem. If $\lambda \vdash n$, $\mu \vdash n$, $\lambda \neq \mu$, then $\chi^\lambda \neq \chi^\mu$. $\square$

Corollary. The characters $\{\chi^\lambda \mid \lambda \vdash n\}$ form a complete set of all irreducible characters of S_n. $\square$

We know that any character of S_n can be expressed as an integral combination of the charaters $\{\chi^\lambda \mid \lambda \vdash n\}$. In fact, any χ^λ can be expressed as a (special type) combination of the characters $\{\xi^\lambda \mid \lambda \vdash n\}$.

A partial ordering of the set of partitions of n

Let $\lambda \vdash n$, $\mu \vdash n$, then $\lambda \geq \mu$ if

$$\begin{array}{ll} \lambda_1 \geq \mu_1 & \\ \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2 & \qquad \text{(we say: ``}\lambda\text{ dominates }\mu\text{'').} \\ \lambda_1 + \cdots + \lambda_r \geq \mu_1 + \cdots + \mu_r & \end{array}$$

Examples: 1) $n = 3 \Rightarrow 3 > 21 > 111$, and $>$ is a total ordering

2) $n = 6$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & 33 \arrow[dr, dash] & & 3111 \arrow[dr, dash] & & & \\
6 \arrow[r, dash] & 51 \arrow[r, dash] & 42 \arrow[ur, dash] \arrow[dr, dash] & & 321 \arrow[ur, dash] \arrow[dr, dash] & & 2211 \arrow[r, dash] & 21111 \arrow[r, dash] & 111111 \\
 & & & 411 \arrow[ur, dash] & & 222 \arrow[ur, dash] & & &
\end{tikzcd}$$

□

Proposition. Let $\lambda \vdash n$. Then for certain $d_{\lambda\mu} \in \mathbf{Z}$ we have

$$\chi^\lambda = \xi^\lambda + \sum_{\mu > \lambda} d_{\lambda\mu} \xi^\mu$$

□

To find $d_{\lambda\mu}$ from the proposition above we use *Giambelli's formula*

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\lambda \vdash n$. Then

$$\chi^\lambda = \begin{vmatrix} \xi^{\lambda_1} & \xi^{\lambda_1+1} & \ldots & \xi^{\lambda_1+r-1} \\ \xi^{\lambda_2-1} & \xi^{\lambda_2} & \ldots & \xi^{\lambda_2+r-2} \\ \ldots & \ldots & \ldots & \ldots \\ \xi^{\lambda_r-r+1} & \xi^{\lambda_r-r+2} & \ldots & \xi^{\lambda_r} \end{vmatrix}$$

where $\xi^0 = 1$, $\xi^{-i} = 0$ for $i > 0$, and $\xi^{i_1}\xi^{i_2} \ldots \xi^{i_r}$ is interpreted as $\xi^{i_1 \ldots i_r}$ □

Examples 1) $S_1 : \chi^1 = |\xi^1| = \xi^1$.

$$\begin{aligned} S_2 : \quad & \chi^2 = |\xi^2| = \xi^2 \ , \\ & \chi^{11} = \begin{vmatrix} \xi^1 & \xi^2 \\ \xi^0 & \xi^1 \end{vmatrix} = \begin{vmatrix} \xi^1 & \xi^2 \\ 1 & \xi^1 \end{vmatrix} = \xi^{11} - \xi^2. \\ S_3 : \quad & \chi^3 = |\xi^3| = \xi^3 \ , \\ & \chi^{21} = \begin{vmatrix} \xi^2 & \xi^3 \\ \xi^0 & \xi^1 \end{vmatrix} = \begin{vmatrix} \xi^2 & \xi^3 \\ 1 & \xi^1 \end{vmatrix} = \xi^{21} - \xi^3 \\ & \chi^{111} = \begin{vmatrix} \xi^1 & \xi^2 & \xi^3 \\ \xi^0 & \xi^1 & \xi^2 \\ \xi^{-1} & \xi^0 & \xi^1 \end{vmatrix} = \begin{vmatrix} \xi^1 & \xi^2 & \xi^3 \\ 1 & \xi^1 & \xi^2 \\ 0 & 1 & \xi^1 \end{vmatrix} = \xi^{111} + \xi^3 - 2\xi^{21} \end{aligned}$$

□

As usual, by d_λ we denote the degree of the irreducible representation ρ^λ (with character χ^λ). Hence, $d_\lambda = \chi^\lambda(1)$. How can we calculate $\{d_\lambda \mid \lambda \vdash n\}$?

Young tableau Let $\lambda \vdash n$. A Young tableau of type λ is the Young diagram of λ with a number from the set $\{1, \ldots, n\}$ in each box.

A standard tableau – this is a Young tableau with entries $1, \ldots, n$ (once each), increasing along rows and down columns.

Examples

$n = 1:$ $\begin{array}{|c|}\hline 1 \\ \hline\end{array}$

$n = 2:$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$

$n = 3:$ $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$

$n = 4:$ $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$

$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$

□

The first degree formula: $d_\lambda = \#\{\text{standard } \lambda\text{-tableaux}\}$. □

Examples S_2: $d_2 = 1$, $d_{11} = 1$
S_3: $d_3 = 1$, $d_{21} = 2$, $d_{111} = 1$
S_4: $d_4 = 1$, $d_{31} = 3$, $d_{22} = 2$, $d_{211} = 3$, $d_{1111} = 1$.

□

Remark. It is obvious also that $|S_2| = 1^2 + 1^2 = d_2^2 + d_{11}^2$, $|S_3| = d_3^2 + d_{21}^2 + d_{111}^2$, $|S_4| = d_4^2 + d_{31}^3 + d_{22}^2 + d_{211}^2 + d_{1111}^2$.

We'll see that this fact is true in general: $n! = \sum\limits_{\lambda \vdash n} d_\lambda^2$. □

Lecture 3

Polynomial invariants of finite reflection groups

Let V be n-dimensional Euclidean space, $\dim_{\mathbf{R}} V = \ell$, with a positive definite scalar product: $u, v \mapsto \langle u, v\rangle \in \mathbf{R}$.

Let $v \in V$, $v \neq 0$, then $H_v := \{x \in V \mid \langle v, x\rangle = 0\}$ is called the hyperplane, orthogonal to v ($H_v \perp v$).

Let $s_v : V \to V$ be the reflection in H_v:

$$s_v(x) = x - 2\frac{\langle v, x\rangle}{\langle v, v\rangle} \cdot v \ .$$

Let W be a finite group generated by reflections, hence $W \subset \operatorname{Aut} V$.

Choose a basis $\{e_1, \ldots, e_\ell\}$ of V; $\forall x \in V, \exists!(x_1, \ldots, x_\ell) \in \mathbf{R}^\ell$

$$x = x_1 e_1 + \cdots + x_\ell e_\ell$$

Let $\mathbf{R}[x_1, \ldots, x_\ell]$ be the polynomial ring of ℓ variables $\{x_1, \ldots, x_\ell\}$. Let $P \in \mathbf{R}[x_1, \ldots, x_\ell]$, then P determines a map $P : V \to \mathbf{R}$ which is a polynomial function on V.

