# COMPUTATIONAL ASPECTS OF ALGEBRAIC GEOMETRY

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## Chapter I. Gröbner basics

**Chapter II. Applications to surfaces in $\mathbb{P}^4$**



These notes follow closely the course I taught at the 1999 summer school in Nordfjordeid's Sophus Lie Conference Centre. I would like to thank the organizers for their invitation and support and the students for plenty of feedback and fun.

# Chapter I. Gröbner basics

## 1. Introduction by historical remarks

Some of the important basic concepts and theorems of computational algebra can be found in 19th century papers on classical invariant theory. The roots of invariant theory can be traced back to Lagrange (1773–1775) and Gauss (1801), who were interested in the problem of representing integers by quadratic binary forms, and who used the discriminant to distinguish between nonequivalent forms. Let
$$F = ax^2 + 2bxy + cy^2$$
be a quadratic binary form over the integers. Then the discriminant $(b^2 - ac)$ is invariant under coordinate transformations with determinant $\pm 1$. Indeed, if a change of coordinates

$$x \to \alpha x + \beta y, \quad y \to \gamma x + \delta y$$

transforms $F$ into the form

$$\tilde{F} = \tilde{a}x^2 + 2\tilde{b}xy + \tilde{c}y^2,$$

then there is the relation

$$\tilde{b}^2 - \tilde{a}\tilde{c} = (b^2 - ac)(\alpha\delta - \beta\gamma)^2.$$

More generally, and in the modern language of representation theory, this result reads as follows: the special linear group $\mathrm{SL}_2(\mathbb{C})$ acts on the vector space $V$ of quadratic binary forms over $\mathbb{C}$ by linear substitution. The dual action on the space of coordinates $a, b, c$ gives rise to an action of $\mathrm{SL}_2(\mathbb{C})$ on the symmetric algebra
$$\mathbb{C}[a,b,c] = \mathbb{C}[V] = S(V^*) = \bigoplus_{d \geq 0} S^d V^*,$$
that is, on the algebra of polynomial functions on $V$. The discriminant $\delta = b^2 - ac$ is an **invariant** under this action. In fact, it generates the ring of all invariants:
$$\mathbb{C}[V]^{\mathrm{SL}_2(\mathbb{C})} = \bigoplus_{d \geq 0} \mathbb{C}[V]_d^{\mathrm{SL}_2(\mathbb{C})} = \mathbb{C}[\delta]\ .$$

Algebraic invariants such as the discriminant naturally show up in algebraic geometry, when one asks for properties of geometric objects, which are invariant under certain classes of transformations. For example, the geometric significance of the discriminant is that a quadratic binary form defines two distinct points on the projective line $\mathbb{P}^1(\mathbb{C})$ if and only if its discriminant is non–zero. People became interested in such invariant properties especially after the introduction of homogeneous coordinates by Möbius (1827) and Plücker

(1830). This was a major impetus for invariant theory.

In the first decades of invariant theory, say between 1840 and 1870, people were mainly concerned with the discovery of particular invariants. The major case of interest was that of forms of degree $d$ in $n$ variables, $d$ and $n$ arbitrary, with $\mathrm{SL}_n(\mathbb{C})$ acting by linear substitution. One way of producing new invariants is to apply certain **covariants** like the Hessian to an invariant already known. More systematic ways of generating invariants are provided by Cayley's **$\Omega$–process** and the **symbolic method** developed by Aronhold, Clebsch and Gordan. A more modern alternative is the **Lie algebra method**. We refer to Sturmfels (1993) for a detailed discussion of these methods.

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**Example 1.1 ($\Omega$–process for quadratic binary forms).** Consider again a form

$$F = ax^2 + 2bxy + cy^2.$$

A general change of coordinates

$$x \to t_{11}x + t_{12}y, \quad y \to t_{21}x + t_{22}y$$

transforms $F$ into the form

$$F(t) = a(t)x^2 + 2b(t)xy + c(t)y^2.$$

Applying the differential operator

$$\Omega_t^d = \left(\frac{\partial^2}{\partial t_{11}\partial t_{22}} - \frac{\partial^2}{\partial t_{12}\partial t_{21}}\right)^d$$

to a monomial

$$a(t)^{d_1} b(t)^{d_2} c(t)^{d_3}$$

of degree $d = d_1 + d_2 + d_3$ yields an invariant of degree $d$ (most likely 0). For example,

$$\Omega_t^1(a(t)) = \Omega_t^1(b(t)) = \Omega_t^1(c(t)) = 0$$

and

$$\frac{1}{4}\Omega_t^2(b(t)^2) = b^2 - ac \,. \qquad \square$$

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As time went on people realized that they could find in many explicit cases such as the example above a **fundamental system of invariants**, that is, a minimal finite system of homogeneous generators of the ring of invariants under consideration. The question whether there always exists a fundamental system of invariants became lateron known as **Hilbert's 14th problem**

(Hilbert, 1900).

Gordan (1868) gave a positive answer to this question in the case of binary forms of any given degree $d$. Gordan's proof, based on the symbolic method, is constructive and allowed him to compute fundamental invariants for binary forms of degree 5 and 6. The results for $d \leq 4$ had been known to Cayley before, who actually claimed that invariant rings of binary forms of degree at least 7 are not (!) finitely generated. See Meyer (1899), Sturmfels (1993) and Olver (1999) for more details.

Sylvester (1897, 1881) produced tables with numerical information on the fundamental invariants and covariants for binary forms of $\leq 10$ and 12. According to these tables, the minimal numbers of generators are as follows:

| deg | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| # inv | 1 | 1 | 2 | 4 | 5 | 26(30) | 9 | 89 | 99 | 109 |
| # cov | 2 | 4 | 5 | 23 | 26 | 124($\geq$ 130) | 69 | 415 | 475 | 948 |

Together with Franklin (1883) Sylvester arrived at his figures by manipulating the **generating functions** of the rings of covariants under consideration following ideas of Cayley and Sylvester. In fact, this approach yields for each given degree only a lower bound on the number of fundamental covariants. In order to conclude that this lower bound is actually the correct number a

> "...fundamental postulate still awaiting demonstration is necessary ... . The validity of the fundamental postulate ... is verified by its conducting to results which have been proved to be accurate for single binary quantics up to sixth order inclusive ..."

Unfortunately, the fundamental postulate proved to be wrong in the next case, that is, in degree 7 (see Hammond, 1883). That there are indeed four more fundamental invariants in this degree than those enumerated by Sylvester was checked by Dixmier and Lazard (1986) with the help of computer algebra. The correct number of covariants, however, seems to be still unknown. Whereas nowadays every reader can easily redo Sylvester's manipulations with his favorite computer algebra system, the contempories of Sylvester had to travel to Baltimore:

> "The manuscript sheets containing the original calculations ... are deposited in the iron safe of the John Hopkins University Baltimore, where they can be seen and examined ..."

Also when constructing invariants explicitly it is helpful to precompute the generating function. In this way we avoid, for example, unnecessary computations such as producing the zeros in odd degrees in Example 1.1. As we will see below, the concept of generating functions was taken up and treated in a systematic way by Hilbert.

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Let $K$ be a field, and let $R = \bigoplus_{d \geq 0} R_d$ be a graded $R_0 = K$–algebra which is finitely generated over $K$. Notice that for every finitely generated, graded $R$–module $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ the graded pieces $M_\nu$ are finite dimensional $K$–vector spaces.

**Definition 1.2.** *Let $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ be a finitely generated, graded $R$–module. Then*

$$H(M, -) : \mathbb{Z} \to \mathbb{Z}, \quad \nu \mapsto \dim_K M_\nu \ ,$$

*is called the* **Hilbert function** *of $M$. Its generating function*

$$H_M(t) := \sum_{\nu \in \mathbb{Z}} H(M, \nu) t^\nu$$

*is called the* **Hilbert series** *of $M$.* □

**Example 1.3.** In the case of quadratic binary forms we obtain

$$H_{\mathbb{C}[V]^{\mathrm{SL}_2(\mathbb{C})}}(t) = \frac{1}{1 - t^2} = 1 + t^2 + t^4 + \dots \ . \qquad \square$$

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Classical invariant theory culminated in two landmark papers of Hilbert (1890 and 1893). The stunning new ideas presented in these papers deeply influenced the development of modern algebra and algebraic geometry: as "lemmas" Hilbert proved

- **the basis theorem**;
- **the Hilbert–Burch theorem (graded case)**;
- **the syzygy theorem (graded case)**;
- **the existence of the Hilbert polynomial**;
- **the Nullstellensatz**;
- **the Hilbert–Mumford criterion**;

- **the Noether normalization theorem (graded case).**

As an application Hilbert generalized Gordan's finiteness result to invariant rings of forms in $n$ variables, $n$ arbitrary (see Section 5.1.3).

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Let $K$ be a field, and let $R = K[x_1, \ldots, x_n]$.

**Hilbert's Basis Theorem 1.4.** *Every ideal in $R$ is finitely generated.* □

**Remark 1.5.** Let $M$ be a finitely generated $R$–module. In order to handle computations in $M$ we not only need to know a set of generators of $M$, say $f_1, \ldots, f_s$, but also the $R$–module relations

$$\sum_{i=1}^{s} g_i f_i = 0, \quad g_i \in R \ ,$$

on the generators. We may think of a relation as an element in the free $R$–module $R^s$ (with basis $e_1, \ldots, e_s$). Indeed, consider the morphism

$$\phi : F_0 = R^s \to M, \quad e_i \mapsto f_i \ .$$

Then the elements of $\ker \phi$ are the relations on the $f_1, \ldots, f_s$ (or **first order syzygies** of $M$). Hilbert's basis theorem implies that $\ker \phi$ is finitely generated. The first order syzygies of $\ker \phi$ are the **second order syzygies** of $M$ and so on. The $i$th order syzygies form the $i$th **syzygy module** of $M$ (note that the syzygy modules are determined up to free summands only). To compute the syzygies means to compute a **free resolution**

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots$$

of $M$. We say that such a resolution has **length** $m$ if $F_m \neq 0$ but $F_i = 0$ for all $i > m$ . □

**Hilbert's Syzygy Theorem 1.6.** *Every finitely generated $R$–module has a free resolution of length $\leq n$ .* □

**Remark and Definition 1.7.** (1) In fact, Hilbert only considered the graded case and showed that every finitely generated, graded $R$–module has a graded free resolution of length $\leq n$ .
(2) Notice that for every $R$–module $M$ as in (1) there exists a unique (graded) **minimal free resolution**

$$0 \leftarrow M \overset{\phi_0}{\leftarrow} \bigoplus_j R(-j)^{\beta_{0j}} \overset{\phi_1}{\leftarrow} \ldots \overset{\phi_n}{\leftarrow} \bigoplus_j R(-j)^{\beta_{nj}} \leftarrow 0 \ .$$

The $\beta_{ij}$ are called the **graded Betti numbers of** $M$. Note that in this case we have uniquely determined graded syzygy modules $\mathrm{Syz}_i(M) := \mathrm{im}\,\phi_i$, $1 \leq i \leq n$ . $\square$

**Example 1.8.** (Hilbert) The variety $X$ given parametrically by

$$\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^3(\mathbb{C}), \quad (s:t) \mapsto (s^3 : s^2t : st^2 : t^3)$$

is defined by the ideal

$$I_X = \langle wy - x^2, xy - wz, xz - y^2 \rangle \subset \mathbb{C}[w, x, y, z] =: S.$$

It has syzygies of type

$$0 \leftarrow S/I_X \leftarrow S \leftarrow 3S(-2) \leftarrow 2S(-3) \leftarrow 0$$

(compare Example 6.13 below). $\square$

By applying his syzygy theorem Hilbert proved:

**Theorem 1.9 (Existence of the Hilbert polynomial).** *Let $M$ be a finitely generated, graded $R$–module of dimension $d$. Then there exists a unique numerical polynomial $P_M(t) \in \mathbb{Q}[t]$ of degree $d-1$ with*

$$H(M, \nu) = P_M(\nu) \quad \textit{for all} \quad \nu >> 0 \ .$$

*$P_M(t)$ is called the* **Hilbert polynomial of** *$M$.* $\square$

The same proof gives the following result (Ostrowski, 1922).

**Theorem 1.10 (Structure of the Hilbert series I).** *Let $M$ be a finitely generated, graded $R$–module with graded Betti numbers $\beta_{ij}$. Then*

$$H_M(t) = \frac{\sum_{i,j} (-1)^i \beta_{ij} t^j}{(1-t)^n} \ . \qquad \square$$

As a corollary of Theorem 1.9 one obtains:

**Theorem 1.11 (Structure of the Hilbert series II).** *Let $M \neq 0$ be a finitely generated, graded $R$–module of dimension $d$. Then there exists a unique Laurent polynomial $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_M(1) \neq 0$ such that*

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d}.$$

*Moreover, if* $Q_M(t) = \sum_\nu h_\nu t^\nu$, *then* $\min\{\nu \mid h_\nu \neq 0\}$ *is the least number such that* $M_\nu \neq 0$. □

**Remark 1.12.** Let $M$ and $d$ be as in Theorem 1.11 and write

$$P_M(t) = \sum_{i=0}^{d-1} (-1)^{d-1-i} e_{d-1-i} \binom{t+i}{i} .$$

Then

$$e_i = \frac{Q_M^{(i)}(1)}{i!}, \quad i = 0, \ldots, d-1 ,$$

where $Q_M^{(i)}$ denotes the $i$th formal derivative. □

**Example 1.13.** In Hilbert's Example 1.8 we have

$$P_{S/I_X}(t) = 3t + 1$$

and

$$H_{S/I_X}(t) = \frac{1 - 3t^2 + 2t^3}{(1-t)^4} = \frac{1+2t}{(1-t)^2} .$$

In particular, $X$ is a curve of degree 3 (**twisted cubic curve**). □

**Remark 1.14.** If $A$ is a finitely generated ring of invariants with generators $f_1, \ldots, f_s$, then the $K$–algebra relations on the generators are the elements of the kernel $I$ of the graded morphism

$$\phi_0 : S = K[z_1, \ldots, z_s] \to A, \quad z_i \mapsto f_i .$$

Here $z_1, \ldots, z_s$ are new variables with $\deg(z_i) = \deg(f_i)$. $I$ is finitely generated by Hilbert's basis theorem, and one can show that in analogy with Theorem 1.11 the Hilbert series of $A = S/I$ is of the form

$$H_A(t) = \frac{Q_A(t)}{(1 - t^{\deg(f_1)}) \cdots (1 - t^{\deg(f_s)})} . \quad \square$$

We refer to Bruns and Herzog (1993) and Eisenbud (1995) for more details. and proofs.

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We will come back to the history of invariant theory later-on. Now we mention that Gordan (1899) gave a new proof of Hilbert's basis theorem (see SIGSAM Bulletin **32**, 47–48, 1998 for a translation into English). In modern terminology he proceeded along the following lines.

- We may order the monomials in $R$ by the **lexicographical order** $>$.

- Every **monomial ideal** in $R$ is finitely generated (**Gordan's Lemma**).
- Every polynomial $f \in R$ can be written in the form
$$f = \mathrm{lc}(f)\mathrm{lm}(f) + \tilde{f}, \quad \mathrm{lc}(f) \in K \setminus \{0\},$$
such that the **leading monomial** $\mathrm{lm}(f)$ is bigger than any monomial appearing in $\tilde{f}$. If $f, g \in R$ with $\mathrm{lm}(g)|\mathrm{lm}(f)$, then the **s–polynomial** of $f$ and $g$ has a smaller leading monomial than $f$, that is,
$$\mathrm{lm}(f - \frac{\mathrm{lc}(f)\mathrm{lm}(f)}{\mathrm{lc}(g)\mathrm{lm}(g)} g) < \mathrm{lm}(f).$$
- Every ideal in $R$ has a **Gröbner basis**.

Macaulay (1927), influenced by the ideas of Hilbert, classified the numerical functions which arise as Hilbert functions of (homogeneous) ideals $I$. His basic idea was to compare with the Hilbert function of the **leading ideal (initial ideal)** of $I$, that is, with the ideal $\mathrm{L}(I)$ generated by all leading monomials $\mathrm{lm}(f)$, $f \in I$. In fact, his proofs give more generally the following results.

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**Theorem 1.15 (Macaulay).** *Let $F$ be a free $R$–module with a finite basis, and let $I$ be a submodule. Let $>$ be a monomial order on $F$ which is a well–ordering. Then the set $\mathcal{B}$ of all monomials not in the* **leading module** $\mathrm{L}_>(I)$ *forms a $K$–vector space basis for $F/I$.* □

**Definition 1.16.** *The monomials in $\mathcal{B}$ above are called* **standard monomials**. □

As a corollary Macaulay obtained the following result.

**Theorem 1.17 (Macaulay).** *Let $M$ be a finitely generated graded $R$–module, given by generators and relations as $M = F/I$, where $F$ is a free $R$–module with a homogeneous basis and $I$ is a graded submodule. Let $>$ be a monomial order on $F$ which is a well–ordering. Then the Hilbert function of $M$ equals the Hilbert function of $F/\mathrm{L}_>(I)$.* □

**Remark 1.18.** The computation of Hilbert functions of monomial ideals is of purely combinatorial nature. □

Again, more details can be found in Bruns and Herzog (1993) and Eisenbud (1995).

Hironoka (1964) and independently Grauert (1972) introduced the notion of **standard bases** (the analogue to Gröbner bases in the case of power series rings), and came up with a division algorithm in their case.

The major computational breakthrough, however, is due to Buchberger (1965 and 1970). His thesis problem, proposed by Gröbner under the influence of Macaulay's work, was to develop tools for computing in affine rings $R/I$. He solved this problem completely. In particular, his ideas allow to compute standard monomials. The essential new ingredient is Buchberger's criterion. This allows to compute **Gröbner bases** by reducing **s–polynomials** (here Buchberger uses a normal form (division) algorithm which relies on single reduction steps as in Gordan's paper). That Buchberger's algorithm can also be used to compute syzygies was remarked by Schreyer (1980) and others. We also mention that standard bases of ideals generated by polynomials in formal power series rings can be computed by a variant of Buchberger's algorithm due to Mora (1982) which is based on a different normal form algorithm.

For many years the computer algebra system

- **Macaulay** (http://www.math.columbia.edu/ bayer/Macaulay.html)

was the only system which was powerful enough to handle interesting examples in algebraic geometry. More modern systems are

- **Cocoa** (http://cocoa.dima.unige.it),
- **Macaulay2** (http://www.math.uiuc.edu/Macaulay2), and
- **Singular** (http://www.singular.uni-kl.de).

At current state, Singular is the most complete system.

## 2. Basic computational problems arising from the ideal–variety correspondence

Let us start by formulating the basic question of algebraic geometry:

- What is the structure of the set of solutions of finitely many polynomial equations in finitely many indeterminates?

The basic idea for attacking this problem is to use the algebraic–geometric dictionary (see Cox, Little and O'Shea, 1996 and 1998 for a nice presentation of what follows). Consider for example an **affine variety**: let $K$ be a field, let $f_1, \dots, f_s$ be polynomials in $K[x] = K[x_1, \dots, x_n]$, and write

$$V = V(f_1, \dots, f_s) = \{p \in K^n \mid f_1(p) = \cdots = f_s(p) = 0\}.$$

Then every linear combination $f = \sum a_i f_i$, $a_i \in K[x]$, also vanishes on $V$ and $V$ equals the common vanishing locus $V(I)$ of all $f$ in the ideal $I = \langle f_1, \dots, f_s \rangle$ generated by $f_1, \dots, f_s$. Conversely, if an ideal $I \subset K[x]$ is given, then its vanishing locus $V(I)$ is an affine variety by Hilbert's basis theorem. It is Hilbert's Nullstellensatz which establishes a successful ideal–variety correspondence.

**Hilbert's Nullstellensatz 2.1.** *Let $K$ be algebraically closed. Let $V \subset K^n$ be an affine variety and denote by*

$$I(V) = \{f \in K[x] \mid f(p) = 0 \textit{ for all } p \in V\}$$

*its* **vanishing ideal**. *Then for every ideal $I \subset K[x]$ the following holds:*

$$V(I) = V \ \Rightarrow\ \sqrt{I} = \mathrm{I}(V) \ ,$$

*where*

$$\sqrt{I} := \{f \in K[x] \mid \exists\, d \textit{ with } f^d \in I\}$$

*is the* **radical** *of $I$.* □

Similarly, we have an ideal–variety correspondence in the case of **projective varieties**. Of course, more generally, by taking multiplicities into account, we are interested in **subschemes of projective spaces**.

**Example 2.2.** We check an explicit example with the help of Singular.

```
// We compute the ideal I of a variety X in P4
// (compare Example 8.8 below)

LIB "random.lib";
LIB "matrix.lib";

ring R=32003,(x(0..4)),dp;
matrix alpha[2][9]=randommat(2,9,maxideal(1),32002);
resolution falpha=mres(alpha,0);
print(betti(falpha),"betti");
```

```
/*
          0     1     2     3     4     5
------------------------------------------
    0:    2     9    15    11     3     -
    1:    -     -     1     3     3     1
------------------------------------------
total:    2     9    16    14     6     1
*/;

resolution kos=mres(maxideal(1),0);
module tkos4=transpose(kos[4]);
module tfalpha3=transpose(falpha[3]);
matrix tpresg=dsum(matrix(tkos4),matrix(tfalpha3));
matrix aaa[16][ncols(tpresg)]=tpresg[1..16,1..ncols(tpresg)];
matrix ran[3][16]=random(32002,3,16);
matrix bbb=ran*aaa;
matrix tkos5m3[15][3]=dsum(matrix(kos[5]),matrix(kos[5]));
tkos5m3=transpose(dsum(matrix(kos[5]),tkos5m3));
matrix tphi=lift(tkos5m3,bbb);
matrix dir=module(transpose(tphi))+module(transpose(tpresg));
dir=transpose(dir);
resolution fdir=mres(dir,2);
ideal I=std(flatten(fdir[2]));

// What type of variety is X?

hilb(I);

/*
codimension = 2
dimension   = 3
degree      = 11
*/;

// So X is a surface.
// Is X smooth? We use the Jacobi-criterion.

matrix JI=jacob(I);
ideal min=minor(JI,2);
hilb(std(min+I));

/*
codimension = 5
dimension   = 0
degree      = 659
*/;
```

```
// So X is a smooth surface.
// What type of surface is X?
// We compute the syzygies of X.

resolution FI=mres(I,0);
print(betti(FI),"betti");

/*
           0     1     2     3     4
------------------------------------
    0:     1     -     -     -     -
    1:     -     -     -     -     -
    2:     -     -     -     -     -
    3:     -     -     -     -     -
    4:     -    10    13     4     -
    5:     -     1     3     3     1
------------------------------------
total:     1    11    16     7     1
*/;

// We detect a distinguished line L (compare Example 9.7).
// Is there any  geometric relation between  L and X?

matrix FI2=FI[2];
matrix L[1][3]=FI2[11,14..16];
ideal P=concat(I,L);
hilb(std(P));

/*
codimension = 4
dimension   = 1
degree      = 6
*/

// So L is a 6-secant line to X.
// Are there other 6-secant lines?

ideal I5=I[1..10];
ideal SSL=quotient(I5,I);
print(betti(mres(SSL,0)),"betti");
```

```
/*
           0     1     2     3
-----------------------------
    0:     1     3     3     1
-----------------------------
total:     1     3     3     1
*/

// So there is precisely one 6-secant line.
// In Example 9.7 below we will use this information to show
// that X is a rational surface.
```

□

By using the ideal–variety correspondence as in the example above we may formulate several geometric problems and their algebraic counterparts (in order to avoid rationality problems one should think of $V(I)$ as being defined as the vanishing locus of $I$ over the algebraic closure of $K$).

- Compute the **dimension** of the variety $V(I)$, that is, compute the **Krull dimension** of the quotient ring $K[x]/I$. If $I$ is homogeneous, compute more generally the Hilbert polynomial (or the Hilbert series) of $K[x]/I$.
- Check whether $V(I)$ is irreducible, and if not, find its **irreducible components**. Algebraically this means to compute the **minimal associated primes** of $I$, or, more generally, to compute a **primary decomposition** of $I$. Another problem in this context is to compute the radical of $I$.
- Compute the union $V(I_1) \cup V(I_2)$ of two varieties, that is, given generators of $I_1$ and $I_2$, find generators of the intersection $I_1 \cap I_2$.
- Given a variety $V(I_1)$ and its union $V(I) = V(I_1) \cup V(I_2)$ with another variety $V(I_2)$ compute $V(I_2)$. Algebraically, compute generators for $I_2$ from generators of $I$ and $I_1$. If $I$ is radical, this amounts to finding generators for the **ideal quotient**

$$I : I_1 = \{f \in K[x] \mid fI_1 \subset I\}.$$

- **Radical membership and ideal membership.** Decide whether a given polynomial $f$ vanishes on $V(I)$, or, algebraically, whether $f$ is in $\sqrt{I}$. Or, decide whether $f$ is an element of $I$.
- Compute the syzygies of a given ideal, or, more generally, of a given submodule of a free module.

- Compute the Zariski closure of the image of an affine variety $V(I)$ under a linear projection onto the last $n-r$ components. Algebraically, eliminate the first $r$ variables from $I$.

- More generally, compute the Zariski closure of the image of an affine or projective variety $V$ under a morphism $f$. Algebraically, if $V$ is defined by the ideal $I \subset K[x_1, \ldots, x_n]$, and if $f$ is given by $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]/I$, then compute the kernel of the map
$$K[y_1, \ldots, y_m] \to K[x_1, \ldots, x_n]/I, \quad y_i \mapsto f_i.$$

- Given an irreducible variety $V(I)$, find its **normalization**. Algebraically, find the **normalization** of $K[x_1, \ldots, x_n]/I$, that is, find the integral closure of $K[x]/I$ in the quotient field of $K[x]/I$ and present this ring as an affine ring $K[y_1, \ldots, y_m]/J$ for some $m$ and $J$.

- Check whether $V(I)$ is **smooth**. If not, study the **singularities** of $V(I)$.

  Studying a variety $V(I)$, $I = \langle f_1, \ldots, f_s \rangle$, locally at a singular point, say the origin of $K^n$, means studying the ideal $IK[x]_{\langle x \rangle}$ generated by $I$ in the local ring
$$K[x]_{\langle x \rangle} = \left\{ \frac{f}{g} \mid f, g \in K[x],\ g \notin \langle x_1, \ldots, x_n \rangle \right\}.$$
  Notice that $K[x]$ is a subring of $K[x]_{\langle x \rangle}$ and that the polynomials $g \in K[x]$ with $g(0) \neq 0$ are the units in $K[x]_{\langle x \rangle}$.

  We can formulate problems analogous to those above for ideals in $K[x]_{\langle x \rangle}$ and modules over $K[x]_{\langle x \rangle}$ instead of $K[x]$ (the geometric problems should be interpreted as concerning properties of the variety in a neighborhood of the given point).

- Compute the projective closure of an affine variety $V(I)$, that is, compute the **homogenization** of $I$ with respect to a new variable $x_0$.

- If $X$ is a closed subscheme of projective $n$–space defined by a homogeneous ideal $I \subset K[x_0, x_1, \ldots, x_n]$ compute the ideal $I_X$ of $X$, that is, compute the **saturation** of $I$ (with respect to the irrelevant ideal):
$$I : \langle x_0, \ldots, x_n \rangle^\infty = \bigcup_{k=1}^{\infty} (I : \langle x_0, \ldots, x_n \rangle^k).$$
  More generally, if ideals $I, J \subset K[x]$ are given, compute the **saturation** of $I$ with respect to $J$:
$$I : J^\infty = \bigcup_{k=1}^{\infty} (I : J^k).$$

Later-on we will see that all these problems have solutions based on Buchberger's algorithm or variants thereof.

## 3. Buchberger's algorithm: how to compute Gröbner bases

In this section we closely follow the forthcoming book of Decker, Greuel and Pfister (2001) to which we refer for details and proofs.

### 3.1. Monomial orders and associated rings

Let $K$ be a field, and let $K[x] = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $K$.

**Definition 3.1.** *A* **monomial order** *on $K[x]$ is a total order $>$ on the set of monomials $\{x^\alpha \mid \alpha \in \mathbb{N}^n\}$ satisfying*

$$x^\alpha > x^\beta \;\Rightarrow\; x^{\alpha+\gamma} > x^{\beta+\gamma} \;\; \textit{for all} \;\; \alpha, \beta, \gamma \in \mathbb{N}^n.$$

*We call a monomial order $>$* **global** *respectively* **local** *respectively* **mixed** *if $x_i > 1$ for all $i$ respectively $x_i < 1$ for all $i$ respectively if there exist $i, j$ so that $x_i > 1$ and $x_j < 1$.* □

This notion is justified by the properties of the associated ring to be defined later-on.

**Remark 3.2.** Let $>$ be a monomial order on $K[x]$.

(1) We often identify the set of monomials with $\mathbb{N}^n$ and write $\alpha > \beta$ if $x^\alpha > x^\beta$.

(2) By a theorem of Robbiano (1986) there exists a (non–unique) matrix $A \in \mathrm{GL}(n, \mathbb{R})$ such that $\alpha > \beta$ iff $A \cdot \alpha$ is bigger than $A \cdot \beta$ with respect to the lexicographical order on $\mathbb{R}^n$. We say then that $>$ **is given by** $A$.