We denote by S the set (algebra) of all polynomial functions on V. Then, W acts on S:

$$P \in S, \quad w \in W \mapsto w(P) \in S, \text{s.t.} \quad w(P)(x) = P(w^{-1}(x)).$$

Thus the action is a left action.

The element $P \in S$ is called an *invariant* of W if

$$w(P) = P \quad \forall w \in W.$$

Denote by J the set of polynomial invariants of W, then J is a subalgebra of S.

Example. $W = S_{\ell+1}$; W acts on the Euclidean space $V' = \{x_1 e_1 + \cdots + x_{\ell+1} e_{\ell+1}\}$ by permuting the coordinates. W does not act faithfully on V' (the element $e_1 + \cdots + e_{n+1}$ is W-invariant). Maschke: $\exists V$ – a complementary invariant subspace in $V' : V' = V \oplus \langle e_1 + \cdots + e_{\ell+1}\rangle\mathbf{R}$.

$V = \{v' \in V' \mid x_1 + \cdots + x_{\ell+1} = 0\}$.

On V ($\dim_{\mathbf{R}} V = \ell$) the group $W(= S_{\ell+1})$ acts faithfully. $S_{\ell+1}$ is generated by the transpositions $(12), (23), \ldots, (\ell, \ell+1)$, which are reflections on V, thus W (acting on V) is a reflection group.

Remark. Suppose $e_1, \ldots, e_{\ell+1}$ is an orthonormal basis of V'; let $v_i = e_i - e_{i+1}$, $i = 1, \ldots, \ell$; then $H_{v_i} \subset V$ is given by

$$H_{v_i} = \Big\{x_1 e_1 + \cdots + x_{\ell+1} e_{\ell+1} \Big| \underbrace{x_1 + \cdots + x_{\ell+1} = 0}_{\Rightarrow \in V}, \underbrace{x_i = x_{i+1}}_{\Rightarrow \perp v_i}\Big\}.$$

Hence, $s_i = (i, i+1)$ is the reflection in H_{v_i}. □

The invariants of $W(= S_{\ell+1})$ are all symmetric polynomials in $\ell+1$ variables, $x_1, \ldots, x_{\ell+1}$, subject to the relation

$$x_1 + \cdots + x_{\ell+1} = 0 \ .$$

Hence, it is isomorphic to $R[T_1, \ldots, T_\ell]$ – the ring of polynomials of ℓ variables. □

Main Theorem (Chevalley). For any finite group generated by reflections, W, the ring $\mathcal{J}$ of polynomial invariants is isomorphic to a polynomial ring in ℓ variables ($\dim_{\mathbf{R}} V = \ell$).

Hint to the proof. First we define the averaging operator

$$A_v \colon S \to \mathcal{J} \ , \qquad A_v^2 = A_v \ ;$$

$P \in S$, $A_v(P) := \frac{1}{|W|} \sum\limits_{w \in W} w(P)$.

Set $S^+ = \{P \in S \mid P(0,\ldots,0) = 0\}$, $\mathcal{J}^+ = \mathcal{J} \cap S^+$, and let

$$S\ \mathcal{J}^+ = \{P_1 J_1 + \cdots + P_k J_k \mid P_i \in S,\ J_i \in \mathcal{J}^+,\ k \in \mathbf{N}\} \ .$$

Since the action of W does not change the degrees of the monomials, the ideal $\mathcal{J}^+$ is homogeneous; hence $S\ \mathcal{J}^+$ is homogeneous ideal in S. By the Hilbert basis theorem there exists a finite set of homogeneous elements of $\mathcal{J}^+$ which generate $S\ \mathcal{J}^+$: $I_1, \ldots, I_n \in \mathcal{J}^+$ – homogeneous of positive degree, and $S\ \mathcal{J}^+ = \langle I_1, \ldots, I_n \rangle S$. Without loss of generality we may assume, that $\{I_1, \ldots I_n\}$ is a minimal set with this property.

By induction we'll prove that $\mathcal{J} = \mathbf{R}[I_1, \ldots, I_n]$. Obviously it suffices to prove this for homogeneous elements of $J \in \mathcal{J}$.

1. *deg* $J = 0 \Rightarrow J \in \mathbf{R}[I_1, \ldots, I_n]$ (we can start off the induction)
2. *deg* $J > 0 \Rightarrow J \in \mathcal{J}^+ \Rightarrow J \in S\ \mathcal{J}^+ \Rightarrow J = P_1 I_1 + \cdots + P_n I_n$,

where $P_1, \ldots, P_n \in S$.

Now, J is homogeneous, $I_1, \ldots, I_n$ – homogeneous, so we can choose $P_1, \ldots, P_n$ homogeneous too, and *deg* P_i + *deg* I_i = *deg* J, $i = 1, \ldots, n$.

Apply the averaging operator:

$$A_v\ J = J = A_v(P_1 I_1) + \cdots + A_v(P_n I_n) \overset{\text{obviously}}{=} (A_v P_1) I_1 + \cdots + (A_v P_n) I_n \ .$$

But $A_v(P_i)$ are invariants, homogeneous, of degree less than *deg* J so the induction applies.

The argument above is applicable to any finite group acting on a k-vector space, where $\operatorname{char} k$ does not divide the order of the group W.

When W is a finite reflection group acting on an Euclidean vector space we have that $I_1, \ldots, I_n$ are algebraically independent, i.e., if $P(I_1, \ldots, I_n) = 0$ for $P \in \mathbf{R}[T_1, \ldots, T_n]$ then $P \equiv 0$ (the proof of this can be found in Bourbaki, Humphreys, Carter).

By using the Galois theory we'll prove that $n = \ell = \dim_{\mathbf{R}} V$. Let $K = \mathbf{R}(x_1, \ldots, x_\ell)$, $k = \mathbf{R}(I_1, \ldots, I_n)$; hence

$$\mathbf{R} \subset k \subset K \ , \quad \text{and} \quad \operatorname{trdeg} K/\mathbf{R} = \ell \quad \operatorname{trdeg} k/\mathbf{R} = n$$
$$\operatorname{trdeg} K/k = \operatorname{trdeg} K/\mathbf{R} - \operatorname{trdeg} k/\mathbf{R}$$

We'll prove that every x_i is algebraic over $\mathbf{k}$. Then, $\operatorname{trdeg} K/k = 0$, and $n = \ell$.

For x_i consider the polynomial $\prod\limits_{w \in W} (t - w(x_i))$. The coefficients of this polynomial lie in $\mathcal{J}$, and x_i is a root of it. Moreover, it has leading coefficient 1. Hence x_i is integrally dependent over $\mathcal{J}$, hence over k. □

A remarkable converse of Chevalley's theorem (Shephard-Todd)

Suppose V is an Euclidean space, G – any finite group of linear transformations of V, and $\dim_{\mathbf{R}} V = \ell$. Suppose the ring $\mathcal{J}$ of invariants of G is generated by ℓ algebraically independent homogeneous polynomials, $I_1, \ldots, I_\ell$. Then G is a reflection group on V. □

Remark. This theorem is true over $\mathbf{C}$ if one considers the unitary reflections, instead of reflections only.