(3) On $\mathbb{N}^n$ we have the **natural partial order** $>_{\mathrm{nat}}$:

$$\alpha \geq_{\mathrm{nat}} \beta \;\Leftrightarrow\; \alpha - \beta \in \mathbb{N}^n \;\Leftrightarrow\; x^\beta | x^\alpha.$$

The following are equivalent:

(a) $>$ is global.

(b) If $>$ is given by a matrix $A$, then the first non–zero entry of each column of $A$ is positive.

(c) $>$ refines the natural partial order, that is

$$\alpha >_{\mathrm{nat}} \beta \;\Rightarrow\; x^\alpha > x^\beta.$$

(d) $>$ is a well–ordering. □

The implication (c) $\Rightarrow$ (d) follows from Gordan's Lemma, which in this context tells us that every subset of $\mathbb{N}^n$ has at most finitely many minimal elements with respect to $>_{\text{nat}}$.

**Definition 3.3.** Let $>$ be a monomial order on $K[x]$. Then every $f \in K[x] \smallsetminus \{0\}$ has a unique representation

$$f = cx^\alpha + \tilde{f}$$

with $c \in K \smallsetminus \{0\}$ and $\alpha > \tilde{\alpha}$ for every non–zero term $\tilde{c}x^{\tilde{\alpha}}$ of $\tilde{f}$. We set

$$\begin{array}{ll} \mathrm{lm}(f) := \mathrm{lm}_>(f) := x^\alpha, & \text{the } \textbf{leading monomial} \text{ of } f, \\ \mathrm{lc}(f) := \mathrm{lc}_>(f) := c, & \text{the } \textbf{leading coefficient} \text{ of } f, \\ \exp(f) := \exp_>(f) := \alpha, & \text{the } \textbf{leading exponent} \text{ of } f. \end{array}$$

For a subset $\mathcal{G} \subset K[x]$ we define the **leading ideal** (**initial ideal**) of $\mathcal{G}$ as

$$\mathrm{L}(\mathcal{G}) := \mathrm{L}_>(\mathcal{G}) := \langle\, \mathrm{lm}(g) \mid g \in \mathcal{G} \smallsetminus \{0\} \rangle_{K[x]} \,. \qquad \square$$

In the following examples we order the variables by $x_1 > \cdots > x_n$, and we use the case of three variables $x > y > z$ for illustrating some important monomial orders: we write down a corresponding matrix $A$ and a particular polynomial $f$ as a sum of monomials in decreasing order.

**Example 3.4.** Typical global orders are:

(1) The **lexicographical order lp**:

$$x^\alpha >_{\text{lp}} x^\beta \;:\Leftrightarrow\; \text{the first non–zero entry of } \alpha - \beta \text{ is positive}\,.$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$f = x^4 + x^3z + x^2y^2 + yz^4 + z^5.$$

(2) The **degree reverse lexicographical order dp**:

$$x^\alpha >_{\text{dp}} x^\beta \;:\Leftrightarrow\; \deg x^\alpha > \deg x^\beta \quad \text{or} \quad \begin{array}{l} \deg x^\alpha = \deg x^\beta \\ \text{and the last non–zero entry} \\ \text{of } \alpha - \beta \text{ is negative.} \end{array}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$f = yz^4 + z^5 + x^4 + x^2y^2 + x^3z \ . \qquad \square$$

**Example 3.5.** Typical local orders are:

(1) The **negative lexicographical order ls**:

$$x^\alpha >_{\mathrm{ls}} x^\beta \ :\Leftrightarrow \ \text{ the first non–zero entry of } \alpha - \beta \text{ is negative .}$$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$f = z^5 + yz^4 + x^2y^2 + x^3z + x^4.$$

(2) The **negative degree reverse lexicographical order ds**:

$$x^\alpha >_{\mathrm{ds}} x^\beta \ :\Leftrightarrow \ \deg x^\alpha < \deg x^\beta \quad \text{or} \quad \begin{array}{l} \deg x^\alpha = \deg x^\beta \\ \text{and the last non–zero entry} \\ \text{of } \alpha - \beta \text{ is negative.} \end{array}$$

$$A = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$f = x^4 + x^2y^2 + x^3z + yz^4 + z^5. \qquad \square$$

**Definition 3.6.** *If a vector $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ is given, then we define the* **weighted degree** *of a monomial $x^\alpha$ as*

$$w\text{– }\deg x^\alpha = \langle w, \alpha \rangle = w_1\alpha_1 + \cdots + w_n\alpha_n \ . \qquad \square$$

**Example 3.7.** Let $w_1, \dots, w_n \in \mathbb{R}$ be strictly positive.

(1) The **weighted degree reverse lexicographical order wp($\mathbf{w_1}, \dots, \mathbf{w_n}$)** is defined as dp with deg replaced by $w$–deg.

$$A = \begin{pmatrix} w_1 & w_2 & w_3 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\mathrm{wp}(5,3,4) : f = x^4 + z^5 + x^3z + yz^4 + x^2y^2.$$

(2) The **negative weighted degree reverse lexicographical order** $\mathbf{ws(w_1, \ldots, w_n)}$ is defined as ds with deg replaced by $w$–deg.

$$A = \begin{pmatrix} -w_1 & -w_2 & -w_3 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\mathrm{ws}(5,5,4) : f = x^3z + x^4 + x^2y^2 + z^5 + yz^4.$$

wp is global and ws is local. □

**Definition 3.8.** *Given two monomial orders* $>_1$ *on* $K[x]$ *and* $>_2$ *on* $K[y]$ *we define the* **product order** *or* **block order** $> = (>_1, >_2)$ *on* $K[x,y]$ *by*

$$x^\alpha y^\beta > x^\gamma y^\delta \;:\Leftrightarrow\; x^\alpha >_1 x^\gamma \;\text{ or }\; x^\alpha = x^\gamma \;\text{ and }\; y^\beta >_2 y^\delta. \qquad \square$$

**Remark 3.9.** If $>_1$ is given by $A_1$ and $>_2$ by $A_2$, then $(>_1, >_2)$ is given by

$$A = \left(\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right). \qquad \square$$

**Example 3.10.**

$$(\mathrm{dp}(1), \mathrm{ls}(2)) : f = x^4 + x^3z + x^2y^2 + z^5 + yz^4,$$

$$(\mathrm{ds}(1), \mathrm{lp}(2)) : f = x^2y^2 + x^3z + x^4 + yz^4 + z^5. \qquad \square$$

**Remark 3.11.** Let $>$ be a monomial order on $K[x]$. Then for all $f, g \in K[x] \setminus \{0\}$ the following holds:

(1) $\mathrm{lm}(gf) = \mathrm{lm}(g)\,\mathrm{lm}(f)$.

(2) $\mathrm{lm}(g+f) \leq \max(\mathrm{lm}(f), \mathrm{lm}(g))$ with inequality holding iff $\mathrm{lm}(g) = \mathrm{lm}(f)$ and $\mathrm{lc}(f) = -\mathrm{lc}(g)$.

In particular,

$$S_> := \{u \in K[x] \setminus \{0\} \mid \mathrm{lm}(u) = 1\}$$

is a multiplicatively closed subset of $K[x]$. □

The localization of $K[x]$ with respect to $S_>$ plays an important role for local and mixed orders.

**Definition 3.12.** *Let* $>$ *be a monomial order on* $K[x]$. *Then we write*

$$\mathrm{Loc}\,K[x] := \mathrm{Loc}_>\,K[x] := S_>^{-1}K[x] = \{\frac{f}{u} \mid f \in K[x], u \in S_>\}$$

*for the localization of $K[x]$ with respect to $S_>$. We call* $\mathrm{Loc}\, K[x]$ *the* **ring associated to $K[x]$ and $>$.** □

**Remark 3.13.** (1) We have

$$K[x] \subset \mathrm{Loc}\, K[x] \subset K[x]_{\langle x \rangle} ,$$

where $K[x]_{\langle x \rangle}$ denotes the localization of $K[x]$ with respect to the maximal ideal $\langle x \rangle = \langle x_1, \ldots, x_n \rangle$. $\mathrm{Loc}\, K[x]$ is noetherian, it is $K[x]$–flat and $K[x]_{\langle x \rangle}$ is $\mathrm{Loc}\, K[x]$–flat.

(2) $\mathrm{Loc}\, K[x] = K[x]$ iff $>$ is global.

(3) $\mathrm{Loc}\, K[x] = K[x]_{\langle x \rangle}$ iff $>$ is local.

(4) Mixed orders typically occur as a product order of two orders with one global and the other local. Many constructions with Gröbner bases in $K[x]$ use a set of auxiliary variables which have to be eliminated at some point. If one wants to perform such constructions in $K[x]_{\langle x \rangle}$ the auxiliary variables must be bigger than 1 (compare Section 4 below). Hence mixed orders occur naturally in this context. □

lm and lc are also defined for elements in the localization.

**Remark and Definition 3.14.** (1) Let $f \in \mathrm{Loc}\, K[x]$. Choose $u \in S_>$ with $\mathrm{lc}(u) = 1$ and $uf \in K[x]$. Then

$$\mathrm{lm}(f) := \mathrm{lm}(uf) \in K[x], \quad \mathrm{lc}(f) := \mathrm{lc}(uf) \in K$$

are defined independently of $u$. Note that $u \in \mathrm{Loc}\, K[x]$ is a unit in $\mathrm{Loc}\, K[x]$ iff $u \in S_>$, that is, iff $\mathrm{lm}(u) = 1$ .

(2) For a subset $\mathcal{G} \subset \mathrm{Loc}\, K[x]$ we define the **leading ideal** of $\mathcal{G}$ as

$$\mathrm{L}(\mathcal{G}) := \langle \mathrm{lm}(g) \mid g \in \mathcal{G} \smallsetminus \{0\} \rangle_{K[x]} \subset K[x] . \qquad \square$$

For applications such as the computation of syzygies we need to extend our definitions from ideals to submodules of free modules.

**Definition 3.15.** *Let $F$ be a free $K[x]$–module with basis $e_1, \ldots, e_r$. Then every $x^\alpha e_i \in F$ is called a* **monomial of $F$ (involving component $i$)**. □

**Definition 3.16.** *A* **monomial order** *on $F$ is a total order $>$ on the set of monomials $\{x^\alpha e_i \mid \alpha \in \mathbb{N}^n, i = 1, \ldots, r\}$ satisfying*

$$x^\alpha e_i > x^\beta e_j \Rightarrow x^{\alpha+\gamma} e_i > x^{\beta+\gamma} e_j \ \text{ for all } \ \alpha, \beta, \gamma \in \mathbb{N}^n, \ i, j = 1, \ldots, r . \qquad \square$$

**Remark and Definition 3.17.** Identify $e_i$ with the $i$th canonical basis vector $e_i$ of $\mathbb{N}^r$. Write $E^r = \{e_1, \ldots, e_r\}$ and consider $\mathbb{N}^n \times E^r \subset \mathbb{N}^n \times \mathbb{N}^r = \mathbb{N}^{n+r}$.

Then the natural partial order on $\mathbb{N}^{n+r}$ induces a partial order $>_{\mathrm{nat}}$ on the set of monomials of $F$, which is given by

$$\begin{aligned} x^\alpha e_i \geq_{\mathrm{nat}} x^\beta e_j \quad &\Leftrightarrow \quad i=j \ \text{ and } \ x^\beta | x^\alpha \\ &\Leftrightarrow: \quad x^\beta e_j | x^\alpha e_i \end{aligned}$$

(in this case we say that $x^\alpha e_i$ is **divisible** by $x^\beta e_j$ with **quotient** $x^{\alpha-\beta} \in K[x]$). Similar to the ideal case, a monomial order on $F$ is a well–ordering iff it refines the natural partial order on $\mathbb{N}^{n+r}$. □

Typically, we consider monomial orders on $F$ of the following form:

**Definition 3.18.** *A monomial order $>_{\mathrm{m}}$ on $F$ is* **compatible** *with a given monomial order $>$ on $K[x]$ if*

$$x^\alpha > x^\beta \Rightarrow x^\alpha e_i >_{\mathrm{m}} x^\beta e_i \textit{ for all } \alpha, \beta \in \mathbb{N}^n, \ i = 1, \ldots, r \ .$$

*In this case we call $>_{\mathrm{m}}$* **global** *respectively* **local** *respectively* **mixed** *if $>$ has the corresponding property.* □

**Remark and Definition 3.19.** Let $>_{\mathrm{m}}$ be a monomial order on $F$.

(1) For $f \in F \smallsetminus \{0\}$ and $\mathcal{G} \subset F$ the leading monomial $\mathrm{lm}(f) \in F$, the leading coefficient $\mathrm{lc}(f) \in K$, and the leading submodule $\mathrm{L}(\mathcal{G}) \subset F$ are defined as in the ideal case.

(2) If $>_{\mathrm{m}}$ is compatible with a monomial order $>$ on $K[x]$ then the notions in (1) carry over to free $\mathrm{Loc}_> K[x]$–modules as in the ideal case.

(3) If $>_{\mathrm{m}}$ is compatible with a monomial order $>$ on $K[x]$ then $>_{\mathrm{m}}$ is global iff $>_{\mathrm{m}}$ is a well–ordering. □

**Example 3.20.** If a monomial order $>$ on $K[x]$ is given we may define compatible monomial orders $>_{\mathrm{m}}$ on $F$ as follows:

$$x^\alpha e_i >_{\mathrm{m}} x^\beta e_j :\Leftrightarrow i > j \ \text{ or } \ i = j \ \text{ and } \ x^\alpha > x^\beta,$$

giving priority to the components, and

$$x^\alpha e_i >_{\mathrm{m}} x^\beta e_j :\Leftrightarrow x^\alpha > x^\beta \ \text{ or } \ x^\alpha = x^\beta \ \text{ and } \ i > j \ ,$$

giving priority to the monomials in $K[x]$. □

In what follows, when considering a monomial order $>$ on a free $K[x]$–module, we always suppose that $>$ is compatible with a monomial order on $K[x]$ which is also denoted by $>$.

### 3.2. Standard bases and normal forms

Let $R = K[x]$ be a polynomial ring as before. We write

$$\check{R} := \mathrm{Loc}_{>} K[x] \ ,$$

and, if $M$ is any $R$–module,

$$\check{M} := \check{R} \otimes_R M \ .$$

We fix a free $R$–module $F$ with basis $e_1, \ldots, e_r$ and consider it as an $R$–submodule of the free $\check{R}$–module $\check{F}$ with basis $e_1, \ldots, e_r$. We also fix a monomial order $>$ on $F$.

**Definition 3.21.** *Let $I \subset \check{F}$ be a submodule. A finite set $\mathcal{G} \subset I$ is called a* **standard basis** *of $I$ if $L(\mathcal{G}) = L(I)$, that is, if for any $f \in I \smallsetminus \{0\}$ there exists a $g \in \mathcal{G}$ with $\mathrm{lm}(g)|\mathrm{lm}(f)$. If $>$ is a well–ordering, then a standard basis of $I$ is also called a* **Gröbner basis** *of $I$.* □

Standard bases always exist since $\check{R}$ is noetherian. Note, however, that standard bases are not uniquely determined by $I$ and $>$.

**Remark and Definition 3.22.** A standard basis $\mathcal{G}$ of $I$ is called **minimal** if $0 \notin \mathcal{G}$ and if $\mathrm{lm}(g) \notin L(\mathcal{G} \smallsetminus \{g\})$ for any $g \in \mathcal{G}$. □

Minimal standard bases are still not unique, but the number of elements in a minimal standard basis is uniquely determined by $I$ and $>$ (see also Remark 3.33 below for reduced standard bases). Note that one obtains a minimal standard basis of $I$ from any given standard basis $\mathcal{G}$ of $I$ by successively deleting those $g \in \mathcal{G}$ with $\mathrm{lm}(h)|\mathrm{lm}(g)$ for some $h \in \mathcal{G} \smallsetminus \{g\}$. In order to compute standard bases we need a **normal form** (or **division**) **algorithm**.

**Definition 3.23.** *Let $\mathcal{G} = \{g_1, \ldots, g_s\} \subset \check{F}$ be a finite subset.*

(1) *A* **standard representation** *of an element $f \in \langle \mathcal{G} \rangle_{\check{R}}$ is an expression of type*

$$f = \sum_{i=1}^{s} a_i g_i, \quad a_i \in \check{R}$$

*with*

$$\mathrm{lm}(f) \geq \mathrm{lm}(a_i g_i)$$

*whenever both sides are defined.*

(2) *Let $f$ and $\mathrm{NF}(f, \mathcal{G})$ be elements in $\check{F}$. Then $\mathrm{NF}(f, \mathcal{G})$ is called a* **normal form** *of $f$ with respect to $\mathcal{G}$ if the following holds:*

(a) $\mathrm{NF}(f,\mathcal{G}) \neq 0 \Rightarrow \mathrm{lm}\big(\mathrm{NF}(f,\mathcal{G})\big) \notin L(\mathcal{G})$ .

(b) $f - \mathrm{NF}(f,\mathcal{G}) \in \langle \mathcal{G} \rangle_{\check{R}}$ .

$\mathrm{NF}(f,\mathcal{G})$ *is called a* **weak normal form** *of $f$ with respect to $\mathcal{G}$ if instead of (b) the following condition holds:*

(b') *There exists a unit $u \in \check{R}$ such that*

$$uf - \mathrm{NF}(f,\mathcal{G}) \in \langle \mathcal{G} \rangle_{\check{R}}.$$

(3) *A map*

$$\mathrm{NF}(-,\mathcal{G}) : \check{F} \to \check{F},\ f \mapsto \mathrm{NF}(f,\mathcal{G})$$

*is called a* **normal form** *(respectively* **weak normal form***) on $\check{F}$ with respect to $\mathcal{G}$ if* $\mathrm{NF}(f,\mathcal{G})$ *is a normal form (respectively weak normal form) of $f$ with respect to $\mathcal{G}$ for all $f \in \check{F}$. If in addition for all $f \in \check{F}$ there exists a standard representation of $f - \mathrm{NF}(f,\mathcal{G})$ (respectively of $uf - \mathrm{NF}(f,\mathcal{G})$ for some unit $u \in \check{R}$) then* $\mathrm{NF}(-,\mathcal{G})$ *is called a* **normal form** *(respectively* **weak normal form***) on $\check{F}$ with* **standard representation** *with respect to $\mathcal{G}$.*

□

In what follows we will describe algorithms for computing normal forms with standard representation. If the given order is not a well-ordering and if we are interested only in the normal form itself (and not also in the standard representation) then the computation of weak normal forms is less costly than the computation of normal forms. For our applications it is enough to consider weak normal forms. This is demonstrated by the following result.

**Lemma 3.24.** *Let $I \subset \check{F}$ be a submodule, let $\mathcal{G} \subset I$ be a standard basis of $I$, and let* $\mathrm{NF}(-,\mathcal{G})$ *be a weak normal form on $\check{F}$ with respect to $\mathcal{G}$. Then the following holds.*

(1) *For any $f \in \check{F}$ we have $f \in I \Leftrightarrow \mathrm{NF}(f,\mathcal{G}) = 0$.*

(2) *If $J \subset \check{F}$ is a submodule with $I \subset J$, then $L(I) = L(J)$ implies $I = J$.*

(3) *$I = \langle \mathcal{G} \rangle_{\check{R}}$, that is, $\mathcal{G}$ generates $I$ as an $\check{R}$–module.* □

For the subsequent algorithms we need the following notation.

**Definition 3.25.** *Let $f, g \in \check{F} \smallsetminus \{0\}$. Write* $\mathrm{lm}(f) =: x^\alpha e_i$, $\mathrm{lm}(g) =: x^\beta e_j$ *and*

$$\gamma := \mathrm{lcm}(\alpha,\beta) := \big(\max(\alpha_1,\beta_1),\ldots,\max(\alpha_n,\beta_n)\big).$$

*for the* **least common multiple** *of $\alpha$ and $\beta$. Then the* **s–polynomial** *of $f$ and $g$ is defined as*

$$\operatorname{spoly}(f,g) := \begin{cases} x^{\gamma-\alpha} f - \dfrac{\operatorname{lc}(f)}{\operatorname{lc}(g)} x^{\gamma-\beta} g & if \;\; i = j, \\ 0 & if \;\; i \neq j. \end{cases} \qquad \square$$

**Remark 3.26.** (1) Of course, $\operatorname{spoly}(f,g) \in \check{F}$ is only a polynomial in the case $f, g \in K[x]$.

(2) If $\operatorname{lm}(g)|\operatorname{lm}(f)$, say $\operatorname{lm}(g) = x^\beta e_i$, $\operatorname{lm}(f) = x^\alpha e_i$ with $\alpha - \beta \geq 0$, then the s–polynomial is especially simple,

$$\operatorname{spoly}(f,g) = f - \frac{lc(f)}{lc(g)} x^{\alpha-\beta} g,$$

and $\operatorname{lm}\big(\operatorname{spoly}(f,g)\big) < \operatorname{lm}(g)$ (compare Gordan's proof of Hilbert's Basis Theorem).

(3) For the normal form algorithms we only need s–polynomials as in (2), while for the subsequent standard basis algorithm we need s–polynomials in the general form above. Our asymetric definition of s–polynomials allows to use the same expression in both algorithms. $\square$

**Buchberger's Normal Form Algorithm 3.27.** Suppose that $>$ is a well–ordering.

**Input:** $f \in F$ and a finite subset $\mathcal{G} \subset F$.
**Output:** a normal form $h \in F$ of $f$ with respect to $\mathcal{G}$.

**procedure NFBuchberger**$(f, \mathcal{G})$
$h = f$;
**while** $(h \neq 0$ **and** $\mathcal{G}_h = \{g \in \mathcal{G} \mid \operatorname{lm}(g)|\operatorname{lm}(h)\} \neq \emptyset)$ **do**
  choose $g \in \mathcal{G}_h$;
  $h = \operatorname{spoly}(h, g)$;
**return** $h$ $\square$

**Remark 3.28.** (1) NFBuchberger terminates since $>$ is a well–ordering.

(2) The output depends on the choices made in the while loop. Deterministic variants of NFBuchberger are obtained by fixing a selection strategy.

(3) By keeping track of the terms $t$ in $\operatorname{spoly}(h,g) = h - tg$ the algorithm also computes a standard expression for $f-$NFBuchberger$(f, \mathcal{G})$. $\square$

If $>$ is not a well–ordering, then NFBuchberger does not necessarily terminate in finite time as shown by the standard example in one variable $x$, with $x < 1$, $f = x$ and $\mathcal{G} = \{g = x - x^2\}$. Let us describe a more general normal form algorithm due to Mora (1982).

**Definition 3.29.** For a monomial $x^\alpha e_i \in F$ set $\deg x^\alpha e_i := \deg x^\alpha$. For an arbitrary element $f \in F$ let $\deg f$ be the maximal degree of all monomials occurring in $f$ and define

$$\mathrm{ecart}(f) := \deg f - \deg \mathrm{lm}(f). \qquad \square$$

**Mora's Normal Form Algorithm 3.30.**

**Input:** $f \in F$ and a finite subset $\mathcal{G} \subset F$.
**Output:** a weak normal form $h \in F$ of $f$ with respect to $\mathcal{G}$.

**procedure NFMora**$(f, \mathcal{G})$
$h = f$; $T = \mathcal{G}$;
**while** $(h \neq 0$ **and** $T_h = \{g \in T \mid \mathrm{lm}(g)|\mathrm{lm}(h)\} \neq \emptyset)$ **do**
  choose $g \in T_h$ with $\mathrm{ecart}(g)$ minimal;
  **if** $(\mathrm{ecart}(g) > \mathrm{ecart}(h))$ **then**
    $T = T \cup \{h\}$;
  $h = \mathrm{spoly}(h, g)$;
**return** $h$ $\square$

**Remark 3.31.** (1) That NFMora terminates is most easily seen by homogenizing.

(2) If the input is homogeneous then the ecart is always zero and NFMora equals NFBuchberger. If $>$ is a well–ordering then $\mathrm{lm}(g)|\mathrm{lm}(h)$ implies that $\mathrm{lm}(g) \leq \mathrm{lm}(h)$, hence $T$ is always equal to $\mathcal{G}$. So in this case NFMora equals NFBuchberger with a special selection strategy in the while loop.

(3) By storing intermediate results as above the algorithm actually computes a unit $u \in \check{R}$ and a standard expression for $uf-\mathrm{NFMora}(f, \mathcal{G})$.

(4) The computation of normal forms with standard representations in the general case $f \in \check{F}$ and $\mathcal{G} \subset \check{F}$ can obviously be reduced to the computation of weak normal forms with standard representations in the special case above. This shows particularly that such normal forms exist. $\square$

In what follows NF will always stand for a weak normal form with standard representation or for an algorithm which computes such a form. We are now ready to present Buchberger's algorithm.

**Buchberger's Algorithm 3.32.**

**Input**: a finite subset $\mathcal{G} \subset \check{F}$.
**Output**: a standard basis of the submodule $I = \langle \mathcal{G} \rangle_{\check{R}} \subset \check{F}$.

**procedure STDBuchberger**$(\mathcal{G})$

$\mathcal{S} = \mathcal{G}$;
$P = \{(f,g) \mid f,g \in \mathcal{S}\} \subset \mathcal{S} \times \mathcal{S}$;
**while** $(P \neq \emptyset)$ **do**
  choose $(f,g) \in P$;
  $P = P \smallsetminus \{(f,g)\}$.
  $h = \mathrm{NF}\big(\mathrm{spoly}(f,g), \mathcal{S}\big)$;
  **if** $(h \neq 0)$ **then**
    $P = P \cup \{(h,f) \mid f \in \mathcal{S}\}$;
    $\mathcal{S} = \mathcal{S} \cup \{h\}$;
**return** $\mathcal{S}$ □

STDBuchberger terminates because of Gordan's Lemma. It is correct due to the following criterion.

**Theorem 3.32 (Buchberger's criterion).** *Let $I \subset \check{F}$ be a submodule, and let $\mathcal{G} = \{g_1, \ldots, g_s\} \subset I$. Then the following are equivalent:*

(1) *$\mathcal{G}$ is a standard basis of $I$.*

(2) $\mathrm{NF}(f, \mathcal{G}) = 0$ *for all $f \in I$.*

(3) *Each $f \in I$ has a standard representation with respect to $\mathcal{G}$.*

(4) *$\mathcal{G}$ generates $I$ and* $\mathrm{NF}\big(\mathrm{spoly}(g_i, g_j), \mathcal{G}\big) = 0$ *for $i, j = 1, \ldots, s$.* □

The crucial part of the proof, that is, the implication (4) $\Rightarrow$ (1) is most easily done by using syzygies. Compare Section 3.3 below.

**Remark 3.33.** One can introduce the notion **reduced Gröbner basis** and show that such a basis is uniquely determined by $I$. There is a straightforward algorithm which computes a reduced Gröbner basis from a minimal one (alternatively one can use in STDBuchberger a variant of NFBuchberger which computes reduced normal forms). More generally, one can consider reduced standard bases, but reduced standard bases do not necessarily exist. □

### 3.3. Syzygies

We explain how to compute syzygies and thus free resolutions following Schreyer (1980). We use the same notations as in Section 3.2.

Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be elements in $\check{F}$, and let $I = \langle \mathcal{G} \rangle_{\check{R}} \subset \check{F}$. Our goal is to compute the module $\mathrm{Syz}\, I$ of (first order) syzygies of $I$, that is, the kernel of the morphism

$$\phi_0 : \check{F}_0 \to \check{F},\ \varepsilon_j \mapsto g_j,$$

where $F_0$ is a free $R$–module with basis $\varepsilon_1, \dots, \varepsilon_s$.

**Remark and Definition 3.34.** Set

$$x^\alpha \varepsilon_i >_S x^\beta \varepsilon_j \;:\Leftrightarrow\; \begin{array}{l} \mathrm{lm}(x^\alpha g_i) > \mathrm{lm}(x^\beta g_j) \;\text{ or} \\ \mathrm{lm}(x^\alpha g_i) = \mathrm{lm}(x^\beta g_j) \;\text{ and }\; i < j. \end{array}$$

Then $>_S$ is a monomial order on $F_0$ which we call the **Schreyer order** on $F_0$ (induced by $\phi_0$ and $>$). □

We introduce some notations. For each pair $i < j$ such that $\mathrm{lm}(g_i)$ and $\mathrm{lm}(g_j)$ involve the same component, say $\mathrm{lm}(g_i) = x^{\alpha_i} e_\nu$, $\mathrm{lm}(g_j) = x^{\alpha_j} e_\nu$, define the monomial

$$m_{ji} := x^{\mathrm{lcm}(\alpha_i, \alpha_j) - \alpha_i} \in K[x].$$

If $c_i = \mathrm{lc}(g_i)$ and $c_j = \mathrm{lc}(g_j)$ then

$$m_{ji} g_i - \frac{c_i}{c_j} m_{ij} g_j = \mathrm{spoly}(g_i, g_j).$$

Let $\mathrm{NF}\big(\mathrm{spoly}(g_i, g_j), \mathcal{G}\big)$ be a weak normal form with standard representation

$$m_{ji} g_i - \frac{c_i}{c_j} m_{ij} g_j - \mathrm{NF}\big(\mathrm{spoly}(g_i, g_j), \mathcal{G}\big) = \sum_{\nu=1}^{s} a_\nu^{(ij)} f_\nu,\; a_\nu^{(ij)} \in \tilde{R},$$

for $\mathrm{spoly}(g_i, g_j)$ with respect to $\mathcal{G}$. Suppose that $\mathrm{NF}\big(\mathrm{spoly}(g_i, g_j), \mathcal{G}\big) = 0$. Then

$$s_{ij} := m_{ji} \varepsilon_i - \frac{c_i}{c_j} m_{ij} \varepsilon_j - \sum_\nu a_\nu^{(ij)} \varepsilon_\nu \in \mathrm{Syz}\, I$$

and

$$\mathrm{lm}(s_{ij}) = m_{ji} \varepsilon_i.$$

With these notations one can state (and prove) Buchberger's criterion and Schreyer's result on syzygies simultaneously:

**Theorem (Schreyer) 3.35.** *With the notations above suppose that* $\mathrm{NF}\big(\mathrm{spoly}(g_i, g_j), \mathcal{G}\big) = 0$ *for* $i < j$. *Then the following holds.*

(1) $\mathcal{G}$ *is a standard basis of* $I$.