Let W be a reflection group on V with $\dim_{\mathbf{R}} = \ell$ and $\mathcal{J} = \mathbf{R}[I_1, \ldots, I_\ell]$ its ring of invariants; I_j is homogeneous, $d_j = deg\ I_j$, $j = 1, \ldots, \ell$. The polynomials $I_1, \ldots, I_\ell$ are not determined uniquely, but $d_1, \ldots, d_\ell$ are. They are called *the degrees of* W. The numbers $m_1 = d_1 - 1, m_2 = d_2 - 1, \ldots, m_\ell = d_\ell - 1$, are called *the exponents of* W.

Example. $W = S_{\ell+1}$ acts on V, $\dim_{\mathbf{R}} V = \ell$. Then $\mathcal{J}$ is the set of symmetric polynomials in $x_1, \ldots, x_{\ell+1}$, subject to $x_1 + \cdots, x_{\ell+1} = 0$. Hence, one can take

$$
\begin{aligned}
& I_1 = x_1x_2 + x_1x_3 + \cdots \\
& I_2 = x_1x_2x_3 + x_1x_2x_4 + \cdots \\
& \cdots\cdots \\
\Rightarrow\quad & d_0 = 2,\ d_2 = 3, \ldots,\ d_\ell = \ell + 1\ ; \quad m_1 = 1,\ m_2 = 2, \ldots,\ m_\ell = \ell\ .
\end{aligned}
$$

□

Properties of the exponents of a reflection group

1) **Shephard-Todd.** Let a_i be the number of elements of W whose fixed point space in V has dimension $\ell - i$. (For example, $a_0 = 1$ since the only element whose fixed point set is V is id, while $a_1 = N$ – the number of reflections of W.) Then,

$$a_0 + a_1t + a_2t^2 + \cdots + a_\ell t^\ell = (1 + m_1t)(1 + m_2t) \cdots (1 + m_\ell t)$$

□

Corollary. $m_1 + \cdots + m_\ell = N$, $d_1d_2 \ldots d_\ell = |W|$.

For the first part we see that the coefficient at t is a_1 from the left hand side, and $m_1 + \cdots + m_\ell$ from the right hand side. For the second part put $t = 1$, then $d_1 \cdots d_\ell = a_1 + \cdots + a_\ell = |W|$. □

Example. $W = S_{\ell+1}$ acting on V, $\dim_{\mathbf{R}} V = \ell$.
$\Rightarrow d_1 \ldots d_\ell = (\ell + 1)! = |W|$, $1 + \ldots + \ell = \frac{\ell(\ell+1)}{2} = N$. □

2) **Chevalley.** Any finite reflection group W is a Coxeter group. Let $s_1, \ldots, s_\ell$ be Coxeter generators of W, $w \in W$, $\ell(w)$ = the shortest length of w as a product of $s_1, \ldots, s_\ell$, $\ell = \dim_{\mathbf{R}} V$. Then

$$\sum_{w \in W} t^{\ell(w)} = \prod_{i=1}^{\ell} (1 + t + t^2 + \cdots + t^{m_i}) = \prod_{i=1}^{\ell} \frac{t^{d_i} - 1}{t - 1}$$

□

Example. $S_3 = G$, $\dim V = 2$; $S_3 = \langle s_1, s_2 \rangle$, $s_1 = (12)$, $s_2 = (23)$
$\Rightarrow S_3 = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\}$ the lengths are $0, 1, 1, 2, 2, 3$; obviously, $1 + 2t + 2t^2 + t^3 = (1 + t)(1 + t + t^2)$, $m_1 = 1$, $m_2 = 2$. □

3) Let W act on V, $\dim_{\mathbf{R}} V = \ell$; $W = \langle s_1, \ldots, s_\ell \rangle$, where $s_1, \ldots, s_\ell$ are Coxeter generators of W. An element of W is called *Coxeter* if it is a product of $s_1, \ldots, s_\ell$ in some order. Any two Coxeter elements are conjugate.

Let h be the order of the Coxeter elements. h is called the *Coxeter number* of W.

Theorem. $h = 2N/\ell$, where $N = a_1 = \#\{\text{reflections in } W\}$. □

Example. $W = S_{\ell+1}$ acts on V, $\dim_{\mathbf{R}} V = \ell$. Coxeter generators are

$$(12), (23), \ldots, (\ell, \ell+1)$$

thus

$$(12)(23)\ldots(\ell, \ell+1) = (12\ldots\ell+1)$$

is a Coxeter element and $h = \ell + 1$. On the other hand, $N = \frac{\ell(\ell+1)}{2}$, and obviously $2N/\ell = h$. □

Consider the eigenvalues of a Coxeter element w on $X_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$.

Theorem (Coleman). Suppose the eigenvalues of a Coxeter element w on $V_{\mathbf{C}}$ are $\zeta^{m_1}, \ldots, \zeta^{m_\ell}$ $(1 \leq m_i < h)$, $\zeta = e^{2\pi i/h}$. Then, $m_1, \ldots, m_\ell$ are the exponents of W. □

(That is the reason $m_1, \ldots, m_\ell$ are called "exponents" of W.)

Example. $W = S_{\ell+1}$ acts on V, $\dim_{\mathbf{R}} V = \ell$.
$w = (12\ldots\ell+1)$ is a Coxeter element. On $V'_{\mathbf{C}}$ it has $1, \zeta, \ldots, \zeta^\ell$ as eigenvalues, so on $V_{\mathbf{C}}$ they are $\zeta, \zeta^2, \ldots, \zeta^\ell$, and we already know that $m_1 = 1, \ldots, m_\ell = \ell$. □

Suppose $I_1, \ldots, I_\ell$ are generators of $\mathcal{J}$, let $d_i = \deg I_i$ $(i = 1, \ldots, \ell)$, ℓ being the minimal possible number. $I_j \in \mathbf{R}[x_1, \ldots, x_\ell]$, so we may consider the Jacobian determinant

$$J = \det(\partial I_i / \partial x_j)\ .$$

J is a polynomial in $x_1, \ldots, x_\ell$, $\deg\ J = N = \#\{\text{reflections in } W\}$.

Theorem. Let $H_\alpha := \{x \mid \langle \alpha, x \rangle = 0\}$ be a reflecting hyperplane for W on V; let L_α be the functional $L_\alpha(x) = \langle \alpha, x \rangle$ on V. Then

$$J = \lambda \prod_\alpha L_\alpha \quad \text{for some } \lambda \in \mathbf{R}$$

□

Example. E_8

$$\begin{array}{cccccccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & | & & & & & & & & \\ & & & & \circ & & & & & & & & \end{array}$$

We already know, from the course of Vinberg, that the lattice E_8 determines a finite reflection group W, acting on $\mathbf{R}^8$. For W we have $N = 120$, hence $h = 30 = 2N/\ell$.
Furthermore, the eigenvalues of a Coxeter element of W are
$\zeta = e^{2\pi i/30}$ and its conjugates:
$\zeta, \zeta^7, \zeta^{11}, \zeta^{13}, \zeta^{17}, \zeta^{19}, \zeta^{23}, \zeta^{29}$; hence the exponents are $m_1 = 1$, $m_2 = 7$, $m_3 = 11$, $m_4 = 13$, $m_5 = 17$, $m_6 = 19$, $m_7 = 23$, $m_8 = 29$. □

Lecture 4

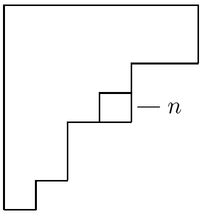
Characters of the symmetric groups

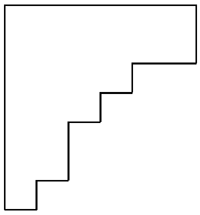
We continue the presentation of the theory of S_n from lecture two.

We are going to prove that for $\lambda \vdash n$

$$d_\lambda = \chi^\lambda(1) = \text{“Number of standard } \lambda\text{-tableaux”}.$$

Let d'_λ denote the number of standard λ-tableaux. Let P be a standard λ-tableau. By removing the box numbered with n we get Q, a standard μ-tableau, where $\mu \vdash n-1$. (This is because n lives in a corner box of P.) In this way we get Q's which differ from P only in a corner box. By adding this box to such Q we obtain a standard λ-tableau. Hence, we get that $d'_\lambda = \sum d'_\mu$, where $\mu \vdash n-1$, and μ has a Young tableau which differ from that of λ in a corner box. (In other words, if $\lambda = (\lambda_1, \dots, \lambda_r)$, $\lambda_1 \geq \dots \geq \lambda_r > 0$, then $\mu = (\mu_1, \dots, \mu_r)$ with $\mu_i = \lambda_i - 1$ for exactly one i, while $\mu_j = \lambda_j$ for $j \neq i$).

P 

 Q

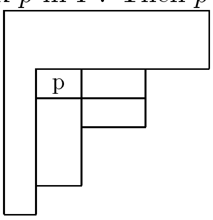
Consider the Young subgroup $S_{n-1} = S_{n-1,1}$ of S_n. If we restrict χ^λ to S_{n-1}, we'll get

$\chi^\lambda|_{S_{n-1}} = \sum \chi^\mu$, where $\mu \vdash n-1$, and μ is obtained from λ by omitting a corner box.
$d_\lambda = \chi^\lambda(1) = \sum \chi^\mu(1) = \sum d_\mu$, μ as above.
By induction we may assume $d_\mu = d'_\mu$ for $\mu \vdash n-1$, thus $d_\lambda = \sum d'_\mu = d'_\lambda$ (μ as above). $\square$

Another way of calculating d_λ is the following

Hook length formula

Let P be a λ-tableau. Fix a box p in P. Then p determines a hook in P: All boxes to the right and downward from p.



By definition

hook length := Number boxes of the hook.

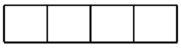
Then we have the formula: for $\lambda \vdash n$

$$d_\lambda = \frac{n!}{\prod(\text{hook lengths in the } \lambda\text{-diagram})}$$

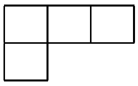
$\square$

(This formula follows from Giambelli's formula.)

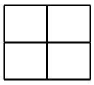
Examples

1) $\lambda = 4$  ; there are 4 hooks with lengths 1,2,3,4:

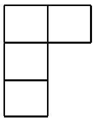
$\Rightarrow d_{(4)} = \frac{4!}{4,3,2,1} = 1$

2) $\lambda = (31)$  ; 4 hooks with lengths 1,2,4,1

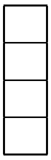
$\Rightarrow d_{(31)} = \frac{4!}{1,1,2,4} = 3$

3) $\lambda = (22)$  ; 4 hooks with lengths 1,2,2,3

$\Rightarrow d_{(22)} = \frac{4!}{2,2,3} = 2$

4) $\lambda = (211)$  ; 4 hooks with lengths 1,1,2,4

$\Rightarrow d_{(211)} = \frac{4!}{2,4} = 3$

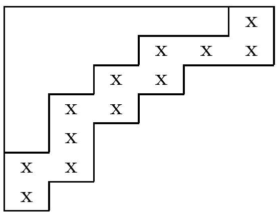
5) $\lambda = (1111)$  ; 4 hooks with lengths 1,2,3,4 $\Rightarrow d_{(1111)} = 1$.

□

Now we turn to an inductive formula for calculating the entries of the character table of S_n. ($\chi^\lambda_\mu = \chi^\lambda(x)$, $x \in K_\mu$)

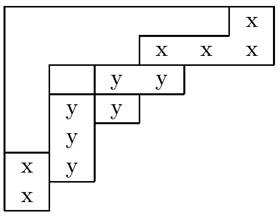
Murnaghan-Nakayama formula

Each Young diagram has its rim: the union of all boxes which have no neighbouring boxes to the right or downward from them.



The rim of P

A *removable worm* of P is a connected subset of boxes in the rim of P, s.t. when we remove the worm, we get another Young diagram. There is 1–1 correspondence between removable worms and hooks (hence of boxes of P).



A removable worm of P

Murnaghan-Nakayama's rule. Suppose $\lambda \vdash n$, $\rho \vdash n$, and one of the parts of ρ is k. Let $\rho' \vdash n-k$ be the partition obtained from ρ by removing k. Then

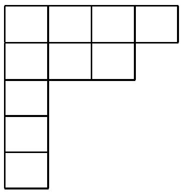
$$\chi^\lambda_\rho = \sum_\omega \varepsilon(\omega)\chi^{\lambda-\omega}_{\rho'}$$

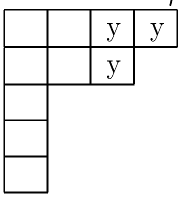
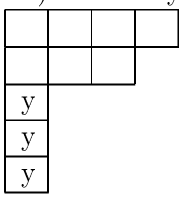
where ω is a removable worm in the λ-diagram of length k. $\lambda - \omega$ denotes the diagram obtained from that of λ by removing the worm ω, and

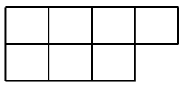
$$\varepsilon(\omega) = \begin{cases} 1 & \text{if } \omega \text{ has an odd number of } \textit{rows} \\ -1 & \text{if } \omega \text{ has an even number of } \textit{rows}. \end{cases}$$

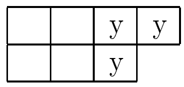
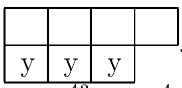
By definition, $\chi^0_0 = 1$. □

Example. By using Murnaghan-Nakayama's rule we'll calculate χ^{43111}_{3322} for S_{10}.

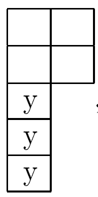
$\lambda = (43111)$ has diagram  ; $\rho = (3322)$, 3 belongs to ρ.

Remove 3 from $\rho : \rho' = (322)$. The only removable worms of length 3 on the λ-diagram are  and  so $\chi^{43111}_{3322} = \chi^{43}_{322} - \chi^{22111}_{322}$.