(2) *The* $\{s_{ij}\}$ *form a standard basis of* $\mathrm{Syz}\, I$ *with respect to the Schreyer order. In particular, the* $\{s_{ij}\}$ *generate* $\mathrm{Syz}\, I$. □

This result gives rise to an algorithm for computing syzygies. It also leads to a simple proof of Hilbert's syzygy theorem which works in the general context of finitely generated $\tilde{R}$–modules.

## 4. Basic applications

Algorithms for primary decomposition and normalization will be discussed in Section 5. Let us now give answers to the other problems posed in Section 2. For proofs and details we refer to Eisenbud (1995) and Decker, Greuel and Pfister (2001).

Since we can compute Gröbner bases we can compute **Hilbert polynomials** and **Hilbert functions** via Macaulay's Theorem 1.17. Moreover, Lemma 3.24 settles the **ideal membership problem**.

**Radical membership.** Let an ideal $I \subset K[x]$ and $f \in K[x]$ be given. Add a slack variable $t$. Then $f \in \sqrt{I}$ iff $1 \in \langle I, tf-1\rangle_{K[x,t]}$. This condition can be checked by a Gröbner basis computation.

We already know how to compute **syzygies**. **Intersections of ideals** and **ideal quotients** can be computed via syzygies (see the exercises) or with the help of slack variables (see Cox, Little and O'Shea, 1997). In particular, by iteration, one can compute **saturations**

$$I : J^\infty = \bigcup_{k=1}^{\infty} (I : J^k).$$

**Elimination.** Let $I$ be an ideal in the polynomial ring

$$K[x,y] = K[x_1, \ldots, x_n, y_1, \ldots y_m].$$

We explain hot to eliminate the variables $x_1, \ldots, x_n$ from $I$, that is, how to compute $I \cap K[y]$ .

**Definition 4.1.** A monomial order $>$ on $K[x,y]$ is an **elimination order** with respect to $x_1, \ldots, x_n$ if the following holds: if $g \in K[x,y]$ with $\operatorname{lm}(g) \in K[y]$, then $g \in K[y]$. □

**Example 4.2.** Any product order $> = (>_1, >_2)$ on $K[x,y]$ with $>_1$ global on $K[x]$ and $>_2$ arbitrary on $K[y]$ is an elimination order with respect to $x_1, \ldots, x_n$. □

**Proposition 4.3.** *Let $>$ be an elimination order on $K[x,y]$ with respect to $x_1, \ldots, x_n$, and let $\mathcal{G}$ be a standard basis of $I$ with respect to $>$. Then $\mathcal{G} \cap K[y]$ is a standard basis of $I \cap K[y]$ with respect to the restriction of $>$ to $K[y]$.* □

The kernel of a morphism

$$\phi : K[y_1, \ldots, y_m] \to K[x_1, \ldots, x_n]/I \ , \quad y_i \mapsto f_i \ ,$$

can be computed via elimination:

**Proposition 4.4.** *Pick polynomial representatives $F_i$ of $f_i$, and let $J$ be the ideal*

$$J = IK[x,y] + \langle F_1 - y_1, \dots, F_m - y_m \rangle_{K[x,y]}.$$

*Then*

$$\ker \phi = J \cap K[y]. \qquad \square$$

Finally, we explain how to compute the homogenization of a given ideal with respect to a new variable.

**Proposition 4.5.** *Let $I \subset K[x_1, \dots, x_n]$ be an ideal. Pick a monomial order $>$ on $K[x_1, \dots, x_n]$ which respects degrees, that is, which satisfies*

$$\deg x^\alpha > \deg x^\beta \;\Rightarrow\; x^\alpha > x^\beta,$$

*and set*

$$x^\alpha x_0^d >_h x^\beta x_0^e \;:\Leftrightarrow\; x^\alpha > x^\beta \;\text{ or }\; (x^\alpha = x^\beta \text{ and } d > e).$$

*Then $>_h$ is a monomial order on $K[x_0, x_1, \dots, x_n]$ and the following holds if we homogenize with respect to $x_0$: if $g_1, \dots, g_s$ is a Gröbner basis of $I$ with respect to $>$, then the homogenized polynomials $g_1^h, \dots, g_s^h$ form a Gröbner basis of the homogenized ideal $I^h$.* $\square$

## 5. More advanced applications: algorithms for invariant theory, primary decomposition and normalization

For details and proofs in this section we refer to Decker, Greuel and Pfister (2001), Decker and de Jong (1998), Decker, Greuel and Pfister (1999), and Decker, de Jong, Greuel and Pfister (1999).

### 5.1. Algorithms for invariant theory

#### 5.1.1. The basic setting of invariant theory

The basic setting of invariant theory consists of a group $G$, a field $K$, a vector space $V$ over $K$ of finite dimension $n$, and a representation $\rho : G \to \mathrm{GL}(V)$ of $G$ on $V$. We suppose that coordinates $x_1, \dots, x_n \in V^*$ are chosen and denote by

$$K[V] := K[x] := \bigoplus_{d \geq 0} K[x_1, \dots, x_n]_d$$

the graded $K$–algebra of polynomial functions on $V$. The action of $G$ on $V$ induces an action of $G$ on $K[V]$:

$$(\pi f)(v) = f(\rho(\pi^{-1})v) \ .$$

The basic objects of study in invariant theory are the fixed point sets of such actions.

**Definition 5.1.** The **ring of invariants** in the above setting is the graded sub-algebra

$$K[V]^G := K[x]^G := \{f \in K[V] \mid \pi f = f \text{ for any } \pi \in G\} \subset K[V] \ . \qquad \square$$

**Definition 5.2.** If $K[V]^G$ is finitely generated as a $K$–algebra then any minimal (that is irredundant) finite system $f_1, \ldots, f_r$ of homogeneous generators is called a **fundamental system of invariants**. $\square$

**Example 5.3.** The symmetric group $S_n$ acts on $K[x_1, \ldots, x_n]$ by permuting the coordinates. The corresponding ring of invariants $K[x]^{S_n}$ is generated by the **elementary symmetric polynomials**

$$\sigma_1(x_1, \ldots, x_n) := x_1 + \ldots + x_n, \ldots, \sigma_n(x_1, \ldots, x_n) := x_1 \cdots x_n \ .$$

In fact,

$$K[x]^{S_n} = K[\sigma_1, \ldots, \sigma_n]$$

is a polynomial algebra. $\square$

In general, $K[V]^G$ is not a polynomial algebra, that is, there are algebraic relations between the elements of a fundamental system of invariants.

- **First fundamental problem of invariant theory**. Find a fundamental system of invariants $f_1, \ldots, f_r$ of $K[V]^G$ .
- **Second fundamental problem of invariant theory**. If a fundamental system of invariants $f_1, \ldots, f_r$ of $K[V]^G$ is given then find the syzygies on the $f_1, \ldots, f_r$ .

The second problem means to compute the kernel $I$ of the graded morphism

$$\phi_0 : K[z_1, \ldots, z_r] \to K[V]^G, \quad z_i \mapsto f_i \ ,$$

where the $z_i$ are new variables with $\deg(z_i) = \deg(f_i)$, and to compute a minimal free resolution of $I$ as a $K[z_1, \ldots, z_r]$–module (see the preceeding sections for the corresponding algorithms).

**Example 5.4.** Klein (1878/1879) studies a natural embedding of the modular curve $X(7)$ of level 7 into the projective plane $\mathbb{P}^2(\mathbb{C})$. In fact, the embedded curve is defined by the quartic invariant

$$f = \lambda^3\mu + \mu^3\nu + \nu^3\lambda$$

of $G = \mathrm{PSL}_2(\mathbb{F}_7)$ acting on the homogeneous coordinate ring of $\mathbb{P}^2(\mathbb{C})$. $G$ is a finite group of order 168, explicitly given as the subgroup of $\mathrm{GL}_3(\mathbb{C})$ generated by the matrices

$$\pi_1 = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma^4 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} A & B & C \\ B & C & A \\ C & A & B \end{pmatrix},$$

where $\gamma = e^{2\pi i/7}$ is a 7th root of unity, and where

$$A = \frac{(\gamma^5 - \gamma^2)}{\sqrt{-7}}, \quad B = \frac{(\gamma^3 - \gamma^4)}{\sqrt{-7}}, \quad C = \frac{(\gamma^6 - \gamma)}{\sqrt{-7}}\ .$$

¿From the single invariant $f$ Klein successively derives further invariants by combining standard ad hoc techniques in invariant theory with geometric arguments. The second invariant he finds is the (suitably normalized) Hessian

$$\nabla = \frac{1}{54} \begin{vmatrix} \frac{\partial^2 f}{\partial\lambda^2} & \frac{\partial^2 f}{\partial\lambda\partial\mu} & \frac{\partial^2 f}{\partial\lambda\partial\nu} \\ \frac{\partial^2 f}{\partial\mu\partial\lambda} & \frac{\partial^2 f}{\partial\mu^2} & \frac{\partial^2 f}{\partial\mu\partial\nu} \\ \frac{\partial^2 f}{\partial\nu\partial\lambda} & \frac{\partial^2 f}{\partial\nu\partial\mu} & \frac{\partial^2 f}{\partial\nu^2} \end{vmatrix}$$

which has degree 6. The plane curves defined by $\nabla$ and $f$ intersect in the 24 flexes of the curve $\{f = 0\}$. Klein shows by using geometric arguments that the flexes form the only set of 24 points on $\{f = 0\}$ which is invariant under the action of $G$. Moreover, there is no such set of lower cardinality. Thus by Bezout's theorem $f$ and $\nabla$ are, up to scalars, the only invariants in degrees $\leq 6$. Similar arguments show that the next possible degree for another generator of the invariant ring is 14 (recall that the curve $\{f = 0\}$ has 28 bitangents). Klein obtains an invariant of degree 14 which is different from $f^2\nabla$ by bordering the Hessian matrix:

$$\mathcal{C} = \frac{1}{9} \begin{vmatrix} \frac{\partial^2 f}{\partial\lambda^2} & \frac{\partial^2 f}{\partial\lambda\partial\mu} & \frac{\partial^2 f}{\partial\lambda\partial\nu} & \frac{\partial\nabla}{\partial\lambda} \\ \frac{\partial^2 f}{\partial\mu\partial\lambda} & \frac{\partial^2 f}{\partial\mu^2} & \frac{\partial^2 f}{\partial\mu\partial\nu} & \frac{\partial\nabla}{\partial\mu} \\ \frac{\partial^2 f}{\partial\nu\partial\lambda} & \frac{\partial^2 f}{\partial\nu\partial\mu} & \frac{\partial^2 f}{\partial\nu^2} & \frac{\partial\nabla}{\partial\nu} \\ \frac{\partial\nabla}{\partial\lambda} & \frac{\partial\nabla}{\partial\mu} & \frac{\partial\nabla}{\partial\nu} & 0 \end{vmatrix}$$

Finally, he finds a fourth invariant by taking the Jacobian of $f$, $\nabla$ and $\mathcal{C}$:

$$\mathcal{K} = \frac{1}{14} \begin{vmatrix} \frac{\partial f}{\partial\lambda} & \frac{\partial\nabla}{\partial\lambda} & \frac{\partial\mathcal{C}}{\partial\lambda} \\ \frac{\partial f}{\partial\mu} & \frac{\partial\nabla}{\partial\mu} & \frac{\partial\mathcal{C}}{\partial\mu} \\ \frac{\partial f}{\partial\nu} & \frac{\partial\nabla}{\partial\nu} & \frac{\partial\mathcal{C}}{\partial\nu} \end{vmatrix}$$

Geometric arguments as above show that these four invariants generate the ring of invariants and that the relation

$$(-\nabla)^7 = (\frac{\mathcal{C}}{12})^3 - 27(\frac{\mathcal{K}}{216})^2$$

generates the (first order) syzygies. □

Klein's arguments show that his ring of invariants is not a polynomial algebra. Finitely generated rings of invariants are, however, always integral over a polynomial algebra by the graded Noether normalization theorem: there exists a **homogeneous system of parameters**, that is, there exist algebraically independent, homogeneous elements $p_1, \ldots, p_m \in K[V]^G$, $m = \dim K[V]^G$, such that $K[V]^G$ is a finitely generated $K[p_1, \ldots, p_m]$–module. Recall that $K[V]^G$ is **Cohen–Macaulay** if $K[V]^G$ is even a free $K[p_1, \ldots, p_m]$–module.