$\lambda' = (43)$ has diagram  ; $\rho = (322)$; remove 3 again

then $\rho' = (22)$ and there are two removable worms of length 3  and . Hence
$\chi^{43}_{322} = \chi^4_{22} - \chi^{22}_{22}$

In the same way we get

$\chi^{22111}_{322} = \chi^{22}_{22}$; the only removable worm of length 3 is  ,

so $\chi^{43111}_{3322} = \chi^4_{22} - 2\chi^{22}_{22} = \chi^2_2 - 2\chi^2_2 + 2\chi^{11}_2 = 2\chi^{11}_2 - \chi^2_2 = -3$, since we can use the character table of S_2

$$\begin{array}{cc} & \begin{array}{cc} K_{11} & K_2 \end{array} \\ \begin{array}{c} \chi^2 \\ \chi^{11} \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

□

Irreducible representations of S_n; Specht modules

Let $\lambda \vdash n$. Our next goal is to show how to obtain an irreducible representation of S_n with character χ^λ.

Consider the set of all λ-tableaux T with distinct entries from $\{1, \ldots, n\}$. For each such tableau T define the polynomial

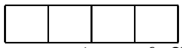
$$p_T(x_1, \ldots, x_n) := \prod (x_i - x_j), \quad \text{where } i, j \text{ are in the same column of } T, \; i\text{-appearing above } j$$

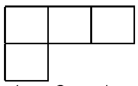
Let $V_\lambda = \{\mathbf{C}\text{-vector space, generated by } p_T \text{ for all such } T\}$. V_λ is an S_n-module: $p_T\sigma = p_{T\sigma}$ $\forall \sigma \in S_n$. V_λ is called *the Specht module* (corresponding to $\lambda \vdash n$).

Theorem. 1) The S_n module V_λ is irreducible.
2) Its character is χ^λ.
3) The polynomials $\{p_T \mid T \text{ a standard } \lambda\text{-tableau}\}$ form a basis for V_λ. □

This theorem shows how to find the irreducible representations of S_n (one for each $\lambda \vdash n$): V_λ with a basis $\{p_T \mid T \text{ a standard } \lambda\text{-tableau}\}$.

Example. Find all irreducible representations of S_4.
$p(4) = 5 : 4,\ 31,\ 22,\ 211,\ 1111$; S_4 is generated by (12),(23),(34).

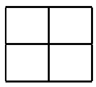
$\lambda = (4)$  ; $p_T = 1$, and this case corresponds to the unit representation of S_4.

$\lambda = (31)$  with standard λ-tableaux $\begin{array}{ccc}1 & 2 & 3\\ 4 & & \end{array}$, $\begin{array}{ccc}1 & 3 & 4\\ 2 & & \end{array}$ and $\begin{array}{ccc}1 & 2 & 4\\ 3 & & \end{array}$

$\Rightarrow V_{(31)} = \langle x_1 - x_2, x_1 - x_3, x_1 - x_4 \rangle$ – 3-dimensional representation

It suffices to find the matrices corresponding to the Coxeter generators of $S_4 : (12), (23)$ and (34) in this representation

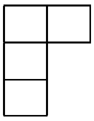
$$(12) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad (23) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (34) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$\lambda = (22)$  ; with standard λ-tableaux $\begin{array}{cc}1 & 3\\ 2 & 4\end{array}$ and $\begin{array}{cc}1 & 2\\ 3 & 4\end{array}$.

$\Rightarrow V_{(22)} = \langle (x_1 - x_3)(x_2 - x_4), (x_1 - x_2)(x_3 - x_4) \rangle$ – 2-dimensional representation

$$(12) \mapsto \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad (23) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (34) \mapsto \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

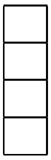
$$(x_1 - x_3)(x_2 - x_4) \overset{(12)}{\mapsto} (x_2 - x_3)(x_1 - x_4) = (x_1 - x_3)(x_2 - x_4) - (x_1 - x_2)(x_3 - x_4)$$
$$(x_1 - x_3)(x_2 - x_4) \overset{(34)}{\mapsto} (x_1 - x_4)(x_2 - x_3) = (x_1 - x_3)(x_2 - x_4) - (x_1 - x_2)(x_3 - x_4)\,.$$

$\lambda = (211)$  ; with standard λ-tableaux $\begin{array}{cc}1&4\\2&\\3&\end{array}$, $\begin{array}{cc}1&3\\2&\\4&\end{array}$, and $\begin{array}{cc}1&2\\3&\\4&\end{array}$.

$\Rightarrow V_{(211)} = \langle (x_1-x_2)(x_1-x_3)(x_2-x_3), (x_1-x_2)(x_1-x_4)(x_2-x_4), (x_1-x_3)(x_1-x_4)(x_3-x_4)\rangle$

$\Rightarrow$ gives a representation of degree 3.

$$(12) \mapsto \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad (23) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (34) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$\lambda = (1111)$  ; with the only standard tableaux $\begin{array}{c}1\\2\\3\\4\end{array}$

$\Rightarrow V_{(1111)} = \langle (x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)\rangle$
– 1-dimensional representation

$$(12) \mapsto (-1)\ , \quad (23) \mapsto (-1)\ , \quad (34) \mapsto (-1)\ .$$

This is the sign representation of S_4. $\square$

Notice that $\rho^\lambda(x)$, $\lambda \vdash n$, $x \in S_n$ have all elements in $\mathbf{Z}$. Hence the characters of S_n have integer values.

Finally, we are coming back to the relation $\sum\limits_{\lambda\vdash n} d_\lambda^2 = n!$, where $|S_n| = n!$ and $d_\lambda =$ Number of standard λ-tableaux.

We'll explain how to prove this relation without using representation theory.

Let T_λ denote the set of all standard λ-tableaux. There is a very nice map $S_n \to \coprod\limits_{\lambda\vdash n} T_\lambda$ defined as follows. Let $\sigma \in S_n$ (for example, $\sigma = \left(\begin{smallmatrix}1&2&3&4&5&6&7&8\\3&6&8&1&5&7&2&4\end{smallmatrix}\right)$)

$1 \overset{\sigma}{\mapsto} 3$, $\Rightarrow$ we put 3 at the first place on the first row;

$2 \overset{\sigma}{\mapsto} 6$, $6 > 3 \Rightarrow 36$: we put 6 from the right of 3.

$3 \overset{\sigma}{\mapsto} 8$, $8 > 6 \Rightarrow 368$ the same rule

$4 \overset{\sigma}{\mapsto} 1$, $1 < 8, 6, 3 \Rightarrow$ we replace 3 by 1 and write $\begin{smallmatrix}1&6&8\\3&&\end{smallmatrix}$

$5 \overset{\sigma}{\mapsto} 5$, $1 < 5 < 6 \Rightarrow$ we replace 6 by 5, and put (under the same rule) 6 on the second row: $\begin{smallmatrix}1&5&8\\3&6&\end{smallmatrix}$

$6 \overset{\sigma}{\mapsto} 7$, $5 < 7 < 8 \Rightarrow$ we replace 8 by 7, and put 8 on the second row: $\begin{smallmatrix}1&5&7\\3&6&8\end{smallmatrix}$

$7 \overset{\sigma}{\mapsto} 2$, $1 < 2 < 5 \Rightarrow$ we replace 5 by 2, and put 5 on the second row: $3 < 5 < 6 \Rightarrow$ we replace 6 by 5 and put 6 on the third row: $\begin{smallmatrix}1&2&7\\3&5&8\\6&&\end{smallmatrix}$