**Definition 5.5.** If $K[V]^G$ is a finitely generated ring of invariants then any homogeneous system of parameters $p_1, \ldots, p_m$ is called a system of **primary invariants**. Any minimal system of homogeneous $s_0 = 1, s_1, \ldots, s_l$ generating $K[V]^G$ as a $K[p_1, \ldots, p_m]$–module is called a system of **secondary invariants**. □

**Remark 5.6.** If $K[V]^G$ is Cohen–Macaulay, $p_1, \ldots, p_m$ form a homogeneous system of parameters, and $s_0, \ldots, s_l$ is a basis of $K[V]^G$ as a free $K[p_1, \ldots, p_m]$–module, then the Hilbert series

$$H_{K[V]^G}(t) = \frac{\sum_{j=0}^{l} t^{\deg(s_j)}}{\prod_{i=1}^{m}(1 - t^{\deg(p_i)})} \,. \qquad \Box$$

**Example 5.7.** In Klein's example above 1 and $\mathcal{K}$ are secondary invariants, if $f$, $\nabla$, and $\mathcal{C}$ are chosen as primary invariants, whereas 1, $\nabla, \ldots, \nabla^6$ are secondary invariants for the primary invariants $f$, $\mathcal{C}$, and $\mathcal{K}$. The Hilbert series

$$H_{\mathbb{C}[\lambda,\mu,\nu]^G}(t) = \frac{1 + t^{21}}{(1 - t^4)(1 - t^6)(1 - t^{14})} = \frac{1 + t^6 + t^{12} + \ldots t^{36}}{(1 - t^4)(1 - t^{14})(1 - t^{21})}$$

$$= 1 + t^4 + t^6 + t^8 + t^{10} + 2t^{12} + 2t^{14} + 2t^{16} + 3t^{18} + 3t^{20} + t^{21} + \ldots \,. \qquad \Box$$

Primary and secondary invariants together form a system of generators $f_1, \ldots, f_r$ of $K[V]^G$ as a $K$–algebra (here we omit $s_0 = 1$). But, as shown by the example, such a system might not be minimal. For each $i$ one can check by linear algebra or via Gröbner bases whether $f_i$ is contained in the $\deg(f_i)$–graded piece of the sub-algebra generated by the remaining $f_j$. In this way superfluous generators can be detected and eliminated. Let us describe an alternative approach.

**Remark 5.8.** Suppose that

$$R := K[p_1, \dots, p_m] \subset A = K[V]^G$$

is a Noether normalization, and let $s_0 = 1, s_1, \dots, s_l$ be a minimal system of homogeneous generators of $A$ as an $R$–module. Then one can describe $A$ as a quotient of a polynomial ring in $m + l$ variables as follows. Let $y_1, \dots, y_l$ be new variables, and consider the $K$–algebra epimorphism

$$\psi : K[p_1, \dots, p_m, y_1, \dots, y_l] \longrightarrow A, \quad p_i \mapsto p_i,\ y_j \mapsto s_j\ .$$

The kernel of $\psi$ describes the algebraic relations between the $p_i$ and $s_j$. Let us single out two types of elements in $\ker \psi$.

(1) Every $R$–module syzygy between the $s_j$, that is, every expression

$$\alpha_0 s_0 + \alpha_1 s_1 + \ldots + \alpha_l s_l = 0\ , \quad \alpha_i \in R\ ,$$

gives an element $\alpha_0 + \alpha_1 y_1 + \ldots + \alpha_l y_l \in \ker \psi$, which we call a **linear relation**.

(2) As $A$ is also an $R$–algebra, one can form the products $s_i s_j$, $1 \le i \le j \le l$. Since $A$ is generated as an $R$–module by $s_0, \dots, s_l$ there exist $\beta_{ijk} \in R$ such that $s_i s_j = \sum_k \beta_{ijk} s_k$. In this way we obtain $\binom{l+1}{2}$ elements $y_i y_j - \beta_{ij0} - \beta_{ij1} y_1 - \cdots - \beta_{ijl} y_l$ in $\ker \psi$ , which we call **quadratic relations**.

It is an easy fact that $\ker \psi$ is generated by the linear and the quadratic relations. Superfluous generators among $p_1, \dots, p_m$, $s_1, \dots, s_l$ correspond to terms in these relations which are linear in either one of the $p_i$ or one of the $s_j$. By eliminating these generators and the corresponding relations we eventually arrive at a fundamental system of invariants, and the remaining linear and quadratic relations generate the first order syzygies between these fundamental invariants. If $A$ is Cohen–Macaulay then there are no linear relations. In any case, the quadratic relations can be computed via linear algebra or via Gröbner bases. $\square$

### 5.1.2. Finite groups

Let $G$ be a finite group explicitly given as a matrix group $G \subset \mathrm{GL}(V^*) \cong \mathrm{GL}_n(K)$ by a finite set of generating matrices, and acting on $K[V] \cong K[x]$ by linear substitution.

- **Emmy Noether's finiteness theorem** (1926) tells us that $K[x]^G$ is finitely generated of dimension n.
- Moreover, $K[x]^G$ is **Cohen–Macaulay**, if the order $|G|$ of the group is invertible in $K$ (**non–modular case**).
- In the non–modular case we are able to precompute the Hilbert series of $K[x]^G$ by the following theorem.

**Molien's Theorem 5.9.** (Molien, 1897) *In the non–modular case the Hilbert series of $K[x]^G$ is given as the Molien series of $G$ which is defined as follows.*

(1) *Let* $\operatorname{char} K = 0$. *Then the Molien series of $G$ is*

$$\frac{1}{|G|}\sum_{\pi\in G}\frac{1}{\det(1-t\pi)}\ .$$

(2) *Let* $\operatorname{char} K = p > 0$, *$|G|$ not divisible by $p$. Fix primitive $|G|$th roots of unity over $K$ and $\mathbb{Q}$ respectively, say $\lambda$ and $\tilde{\lambda}$. For $\pi \in G$ let $\lambda^{k_1(\pi)}, \ldots,$ $\lambda^{k_n(\pi)}$ be the eigenvalues of $\pi$ and set*

$$\Phi_\pi(t) := \prod_{i=1}^{n}(1 - t\tilde{\lambda}^{k_i(\pi)})\ .$$

*Then the Molien series of $G$ is*

$$\frac{1}{|G|}\sum_{\pi\in G}\frac{1}{\Phi_\pi(t)}\ . \qquad \square$$

How to compute a fundamental system of invariants? As in Klein's Example 5.4 the basic strategy is to proceed degree by degree. Thus the first task is to calculate for a given degree $d$ a $K$–basis of the graded piece $K[x]^G_d$. There are two basic methods. To be invariant under just one element of $G$ imposes a linear condition on the homogeneous polynomials of a given degree $d$. Therefore, for each $d$, we may compute a $K$–basis of $K[x]^G_d$ by solving a linear system of equations which depends on the given set of generators of $G$. In the non–modular case we may evaluate the Reynolds operator $R$ in the monomials of degree $d$ in order to calculate a $K$–basis of $K[x]^G_d$:

- If $|G|$ is invertible in $K$ then there is the well–defined **Reynolds operator**

$$R : K[V] \to K[V]\ , \quad f(x) \mapsto \frac{1}{|G|}\sum_{\pi\in G}(\pi f)(x)\ ,$$

  which projects $K[V]$ onto $K[V]^G$, and which is a $K[V]^G$–module homomorphism.

Summing up what we have so far gives a first algorithm in the non–modular case: compute invariants degree by degree until the Hilbert series of the $K$–sub-algebra of $K[x]^G$ generated by the invariants computed so far equals the Molien series. Or, again in the non–modular case, use degree bounds such as Noether's degree bound (1916) as criteria for termination.

Another idea, yielding more effective algorithms, is to compute a Noether normalization of $K[x]^G$ first.

**1. Step**. The computation of primary invariants is based on the following well–known lemma.

**Lemma 5.10.** *Let $p_1, \ldots, p_i$ be homogeneous elements of $K[x]^G$. Then*

$$\dim \langle p_1, \ldots, p_i \rangle \geq n - i$$

*and equality holds if and only if $p_1, \ldots, p_i$ can be extended to a homogeneous system of parameters.* □

**Algorithm for primary invariants 5.11.**
(Decker, Heydtmann and Schreyer, 1998)

**Input**: a set $\Delta$ of generators of $G$.
**Output**: primary invariants for $K[x]^G$.

**Procedure PrimInv**($\Delta$)
$d = 0$; $i = 0$;
**repeat**
  $d = d + 1$;
  compute a basis $b_1, \ldots, b_{c_d}$ of $K[x_1, \ldots, x_n]_d^G$;
  compute $m = \dim(\langle p_1, \ldots, p_i, b_1, \ldots, b_{c_d} \rangle)$ via Gröbner bases;
  **while** $n - i > m$ **do**
    **if** $K$ is infinite **then**
      $i = i + 1$;
      find a $K$–linear combination $p_i$ of $b_1, \ldots, b_{c_d}$ such that
        $\dim \langle p_1, \ldots, p_{i-1} \rangle > \dim \langle p_1, \ldots, p_i \rangle$;
    **else**
      find a new $K$–linear combination $q$ of $b_1, \ldots, b_{c_d}$;
      **if** $\dim \langle p_1, \ldots, p_i \rangle > \dim \langle p_1, \ldots, p_i, q \rangle$ **then**
        $i = i + 1$; $p_i = q$;
      **if** all $K$–linear combinations of $b_1, \ldots, b_{c_d}$ have been tested **then**
        **break** [of **while**–loop]
**until** $i = n$;
**return** $p_1, \ldots, p_n$ □

**2. Step**. There is an obvious algorithm for secondary invariants in the non–modular case. It is based on the first part of the following trivial remark.

**Remark 5.12.** Suppose that primary invariants $p_1, \dots, p_n$ for $K[x]^G$ are given. Then the following holds.

(1) In the non–modular case the degrees $d_j$ in which secondary invariants occur respectively their number $\mu_j$ in degree $d_j$ can be precomputed as the exponents respectively the coefficients of the univariate polynomial

$$\sum_{j=0}^{b} \mu_j t^{d_j} := H_{K[x]^G}(t) \prod_{i=1}^{n} (1 - t^{\deg(p_i)}) \ .$$

(2) The total number of secondary invariants equals

$$\frac{1}{|G|} \prod_{i=1}^{n} \deg(p_i) \ . \qquad \square$$

The following algorithm singles out those secondary invariants which are not power products of secondary invariants of lower degree. We call these secondary invariants **irreducible** and the others **reducible**.

**Algorithm for secondary invariants in the non–modular case 5.13.** (Kemper and Steel, 1997)

**Input**: the Hilbert series $H = H_{K[x]^G}$, a set $\mathcal{P} = \{p_1, \dots, p_n\}$ of primary invariants for $K[x]^G$, a Gröbner basis $\mathcal{G}$ of $\langle p_1, \dots, p_n \rangle$, the Reynolds operator $R$.

**Output**: secondary invariants for $K[x]^G$ (**irreducible** and **reducible**).

**Procedure SecInv**($H$, $\mathcal{P}$, $\mathcal{G}$, $R$)

Compute the polynomial

$$\sum_{j=0}^{b} \mu_j t^{d_j} = H(t) \prod_{i=1}^{n} (1 - t^{\deg(p_i)});$$

$I = \emptyset$; $P = \{1\}$;
**for** $j = 1, \dots, b$ **do**
  i=0;
  **while** $i < \mu_j$ **do**
    **for** all power products $q$ of elements of $I$ having degree $d_j$ **do**
      **if** $\mathrm{NF}(s_1, \mathcal{G}), \dots, \mathrm{NF}(s_i, \mathcal{G}), \mathrm{NF}(q, \mathcal{G})$ are linearly independent
          modulo $\langle p_1, \dots, p_n \rangle$ **then**
        $i = i + 1$; $s_i = q$; $P = P \cup \{s_i\}$;
    compute the monomials $m_1, \dots, m_c$ of degree $d_j$ which are not

in the initial ideal of $\langle p_1, \dots, p_n\rangle$;
**for** $r = 1, \dots, c$ **do**
**if** $\mathrm{NF}(s_1, \mathcal{G}), \dots, \mathrm{NF}(s_i, \mathcal{G}), \mathrm{NF}(R(m_r), \mathcal{G})$ are linearly independent modulo $\langle p_1, \dots, p_n\rangle$ **then**
$i = i + 1$; $s_i = R(m_r)$; $I = I \cup \{s_i\}$;
**return** I, P $\square$

It is easy to see that for the calculation of fundamental invariants and syzygies as in Remark 5.8 it is enough to check the products of type $s_i s_j$, $s_i \in I$ and $s_j \in I \cup (P \setminus \{1\})$.

**3. Step**. Finally, due to an idea of Kemper (1996), the computation of secondary invariants in the modular case can be reduced to the non–modular case by considering a subgroup $H$ whose order is invertible in $K$ (for example, the trivial subgroup will do). The given primary invariants $p_1, \dots, p_n$ for $G$ are also primary invariants for $H$. Write $A = K[p_1, \dots, p_n]$. Compute secondary invariants $t_1, \dots, t_m$ for $H$ with respect to $p_1, \dots, p_n$. Choose a subset $\Delta \subset G$ such that the elements of $\Delta \cup H$ generate $G$. Then there is a commutative diagram of graded $K[p_1, \dots, p_n]$–modules with exact rows and columns,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & K[x]^G & \rightarrow & K[x]^H & \xrightarrow{\alpha} & \bigoplus_{\pi\in\Delta} K[x] \\
 & & \uparrow & & \beta\uparrow & & \uparrow \\
0 & \rightarrow & M & \longrightarrow & A^m & \xrightarrow{\tilde{\alpha}} & A^r \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
\end{array}$$

where $\alpha$ is defined by $f \mapsto (\pi f - f)_{\pi\in\Delta}$, $\beta$ is the map $(f_1, \dots, f_m) \mapsto \sum_{j=1}^m f_j t_j$, and $r = |\Delta| \cdot \prod_{i=1}^n \deg(p_i)$. Indeed, $K[x]$ is a free $K[p_1, \dots, p_n]$–module of rank $\prod_{i=1}^n \deg(p_i)$ with a basis $\mathcal{B}$ given by those monomials which are not in the leading ideal of $\langle p_1, \dots, p_n\rangle$ (with respect to the chosen monomial order). The map $\tilde{\alpha}$, that is, the representation of each $(\pi t_j - t_j)$ in terms of $\mathcal{B}$, can be computed by linear algebra or via Gröbner bases. In this way the computation of secondary invariants for $G$ and of the $K[p_1, \dots, p_n]$–module relations between the secondary invariants can be reduced to the computation of syzygies of submodules of free $K[p_1, \dots, p_n]$–modules. $\square$

**Implementations** are available in Maple (INVAR package by Kemper, 1993), Singular (library finvar.lib by Heydtmann) and Magma (see Kemper and Steel, 1999).

### 5.1.3. Linearly reductive groups

In this section we suppose that our ground field $K$ is algebraically closed.

Linearly reductive groups are linear algebraic groups, and they are characterized by the fact that every rational representation comes with a unique Reynolds operator.

**Examples of linearly reductive groups 5.14.** (1) (Maschke, 1899) Finite groups in the non–modular case.
(2) **Tori** $(K^*)^r$.
(3) (Weyl, 1946) The **classical groups** $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ and $\mathrm{Sp}_n(\mathbb{C})$. □

Hilbert (1890) showed that the ring of invariants in the case of $\mathrm{SL}_n(\mathbb{C})$ acting on $n$–forms by linear substitution is finitely generated. In fact, his proof gives the following result.

**Hilbert's Finiteness Theorem 5.15.** *Let $G$ be a linearly reductive group, acting rationally on a finite dimensional vector space $V$. Then the ring of invariants $K[V]^G$ is finitely generated as a $K$–algebra.*

Proof. Consider the ideal $I_{\mathcal{N}}$ in $K[V]$ generated by the homogeneous invariants of positive degree. Hilbert's basis theorem implies that already finitely many of those generate $I_{\mathcal{N}}$ :

$$I_{\mathcal{N}} = \langle f_1, \dots , f_r\rangle.$$

We claim that the invariants are generated as a $K$–algebra by $f_1, \dots , f_r$. If $f$ is any homogeneous invariant of positive degree, then $f \in I$, so we can write $f = \sum h_i f_i$. Applying the Reynolds operator $R$ yields:

$$f = \sum R(h_i) f_i.$$

The $R(h_i)$ are invariants, which we may assume to be of lower degree than $f$. By induction, the $R(h_i)$ are in the sub-algebra generated by $f_1, \dots , f_r$. Hence so is $f$. □

The proof, as it stands, is non–constructive, since one has to find generators of $I_{\mathcal{N}}$. Because of this, Hilbert was harshly criticized. (Gordan: "Das ist Theologie und keine Mathematik"). Let us quote from a letter of Minkowski to Hilbert from 1892 (see Minkowski, 1973):

(...) Dass es nur eine Frage der Zeit sein konnte, wann Du die alten Invariantenfragen soweit erledigt haben würdest, dass kaum noch das Tüpfelchem auf dem i fehlt, war mir eigentlich schon seit lange nicht zweifelhaft. Dass es aber damit so schnell geht, und alles so überraschend einfach gelingt, hat mich aufrichtig gefreut, und beglückwünsche ich Dich dazu. Jetzt, wo Du in Deinem letzten Satze sogar das rauchlose Pulver gefunden hast, nachdem schon Theorem I nur noch vor Gordans Augen Dampf gab, ist es wirklich an der Zeit, dass die Burgen der Raubritter STROH, GORDAN, STEPHANOS und wie sie alle heissen mögen, welche die einzelreisenden Invarianten überfielen und in's Burgverlies sperrten, dem Erdboden gleichgemacht werden, auf die Gefahr hin, dass aus diesen Ruinen niemals wieder neues Leben spriesst. (...)

Later-on Gordan admitted, that "theology also has its advantages". In particular, he gave the new proof of Hilbert's basis theorem which we discussed in Section 1.

How to overcome the non–contructiviness of Hilbert's proof?

Taking the above historical discussion into account, it came with a big surprise that Derksen (1997) could refine the ideas of Hilbert (1890) and turn them into an algorithm. In fact, he proceeds as follows.

- He first shows: if $I_{\mathcal{N}} = \langle f_1, \ldots, f_r \rangle$ with arbitrary polynomials $f_1, \ldots, f_r$, then $R(f_1), \ldots, R(f_r)$ generate $K[V]^G$.
- To find generators of $I_{\mathcal{N}}$ Derksen considers the map
$$\psi : G \times V \longrightarrow V \times V \ , \qquad (\pi, v) \mapsto (v, \pi v) \ .$$
Let $B$ be the Zariski closure of the image of $\psi$, with defining ideal $J \subset k[V \times V] \simeq k[x, y]$, where $y = y_1, \ldots, y_n$ is a new set of variables. Derksen proves that one can get generators for $I_{\mathcal{N}}$ by substituting $y = 0$ in a generating set of $J$ (such a set can be computed via elimination).

Hilbert (1893) himself reacted to the criticism by writing a second paper in which he gave a constructive way of finding generators for the ring of invariants in the case of $\mathrm{SL}_n(\mathbb{C})$ acting on $n$–forms by linear substitution. Instead of looking for generators of the ideal $I_{\mathcal{N}}$, Hilbert asks for homogeneous invariants cutting out the variety defined by $I_{\mathcal{N}}$. This variety is called the **Null–cone** by Hilbert. He first shows the following result.

**Theorem 5.16.** *Let $f_1, \ldots, f_r$ be homogeneous invariants whose zero–set is the Null–cone. Then $K[V]^G$ is a finitely generated $K[f_1, \ldots, f_r]$–module.*

The proof uses the Nullstellensatz, which was proved for this purpose, and the Reynolds operator.

Next Hilbert gives a method to find such $f_1, \ldots, f_r$ via the Hilbert–Mumford criterion, which tells us that the Null–cone is the $\mathrm{SL}_n(\mathbb{C})$–orbit of the **canonical cone**. Here, by definition, the canonical cone is the variety defined by all homogeneous invariants of positive degree of the maximal torus in $\mathrm{SL}_n(\mathbb{C})$. These invariants in turn can be computed via integer programming (see Sturmfels, 1993).

At this point, Hilbert gives two costly alternatives how to proceed. Both methods reduce the problem to the computation of a normalization. An algorithm for computing normalizations will be discussed in Section 5.3 below.

Alternatively, at this point, Hilbert's ideas allow to compute the dimension of $K[V]^G$, and we may proceed as in the case of finite groups. Indeed, invariant rings arising from rational representations of linearly reductive groups are Cohen–Macaulay by a deep theorem of Hochster and Roberts (1974), and it is also possible to precompute partial expansions of the Hilbert series (see Sturmfels, 1993).

**Implementations** are available in Singular by Derksen, Kendzior, and Peitz. Future projects are on their way (Croeni).

### 5.2. Algorithms for primary decomposition

We are interested in algorithms which compute for any given ideal $I \subset R := K[x_1, \ldots, x_n]$ a **minimal primary decomposition**

$$I = Q_1 \cap \cdots \cap Q_s \cap Q_{s+1} \cap \cdots \cap Q_r,$$

where $Q_1, \ldots, Q_s$ are the primary components belonging to the minimal associated primes of $I$. If such a decomposition is given we write

$$\begin{aligned} P_i &:= \sqrt{Q_i}, \quad i = 1, \ldots, r, \\ \mathrm{Ass}(I) &:= \{P_1, \ldots, P_s, P_{s+1}, \ldots, P_r\}, \\ \mathrm{minAss}(I) &:= \{P_1, \ldots, P_s\}, \end{aligned}$$

and

$$E_v(I) := \bigcap_{\mathrm{codim}(Q_i)=v} Q_i \quad (\text{if } \mathrm{codim}(Q_i) \neq v \text{ for all } i \text{ let } E_v(I) := R).$$

For many applications one does not need to compute a complete primary decomposition, it might actually be enough to to solve one of the following problems.

(1) Compute the **equidimensional hull** $E_c(I)$, $c = \operatorname{codim}(I)$.

(2) More generally, compute the **equidimensional decomposition** of $I$, that is, compute $E_v(I)$ for all $v \geq c$.

(2)' To solve a weaker problem, compute for all $v \geq c$ equidimensional ideals $I_v$ such that $\sqrt{I_v} = \sqrt{E_v(I)}$.

(3) Compute $\operatorname{Ass}(I)$ and/or $\operatorname{minAss}(I)$.

(4) Compute the radical $\sqrt{I} = \bigcap_{i=1}^{s} P_i$ and/or the equidimensional radical $\sqrt[\text{equi}]{I} = \sqrt{E_c(I)}$.

(5) Compute, for $I$ radical, the **normalization** of $R/I$, that is, the integral closure of $R/I$ in its total quotient ring $Q(R/I)$.

That primary decomposition can be computed effectively was first shown by Grete Hermann (1926). In modern times, quite a number of people presented algorithms which solve one or another of the above problems. Algorithms which essentially compute a complete primary decomposition are due to

- Gianni, Trager and Zacharias (1988);
- Wang (1989, computes the minimal associated primes only);
- Eisenbud, Huneke and Vasconcelos (1992, assumption: an algorithm is available, which computes the normalization as above in the case of radical ideals);
- Shimoyama and Yokoyama (1996, assumption: an algorithm is available, which computes the minimal associated primes)

These algorithms reduce the general problem to the special case of a hypersurface, that is, to polynomial factorization. The reduction relies on Gröbner basis techniques, and, in the case of Wang, also on the Wu–Ritt–principle for computing characteristic sets. The only system where all algorithms are implemented is Singular.

In what follows we refer to the above papers as GTZ respectively WWR respectively EHV respectively SY.

### 5.2.1. Splitting tools

Splitting tools are major ingredients of some algorithms. They may also be used in combination with other algorithms in order to simplify the given decomposition problem. Another useful tool for preprocessing is a factorizing

Buchberger algorithm (see Gräbe, 1997).

**Lemma 5.17.** *Let $I \subset R$ be an ideal. Then the following holds.*

(1) *If $I : f = I : f^2$ for some $f \in R$, then*

$$I = (I : f) \cap \langle I, f \rangle.$$

(2) *If $f \cdot g \in I$, and $\langle f, g \rangle = R$, then $I = \langle I, f \rangle \cap \langle I, g \rangle$.*

(3) *If $f \cdot g \in I$, then $\sqrt{I} = \sqrt{\langle I, f \rangle} \cap \sqrt{\langle I, g \rangle}$.*

(4) *If $f^n \in I$, then $\sqrt{I} = \sqrt{\langle I, f \rangle}$.*

(5) *If $J \subset R$ is an ideal, then*

$$\sqrt{I} = \sqrt{I : J} \cap \sqrt{I + J} = \sqrt{I : J} \cap \sqrt{I : (I : J)}. \qquad \square$$

Polynomials $f$ as in (1) can be found via saturation:

**Lemma 5.18.** *If $h \in R$ is a polynomial, and if $I : h^\infty = I : h^N$, then*

$$I = (I : h^N) \cap \langle I, h^N \rangle. \qquad \square$$

**Remark 5.19.** If $h_1 \cdots h_s$ is the square–free part of $h$, then we may replace $h^N$ by $h_1^{k_1} \cdots h_s^{k_s}$, where $I : h^\infty = I : (h_1^{k_1} \cdots h_s^{k_s})$. In fact, we may compute $I : h^\infty$ via ideal quotients by successively increasing the powers of the $h_i$. $\square$

This idea applies, in particular, in the following case.

**Remark 5.20.** Let $u \subset x = \{x_1, \dots, x_n\}$ be a subset of variables. Fix a block order $>$ on $K[x]$ with $x \setminus u >> u$, that is, with $x_b > x_a$ whenever $x_b \in x \setminus u$ and $x_a \in u$. Let $\mathcal{G}$ be a Gröbner basis of $I$ with respect to $>$. Then $\mathcal{G}$ is a (not necessarily minimal) Gröbner basis of the **extended ideal** $I^{\mathrm{e}} := IK(u)[x \setminus u]$ with respect to the order $>$ restricted to $x \setminus u$. Set $h = \mathrm{lcm}\{\mathrm{lc}(g) \mid g \in \mathcal{G}\}$, where $\mathrm{lc}(g) \in K[u]$ is taken of $g$ as a polynomial in $K(u)[x \setminus u]$. Then:

(1) The **contracted ideal** $I^{\mathrm{ec}} := IK(u)[x \setminus u] \cap K[x] = (I : h^\infty)$.

(2) If $I^{\mathrm{e}}$ is radical, then $\sqrt{I} = (I : h^\infty) \cap \sqrt{\langle I, h \rangle}$. $\square$

Recall that

$$\dim(I) = \max\{\#u \mid u \subset x, u \text{ is independent mod } I\},$$

where $u$ is **independent mod** $I$ if $I \cap k[u] = \{0\}$. In particular, $I$ is zero–dimensional iff $I$ contains for each $i$ a non–constant polynomial in the variable $x_i$. If $u$ is **maximal independent** mod $I$ with $\#u = \dim(I)$ then $I^{\mathrm{e}}$ is zero–dimensional, and $I^{\mathrm{ec}}$ is equidimensional of dimension $\dim(I)$. Notice that instead of computing

$$\mathcal{M}_I := \{u \subseteq x \mid u \text{ is a maximal independent set mod } I \text{ with } \#u = \dim(I)\}$$

it is much easier to compute the (possibly proper) subset $\mathcal{M}_{\mathrm{L}(I)}$, where $\mathrm{L}(I)$ is the leading ideal of $I$ with respect to a monomial order on $R$.

### 5.2.2. Algorithms

We describe some of the main ideas behind the algorithms and leave it to the reader to actually write down the algorithms.

The remarks in the above section yield GTZ–type algorithms for the computation of equidimensional decompositions, and, provided that we can handle the zero–dimensional case, for radicals and primary decompositions via extensions and contractions.

An algorithm for radicals, due to Krick and Logar (1991), relies on the following result.

**Proposition 5.21.** Let $I \subset K[x_1, \ldots, x_n]$ be a zero–dimensional ideal. For $i = 1, \ldots, n$ let $F_i(x_i) \in K[x_i] \cap I$ be a polynomial of minimal degree, and let $L_i(x_i)$ be the square free part of $F_i$. Then $\sqrt{I} = I + \langle L_1, \ldots, L_n\rangle$. $\square$

The general GTZ–algorithm is based on the notion of **general position** for zero–dimensional ideals and the following result.

**Theorem 5.22.** *Let $I \subset K[x_1, \ldots, x_n]$ be a zero–dimensional ideal in general position with respect to the lexicographical order induced from $x_1 > \cdots > x_n$, let $\mathcal{G}$ be a corresponding minimal Gröbner basis of $I$, let $f$ be the unique polynomial in $\mathcal{G} \cap K[x_n]$, and let $f = f_1^{\rho_1} \cdot \ldots \cdot f_s^{\rho_s}$ be the decomposition of $f$ into a power product of pairwise non–associated irreducible factors $f_k$. Then the minimal primary decomposition of $I$ is given by*

$$I = \bigcap_{k=1}^{s} (I, f_k^{\rho_k}) \ . \qquad \square$$

The variant of SY relies on two easy lemmas.

**Lemma 5.23.** *Let $I \subset K[x_1, \dots, x_n]$ be an ideal, and let*

$$\operatorname{minAss}(I) = \{P_1, \dots, P_s\}.$$

*Let $f_1, \dots, f_s \in K[x_1, \dots, x_n]$ be chosen such that*

$$f_i \in \bigcap_{j \neq i} P_j \quad \text{but} \quad f_i \notin P_i.$$

*Let $k_i$ and $\bar{Q}_i$ be defined by $I : f_i^\infty = I : f_i^{k_i} =: \bar{Q}_i$, and let $J := I + \langle f_1^{k_1}, \dots, f_s^{k_s} \rangle$. Then:*

(1) $\sqrt{\bar{Q}_i} = P_i$, *that is, $\bar{Q}_i$ is* **pseudo–primary** *with associated prime $P_i$.*

(2) $I = \bigcap_{i=1}^s \bar{Q}_i \cap J$.