$8 \overset{\sigma}{\mapsto} 4$, $2 < 4 < 7 \Rightarrow$ we replace 7 by 4, and put 7 on the second row: $5 < 7 < 8 \Rightarrow$ we replace 8 by 7 and put 8 on the third row: $\begin{smallmatrix}1&2&4\\3&5&7\\6&8&\end{smallmatrix}$

$$\Rightarrow \sigma \mapsto A(\sigma) = \begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & 8 \end{smallmatrix} \in T_\lambda, \quad \lambda = (332).$$

There is another map $S_n \to \coprod_{\lambda \vdash n} T_\lambda$, $\sigma \mapsto B(\sigma)$ such that the entries of $B(\sigma)$ give the order in which the positions in $A(\sigma)$ are filled. Thus if $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 8 & 1 & 5 & 7 & 2 & 4 \end{smallmatrix}\right)$ as above we have $B(\sigma) = \begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 \end{smallmatrix}$:

Theorem. $\forall \sigma \in S_n \Rightarrow B(\sigma) = A(\sigma^{-1})$. □

Theorem (Robinson-Schensted). The map $S_n \to \coprod_{\lambda \vdash n} T_\lambda \times T_\lambda$. $\sigma \mapsto (A(\sigma), B(\sigma))$, is a bijection. □

Thus

$$|S_n| = n! = \left| \coprod_{\lambda \vdash n} T_\lambda \times T_\lambda \right| = \sum_{\lambda \vdash n} |T_\lambda \times T_\lambda| = \sum_{\lambda \vdash n} d_\lambda^2 \,.$$

We say that σ is an involution, if $\sigma^2 = \mathrm{id}$ (including $\sigma = \mathrm{id}$).

Corollary. There is a 1–1 correspondence between involutions and standard tableaux: $\sigma \mapsto A(\sigma)$ where $\sigma^2 = \mathrm{id}$.

$$\begin{aligned} \sigma^2 = \mathrm{id} \Leftrightarrow \sigma = \sigma^{-1} &\Leftrightarrow (A(\sigma), B(\sigma)) = (A(\sigma^{-1}), B(\sigma^{-1})) = (B(\sigma), A(\sigma)) \\ &\Leftrightarrow A(\sigma) = B(\sigma) \end{aligned}$$

□

The Robinson-Schensted map gives a nice way of splitting the symmetric group S_n.

$\sigma, \sigma' \in S_n$ lie in the same left cell, if $A(\sigma) = A(\sigma')$.
$\sigma, \sigma' \in S_n$ lie in the same right cell, if $B(\sigma) = B(\sigma')$.

By the theorem above, every left cell contains d_λ elements, and there are d_λ left cells of type λ (the number of standard λ-tableau). Obviously,

"Number of left cells"$= \sum_{\lambda \vdash n} d_\lambda =$"Number of right cells".

$\sigma, \sigma' \in S_n$ lie in the same two-sided cell, if $A(\sigma), A(\sigma') \in T_\lambda$ for some $\lambda \vdash n$ (i.e. have the same shape).

Let C_λ denote the two-sided cell corresponding to $\lambda \vdash n$. C_λ is a union of d_λ left cells each with d_λ elements (equivalently, of d_λ right cells each with d_λ elements), hence

$|C_\lambda| = d_\lambda^2$.

As a consequence of the properties of the Robinson-Schensted map one can show that "Every left cell in C_λ intersects every right cell in C_λ in exactly one element."

Example. S_3: (we'll write $i_1 i_2 i_3$ instead of $\begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix}$)

σ	$A(\sigma)$	$B(\sigma)$
$1\,2\,3$	$1\,2\,3$	$1\,2\,3$
$1\,3\,2$	$\begin{matrix} 1\,2 \\ 3 \end{matrix}$	$\begin{matrix} 1\,2 \\ 3 \end{matrix}$
$2\,1\,3$	$\begin{matrix} 1\,3 \\ 2 \end{matrix}$	$\begin{matrix} 1\,3 \\ 2 \end{matrix}$
$2\,3\,1$	$\begin{matrix} 1\,3 \\ 2 \end{matrix}$	$\begin{matrix} 1\,2 \\ 3 \end{matrix}$
$3\,1\,2$	$\begin{matrix} 1\,2 \\ 3 \end{matrix}$	$\begin{matrix} 1\,3 \\ 2 \end{matrix}$
$3\,2\,1$	$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$	$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$

left cells: $\{123\}, \{132, 312\}, \{213, 231\}$, and $\{321\}$,

right cells: $\{123\}, \{132, 231\}, \{213, 312\}$, and $\{321\}$,

two-sided cells: $\{123\}, \{132, 213, 231, 312\}$, and $\{321\}$,

Hence, the assertion above is easily verified in the case of S_3. □

Corollary. Each left (right) cell contains just one involution.
Let $P \in T_\lambda$ determine the left (right) cell; by the Robinson-Schensted theorem $\exists!\ \sigma \in S_n$: $(A(\sigma), B(\sigma)) = (P, P)$. This σ is the involution in the left (right) cell. □

Remark. The notions of left, right, 2-sided cells have been generalized to the case of any Coxeter group by Kazhdan-Lusztig. However not all the above properties of cells hold in this more general case.

An alternative description of the Robinson-Schensted map

Given $\sigma \in S_n$, $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$, mark the points in $\mathbf{N}^2$ with coordinates $(i, \sigma(i))$. For $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 8 & 1 & 5 & 7 & 2 & 4 \end{pmatrix}$ we have the points from the left marked by fat dots $\bullet$. Then we start drawing paths according to the following rule.

We start with the point $(1, \sigma((1))$. We draw first a vertical line from $(1, \sigma(1))$ up to $(1, n)$; then a horizontal line from $(1, \sigma(1))$ to the right up to $(i, \sigma(1))$, where i is the least number with $\sigma(i) < \sigma(1)$; then vertical line down to $(i, \sigma(i))$; we continue drawing lines to the right and downward in the same way: to the right up to $(j, \sigma(i))$ where j is the least number, greater than i with $\sigma(j) < \sigma(i)$; then downward to $(j, \sigma(j))$, and so on until we reach the n-th column.

Denote by Σ_0 the set of points $(i, \sigma(i))$, $i = 1, \dots, n$. We get a connected path, and all points of Σ_0 which don't lie on this path are to the right and above it.

Starting with the point $(j, \sigma(j))$, which does not lie in the path with least possible **j** we do the same repeating the process until all points of Σ_0 will lie on such paths. All such paths have left corners and right corners. The left corners are exactly the points from Σ_0. Let Σ_1 be the set of right corners – the first derived set. (In our example the points from Σ_1 are marked by "0".)

We do the same with Σ_1, and get the second derived set Σ_2 (marked by "x" in our example). And so on, until the last paths we draw have no right corners at all. Let $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \dots$ be the set of all points; initial and derived.

Denote by i_k the number of points of Σ lying in k-th row, and by j_k the number of points of Σ lying in k-th column. Thus we get $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$.

In our example $I = (1, 1, 2, 1, 2, 3, 2, 3)$, $J = (1, 1, 1, 2, 2, 2, 3, 3)$].

Now $A(\sigma)$ $(B(\sigma))$ is defined to be the standard tableaux the first row of which consists of all numbers (in increasing order) k for which $i_k = 1$ $(j_k = 1)$; the second row – all

$k : i_k = 2$ ($j_k = 2$), and so on. In our example

$$A(\sigma) = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & 8 & \end{matrix} \qquad B(\sigma) = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \end{matrix} \;,$$

which we know from the preceding examples.) □

There is a third way of defining $A(\sigma)$ and $B(\sigma)$ – a Freudian one ("jeu de taquin"). But we stop here.

Lecture 5

Primitive ideals in universal enveloping algebras

1. Let $g = \mathrm{sl}_n(\mathbf{C})$ be the simple Lie algebra of all $n \times n$ matrices of trace 0 under $[AB] = AB - BA$. Consider the decomposition

$$g = n_+ \oplus h \oplus n_-, \quad n_+ = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, \quad h = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}, \quad n_- = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix}$$

$b = n_+ \oplus h$, $g = b \oplus n_-$

Consider $\mathcal{U}(g)$ – universal enveloping algebra of g (associative algebra),

$b_1, \ldots, b_n$ – basis of g

$\mathcal{U}(g)$ is the associative algebra generated by $1, b_1, \ldots, b_n$ subject to relations

$$b_i \cdot b_j - b_j \cdot b_i = [b_i b_j].$$

Then $\dim \mathcal{U}(g) = \infty$ and every g-module is a $\mathcal{U}(g)$-module, and every $\mathcal{U}(g)$-module is a g-module

Problem. Determine all irreducible $\mathcal{U}(g)$-modules, finite or infinite dimensional.

There are so many such modules that it appears impossible to classify them.

A *primitive ideal* of $\mathcal{U}(g)$ is the kernel of some irreducible representation of $\mathcal{U}(g)$.

Problem. Find all the primitive ideals of $\mathcal{U}(g)$. (Note that different irreducible representation may have the same kernel.) This is a more tractible problem.

$$h^* = \mathrm{Hom}(h, \mathbf{C}) \qquad h = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \middle| x_1 + \cdots + x_n = 0 \right\}$$

$$\alpha_{ij} \in h^* \qquad \alpha_{ij}\left(\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \right) = x_i - x_j, \qquad i \neq j$$

$\Phi = \{\alpha_{ij} \mid i \neq j\}$ is the set of roots.

Let $\lambda \in h^*$. λ is a *weight*

λ gives a 1-dim h-module, and a 1-dim b-module. Note that n_+ is an ideal of b and that this ideal lies in the kernel of λ.

λ gives a 1-dim $\mathcal{U}(b)$-module, $\mathbf{C}v_\lambda$. Now, $g = b \oplus n_- \quad \mathcal{U}(g) = \mathcal{U}(b) \otimes \mathcal{U}(n_-)$, so let V_λ be the induced $\mathcal{U}(g)$-module. Then

$$V_\lambda = \mathbf{C}v_\lambda \otimes_{\mathcal{U}(b)} \mathcal{U}(g)$$

$V_\lambda \cong \mathcal{U}(g)/I_\lambda$ where $I_\lambda = \mathcal{U}(g)n_+ + \sum\limits_{x \in h} \mathcal{U}(g)(x - \lambda(x))$.

V_λ is the *Verma module* with highest weight λ. $\dim V_\lambda = \infty$, and V_λ has a unique maximal submodule $\mathcal{M}_\lambda$. Let $\mathcal{L}_\lambda = V_\lambda / \mathcal{M}_\lambda$. Then $\mathcal{L}_\lambda$ is an irreducible $\mathcal{U}(g)$-module. Let $J_\lambda =$ Kernel of $\mathcal{L}_\lambda = \operatorname*{Ann}_{\mathcal{U}(g)} \mathcal{L}_\lambda$. Then J_λ is a primitive ideal of $\mathcal{U}(g)$.

Main theorem (Duflo). Every primitive ideal of $\mathcal{U}(g)$ has form J_λ for some $\lambda \in h^*$.

When is $J_\lambda = J_\mu$?

$W = S_n$ acts on h by permuting $x_1, \dots, x_n$

$$\begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{pmatrix} \qquad x_1 + \cdots + x_n = 0$$

So W acts on h^*.

This gives the reflection representation of S_n of dimension $n-1$.

Let $\lambda, \mu \in h^*$. Let $Z(g)$ be the centre of $\mathcal{U}(g)$. $Z(g)$ is isomorphic to the ring of invariants of S_n on h^*. So $Z(g) \cong \mathbf{C}[\chi_1, \dots, \chi_\ell]$, $\ell = n-1$. Each $z \in Z(g)$ acts on V_λ (so on $\mathcal{L}_\lambda$) as a scalar multiplication. This gives a 1-dimensional representation of $Z(g)$, called the central character χ_λ.

When is $\chi_\lambda = \chi_\mu$?

If $J_\lambda = J_\mu$, then $\chi_\lambda = \chi_\mu$ (but not conversely).

$J_\lambda \cap Z(g)$ is a maximal ideal of $Z(g)$ since

$$\dim Z(g)/J_\lambda \cap Z(g) = 1$$

Proposition. Let $\lambda, \mu \in h^*$. Then $\chi_\lambda = \chi_\mu$ if and only if $\lambda + \rho = w(\mu + \rho)$ for some $w \in W$ where $\rho \in h^*$ is given by $\rho = \frac{1}{2} \sum\limits_{\alpha \in \Phi^+} \alpha$. $\Phi^+ = \{\alpha_{ij} \mid i < j\}$.

Define $w \cdot \lambda = w(\lambda + \rho) - \rho$, then $\chi_\lambda = \chi_\mu \Leftrightarrow \lambda = w \cdot \mu$ for some $w \in W$

Let X be the set of primitive ideals of $\mathcal{U}(g)$ and let

$X_\lambda = \{J_\mu \mid \mu = w.\lambda$ for some $w \in W\}$. Then

$X_\lambda = X_\mu \Leftrightarrow \lambda = w.\mu$ for some $w \in W$. Thus $X = \cup X_\lambda$ taking one λ in each W-orbit on h^*.

Now $|X_\lambda|$ is finite. The X_λ are called the fibres of X.

We consider the regular integral fibres. λ is regular if $w.\lambda = \lambda \Rightarrow w = 1$, λ is integral if $\lambda(h_\alpha) \in \mathbf{Z}$ for all $\alpha \in \Phi$

Here

$$h_{\alpha_{ij}} = \begin{array}{c} \begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & -1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \end{array} .$$

Let X_λ be a regular integral fibre, and let $X_\lambda = \{J_{w\cdot\lambda} \mid w \in W\}$.

Theorem (A. Joseph). $J_{w\cdot\lambda} = J_{w'\cdot\lambda}$ if and only if $A(w) = A(w')$ where $w \to A(w)$ is the Robinson-Schensted map. So w, w' give rise to the same primitive ideal in X_λ if and only if they lie in the same left cell of W.

Corollary. There is a natural $1-1$ correspondence between primitive ideals in a regular integral fibre and left cells in W.

Similar results hold if g is any finite dimensional simple Lie-algebra over $\mathbf{C}$, but the details are more complicated.

Finite irreducible reflection groups

Type	**Diagram**	**Degrees of basic invariants**
A_ℓ	$\circ - \circ - \circ - \circ - \circ - \circ$	$2, 3, \dots \ell+1$
$B_\ell = C_\ell$	$\circ - \circ - \circ - \circ - \circ = \circ$	$2, 4, 6, \dots 2\ell$
D_ℓ	$\circ - \circ - \circ - \circ <\begin{smallmatrix}\circ\\ \circ\end{smallmatrix}$	$2, 4, 6, \dots 2\ell-2, \ell$
E_6	$\begin{array}{c}\circ - \circ - \circ - \circ - \circ \\ \mid \\ \circ\end{array}$	$2, 5, 6, 8, 9, 12$
E_7	$\begin{array}{c}\circ - \circ - \circ - \circ - \circ - \circ \\ \qquad\quad \mid \\ \qquad\quad \circ\end{array}$	$2, 6, 8, 10, 12, 14, 18$
E_8	$\begin{array}{c}\circ - \circ - \circ - \circ - \circ - \circ - \circ \\ \qquad\qquad \mid \\ \qquad\qquad \circ\end{array}$	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$\circ - \circ = \circ - \circ$	$2, 6, 8, 12$
G_2	$\circ \equiv \circ$	$2, 6$
H_3	$\circ - \circ \underset{5}{-} \circ$	$2, 6, 10$
H_4	$\circ - \circ - \circ \underset{5}{-} \circ$	$2, 12, 20, 30$
$I_2(m)$	$\circ \underset{m}{-} \circ$	$2, m$

$A_2 = I_2(3) \qquad B_2 = I_2(4) \qquad G_2 = I_2(6)$

$\circ \quad \circ$ means $\circ \underset{2}{-} \circ$

$\circ - \circ$ means $\circ \underset{3}{-} \circ$

$\circ = \circ$ means $\circ \underset{4}{-} \circ$

$\circ \equiv \circ$ means $\circ \underset{6}{-} \circ$

Problems to lecture 1

1. Show that all the irreducible representations of G have degree 1 if and only if G is abelian.

2. Let $G = D_8$ be the dihedral group of order 8. Thus

$$G = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

where $a^4 = 1$, $b^2 = 1$, $ba = a^3b$.

(a) Find the conjugacy classes of G
(b) Find the 1-dimensional representations of G
(c) Find the character table of G

3. Let $G = D_{10}$ be the dihedral group of order 10, given by

$$G = \langle a, b;\ a^5 = 1,\ b^2 = 1,\ ba = a^4b\rangle$$

The conjugacy classes of G are
(1) $(a\ a^4)$ $(a^2\ a^3)$ $(b\ ab\ a^2b\ a^3b\ a^4b)$

(a) Find the 1-dimensional representations of G.

(b) Let H be the subgroup of G generated by a. So $H = \{1, a, a^2, a^3, a^4\}$. Let σ be the 1-dimensional representation of H satisfying $\sigma(a) = \omega$ where $\omega = e^{2\pi i/5}$.
Find the character of the induced representation of G.

(c) Show that the induced character is irreducible (check that it satisfies $\langle\chi, \chi\rangle = 1$).

(d) Find the character table of G.

4. Let G be the group of order 21 defined by generators and relations

$$G = \langle a, b;\ a^7 = 1,\ b^3 = 1,\ ba = a^2b\rangle\ .$$

The elements of G have form a^ib^j where $0 \le i \le 6$, $0 \le j \le 2$ and the conjugacy classes of G are

$$\begin{array}{l} (1)\quad (a\ a^2\ a^4)\quad (a^3\ a^5\ a^6) \\ (b\ ab\ a^2b\ a^3b\ a^4b\ a^5b\ a^6b) \\ (b^2\ ab^2\ a^2b^2\ a^3b^2\ a^4b^2\ a^5b^2\ a^6b^2) \end{array}$$

Find the character table of G.

Problems to lecture 2

1. Let G be the symmetric group S_4.

(a) For each partition λ of 4 find the number of elements in the conjugacy class K_λ of G.

(b) For each partition λ of 4 calculate the induced character ξ^λ of G given by

$$\xi^\lambda = (1_{S_\lambda})^{S_4}\ .$$

(c) For each partition λ of 4 use Giambelli's formula to express the irreducible character χ^λ of G as a **Z**-combination of induced character ξ^μ.

(d) Write down the character table of G.

(e) Check that the character table you have calculated satisfies the row and column orthogonally relations.
(If it doesn't, try again!)

2. Do the same for the symmetric group S_5.

Problems to lecture 3

1. Suppose $I_1, \ldots, I_\ell$ and $I'_1, \ldots, I'_\ell$ are two sets of basic polynomial invariants for the finite reflection group W. Prove that the numbering can be chosen so that

$$\deg I_i = \deg I'_i\ .$$

(Hint. Consider the pair of inverse matrices

$$(\partial I_i/\partial I'_i) \qquad (\partial I'_i/\partial I_j)\ ).$$

2. Let $W = \langle s_1, \ldots, s_\ell \rangle$ be a finite reflection group of type B_ℓ acting on a Euclidean space V with orthonormal basis $e_1, \ldots, e_\ell$. The elements $s_1, \ldots, s_{\ell-1}$ permute the basis elements by permutations $(12), \ldots (\ell - 1\ \ell)$ respectively and s_ℓ fixes $e_1, \ldots, e_{\ell-1}$ and transforms e_ℓ to $-e_\ell$. Find a set $I_1, \ldots I_\ell$ of basic polynomial invariants and hence determine the degrees $d_1, \ldots, d_\ell$.

3. Calculate the Jacobian determinant

$$J = \det(\partial I_i/\partial x_j)$$

of the set $I_1, \ldots I_\ell$ of basic invariants determined in $Q2$. Show that J factorises into a product of linear factors which give the equations of all the reflecting hyperplanes of $W(B_\ell)$.

4. Find the matrix representing a Coxeter element of $W(B_\ell)$ with respect to the basis $e_1, \ldots e_\ell$.
What are the eigenvalues of this matrix?
Use these to determine the exponents $m_1, \ldots m_\ell$.

Problems to lecture 4

1. Show that $\chi^\lambda_{(n)}$ always takes the value $1, -1$ or 0. For which λ is it 1? For which λ is it -1?

2. Show that for any partition $\rho \vdash n$ with $\rho \neq (n)$ $\chi^n_\rho - \chi^{n-1,1}_\rho + \chi^{n-2,1,1}_\rho - \ldots + (-1)^n \chi^{1\ 1 \ldots 1}_\rho = 0$.

3.Show that $X^{n-1,1}_\rho = k - 1$ where k is the number of parts of ρ equal to 1.

4. Determine the Robinson-Schensted map $\sigma \to (A(\sigma), B(\sigma))$ for the symmetric group S_4.

5. Prove that the Robinson-Schensted map

$$S_n \to \bigcup_{\lambda \vdash n} (T_\lambda \times T_\lambda)$$

is bijective. Describe the inverse map.

6. Show that $B(\sigma) = A(\sigma^{-1})$.

7. Show that the tableaux $A(\sigma), B(\sigma)$ obtained from $\sigma \in S_n$ by the method of derived sets of points coincide with $A(\sigma), B(\sigma)$ obtained by the Robinson-Schensted algorithm.