(3) $\operatorname{codim}(J) > \operatorname{codim}(I)$. □

**Lemma 5.24.** *Let $\bar{Q}$ be pseudo–primary with $\sqrt{\bar{Q}} = P$ prime, and let $u \subset x$ be a maximal independent set mod $\bar{Q}$. Then $\bar{Q}K(u)[x \smallsetminus u] \cap K[x] =: Q$ is $P$–primary. Let $h \in K[u]$ be chosen such that $Q = \bar{Q} : h^\infty$, and set $J := \langle \bar{Q}, h \rangle$. Then:*

(1) $\bar{Q} = Q \cap J$.

(2) $\operatorname{codim} J > \operatorname{codim}(\bar{Q})$. □

Completely different approaches are due to EHV. The EHV–type algorithms are based on deeper theorems in commutative algebra, in particular syzygy theory.

One of the EHV–algorithms for radicals relies on a theorem of Scheja–Storch (1975). In fact, we need the following corollary to this theorem.

**Proposition 5.25.** *Let $I \subset K[x_1, \dots, x_n]$ be a complete intersection of dimension $d$. Then $\sqrt{I} = I : J_d(I)$.* □

Here $J_d(I)$ is the $d$th **Jacobian ideal** of $I$, that is, if $I = \langle f_1, \dots, f_s \rangle$, then $J_d(I) = I +$ the ideal generated by the $(n-d)$–minors of the Jacobian matrix of $f_1, \dots, f_s$.

**Corollary 5.26.** *Let $I \subset K[x_1, \dots, x_n]$ be an equidimensional ideal of codimension $c$, and let $f_1, \dots, f_c \in I$ be a regular sequence. If $I_0 = \langle f_1, \dots, f_c \rangle$, then*

$$\sqrt{I} = \sqrt{I_0} : (\sqrt{I_0} : I) \ . \qquad \square$$

This yields an algorithm for the computation of equidimensional radicals whose main drawback is the computation of regular sequences via random linear combinations.

The computation of the equidimensional hull following EHV is based on the following result.

**Proposition 5.27.** *Let $I \subset R = K[x_1, \dots, x_n]$ be an ideal of codimension $c$. For $v \geq c$ let $E_v(I)$ be the equidimensional part of $I$ of codimension $v$. Then the following holds.*

(1) $E_c(I) = \operatorname{ann} \operatorname{Ext}^c_R(R/I, R)$.

(2) *If $I_0 \subset I$ is a complete intersection of codimension $c$, then*

$$E_c(I) = I_0 : (I_0 : I) \ .$$

(3) for $v \geq c$

$$\sqrt{E_v(I)} = \sqrt{E_v\big(\operatorname{ann}\left(\operatorname{Ext}^v_R(R/I, R)\right)\big)} \ . \qquad \square$$

This proposition also yields an algorithm which computes a weak equidimensional decomposition.

By combining the above ideas with localization techniques, EHV reduce the computation of a primary decomposition for an arbitrary ideal to that of an equidimensional radical ideal $I$ (which can be assumed to be homogeneous). Thus one needs to find the minimal primes of the graded, reduced ring $S = K[x_1, \dots, x_n]/I$. These in turn can be obtained by intersecting the minimal primes of the normalization $\bar{S}$ of $S$ with $S$. The latter ideals are in one to one correspondence with the idempotents of $\bar{S}$ and can therefore be computed. In this way, primary decomposition is reduced to normalization. An algorithm for computing normalizations will be discussed in Section 5.3 below.

Let us shift gears again: the concept of **characteristic sets** goes back to Ritt (1932) and Wu (1984). In our context, when applying this concept, the basic strategy is the following.

Let $X$ be a finite set of generators for the given ideal $I \subset K[x_1, \ldots, x_n]$. Compute a characteristic set of $X$ with the help of pseudo–division. Successively extend this characteristic set via pseudo–division. Split the radical of $I$ with the help of the extended characteristic set $\mathcal{F}$ (without computing the radical). Distinguish two different types of splitting, depending on whether $\mathcal{F}$ is irreducible (then $\mathcal{F}$ corresponds to a prime ideal which can be computed via elimination, that is, via Gröbner bases) or not. When applying the above idea recursively, the prime ideals corresponding to the irreducible extended characteristic sets provide a not necessarily minimal prime decomposition of $\sqrt{I}$.

The running times of the examples presented in Decker, Greuel and Pfister (1999) underline the fact that there is no universal answer to the question for the "best algorithm".

### 5.3. Normalization

In algebraic geometry normalization is an important step towards a resolution of singularities, and provides in fact a resolution of singularities in the case of curves.

We recall the corresponding definitions in the algebraic setting. Let $R$ be a reduced ring. The **normalization** $\widetilde{R}$ of $R$ is the integral closure of $R$ in its total quotient ring $Q(R)$. That is, $\widetilde{R}$ contains the elements $a \in Q(R)$ satisfying an equation

$$a^n + \alpha_{n-1}a^{n-1} + \cdots + \alpha_0 = 0, \quad \alpha_i \in R.$$

$R$ is **normal** if $R = \widetilde{R}$.

¿From now on suppose that $R$ is reduced and noetherian. We will describe an algorithm which computes the normalization in this case. The algorithm is essentially due to Grauert and Remmert (1971 and 1984), who needed a normality criterion in order to give a nice and simple proof of a theorem of Oka. (Oka showed, that the non–normal points of an analytic space form an analytic space themselves.) That the criterion can be turned into an effective algorithm was pointed out by de Jong (1998).

Let $I$ be an ideal of $R$ containing a non–zerodivisor. Then we have canonical inclusions

$$R \subset \mathrm{Hom}_R(I, I) \subset \widetilde{R}.$$

The first inclusion is the map which sends an element of $R$ to multiplication with this element. For the second inclusion we pick a non–zerodivisor $f$ in $I$ and map $\phi \in \mathrm{Hom}_R(I, I)$ to $\frac{\phi(f)}{f}$. It is easily checked that this map is independent of the choice of $f$. That we in fact land in $\widetilde{R}$ is a consequence of the Theorem of Cayley and Hamilton.

**Theorem 5.28 (Grauert–Remmert).** *Let $I$ be an ideal in $R$ containing a non–zerodivisor. Suppose that the vanishing locus of $I$,*

$$V(I) = \{P \in \operatorname{Spec}(R) \mid I \subset P\},$$

*contains the non–normal locus of $R$,*

$$\mathrm{NNL} := \{P \in \operatorname{Spec}(R) \mid R_P \text{ is not normal}\}.$$

*Moreover suppose that $I$ has the property*

$$\operatorname{Hom}_R(I, I) = \operatorname{Hom}_R(I, R) \cap \widetilde{R} \quad (*).$$

*Then*

$$R \textit{ is normal} \iff R = \operatorname{Hom}_R(I, I). \qquad \square$$

**Theorem 5.29 (Grauert–Remmert).** *Every radical ideal $I$ containing a non–zerodivisor satisfies (*).* $\square$

These results give rise an algorithm whose major steps are as follows.

**Algorithm 5.30 (de Jong).**

**Input**: a reduced noetherian ring $R$.
**Output**: the normalization $\widetilde{R}$ of $R$.
STEP 1: determine an ideal $I$ with $NNL \subset V(I)$;
STEP 2: $I = \sqrt{I}$;
STEP 3: pick an $f \in I$; $J = \operatorname{ann}\langle f\rangle$;
  **if** $J = 0$ **then goto** STEP 5;
STEP 4: $R = R/\langle f\rangle \oplus R/J$; **goto** STEP 1;
STEP 5: compute $\operatorname{Hom}_R(I, I)$;
  **if** $R = \operatorname{Hom}_R(I, I)$ **then return** R;
STEP 6: $R = \operatorname{Hom}_R(I, I)$; **goto** STEP 1. $\square$

The algorithm terminates exactly when the normalization $\widetilde{R}$ is finitely generated as an $R$–module. In particular it terminates for affine rings.

# Chapter II. Applications to surfaces in $\mathbb{P}^4$

In this chapter we suppose for simplicity that our base field is $\mathbb{C}$. We refer to Decker and Schreyer (2000) for a discussion of the problems arising from the fact that actual computer algebra computations are typically performed over a finite prime field.

## 6. General structure theorems

### 6.1. The functors $\tilde{}$ and $\Gamma_*$, cohomology of sheaves

Let $S = \mathbb{C}[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. There are well–known functors

$$M \text{ graded } S\text{–module} \rightsquigarrow \widetilde{M} \text{ quasi–coherent sheaf on } \mathbb{P}^n,$$

$$\mathcal{F} \text{ quasi–coherent sheaf on } \mathbb{P}^n \rightsquigarrow \text{graded } S\text{–module } \Gamma_*\mathcal{F} := \bigoplus_{\nu\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{F}(\nu)).$$

$\sim$ is exact, $\Gamma_*$ is left exact. We summarize further properties of these functors and recall some basic theorems on the cohomology of coherent sheaves on projective spaces (for details and proofs see Serre, 1955; Hartshorne, 1977; Okonek, Schneider and Spindler, 1980).

**Definition 6.1.**

(1) We write $M \sim M'$ if $M_{\geq\nu_0} \cong M'_{\geq\nu_0}$ for some $\nu_0$. Here $M_{\geq\nu_0} = \bigoplus_{\nu\geq\nu_0} M_\nu$ if $M = \bigoplus_{\nu\in\mathbb{Z}} M_\nu$ .

(2) $M$ is **quasi–finitely generated** if $M \sim M'$ with $M'$ finitely generated. □

**Theorem 6.2.**

(1) *If $M$ is a graded, finitely generated $S$–module then $\tilde{M}$ is coherent. Every coherent sheaf on $\mathbb{P}^n$ arizes in this way. In fact, $\sim$ and $\Gamma_*$ induce an equivalence of categories between quasi–finitely generated, graded $S$–modules modulo $\sim$ and coherent sheaves on $\mathbb{P}^n$. In particular, $\tilde{M} = 0$ if $M$ is of finite length.*

(2) $\widetilde{S(\nu)} \cong \mathcal{O}_{\mathbb{P}^n}(\nu)$, $\quad \widetilde{M(\nu)} \cong \tilde{M}(\nu) := \tilde{M} \otimes \mathcal{O}_{\mathbb{P}^n}(\nu)$ .

(3) *For every quasi–coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ there is a natural isomorphism*

$$\widetilde{\Gamma_*\mathcal{F}} \cong \mathcal{F} .$$

(4) *If $X \subset \mathbb{P}^n$ is a closed subscheme with ideal sheaf $\mathcal{J}_X$ then $\Gamma_*(\mathcal{J}_X)$ is the saturated, homogeneous ideal $I_X$ defining $X$.*

(5) **Serre's Theorem A.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then there exists an integer $\nu_0$ such that for every $\nu \geq \nu_0$ the twisted sheaf $\mathcal{F}(\nu)$ is generated by finitely many global sections, that is, there exists a surjective morphism of type*
$$0 \leftarrow \mathcal{F}(\nu) \leftarrow \mathcal{O}_{\mathbb{P}^n}^s .$$

(6) **Serre's Theorem B.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then $H^i(\mathbb{P}^n, \mathcal{F})$ is a finite dimensional $\mathbb{C}$–vector space $\forall\ i \geq 0$. Moreover, there exists an integer $\nu_0$ such that*
$$H^i(\mathbb{P}^n, \mathcal{F}(\nu)) = 0 \quad \forall\ \nu \geq \nu_0, \quad \forall\ i > 0. \quad \square$$

More generally, if $X \subset \mathbb{P}^n$ is any closed subscheme with defining ideal $I_X$ then there exists a correspondence between graded $(S/I_X)$–modules and quasi–coherent sheaves on $X$. The analogous results hold.

**Definition 6.3.** Let $X \subset \mathbb{P}^n$ be a closed subscheme of codimension $r$. Then the **dualizing sheaf** of $X$ is the coherent sheaf
$$\omega_X := \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \omega_{\mathbb{P}^n}),$$
where $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$. $\square$

**Theorem (Serre–Duality) 6.4.** *Let $X \subset \mathbb{P}^n$ be a closed subscheme which is locally Cohen–Macaulay and has equidimension $m$. Then the following holds.*

(1) *For any coherent sheaf $\mathcal{F}$ on $X$ there exist natural isomorphisms*
$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \cong H^{m-i}(X, \mathcal{F})^*, \qquad i \geq 0 .$$

(2) *For any locally free sheaf $\mathcal{F}$ on $X$ there exist natural isomorphisms*
$$H^i(X, \mathcal{F}^* \otimes \omega_X) \cong H^{m-i}(X, \mathcal{F})^*, \qquad i \geq 0 .$$

(3) *If $X$ is a locally complete intersection, then the dualizing sheaf is isomorphic to the* **canonical sheaf** *of $X$:*
$$\omega_X \cong \omega_{\mathbb{P}^n} \otimes \bigwedge^{n-m} (\mathcal{J}_X/\mathcal{J}_X^2)^* .$$
*In particular, $\omega_X$ is locally free of rank 1.* $\square$

If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, then the graded $S$–modules

$$H^i_*\mathcal{F} := \bigoplus_{\nu \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(\nu)), \quad i \geq 0 ,$$

are called the **cohomology modules** of $\mathcal{F}$ (for the $S$–module structure note that

$$H^0_*\mathcal{O}_{\mathbb{P}^n} \cong \Gamma_*\mathcal{O}_{\mathbb{P}^n} \cong S) .$$

Properties of $\mathcal{F}$ are reflected by the cohomology modules of $\mathcal{F}$. For example, we have the following theorem (as usual we do not distinguish between locally free sheaves and vector bundles).

**Theorem 6.5.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$.*

(1) *$\mathcal{F}$ is locally free (a vector bundle) iff*

$$H^i(\mathbb{P}^n, \mathcal{F}(-\nu)) = 0 \quad \text{for} \quad i < n \quad \text{and} \quad \nu \gg 0$$

*iff $\Gamma_*(\mathcal{F})$ is fintely generated and the* **intermediate cohomology modules**

$$H^i_*\mathcal{F}, \quad 1 \leq i \leq n-1,$$

*are of finite length.*

(2) **Splitting criterion of Horrocks.** *$\mathcal{F}$ is a direct sum of locally free sheaves of rank 1 (line bundles),*

$$\mathcal{F} \cong \bigoplus \mathcal{O}_{\mathbb{P}^n}(\nu_i) ,$$

*iff $\Gamma_*(\mathcal{F})$ is fintely generated and $\mathcal{F}$ has no intermediate cohomology:*

$$H^i_*\mathcal{F} = 0, \quad 1 \leq i \leq n-1 . \quad \square$$

For ideal sheaves we obtain from the exact sequence

$$0 \longleftarrow \mathcal{O}_X \longleftarrow \mathcal{O}_{\mathbb{P}^n} \longleftarrow \mathcal{J}_X \longleftarrow 0 \;:$$

**Proposition 6.6.** *If $\mathcal{J}_X$ is the ideal sheaf of a closed subscheme $X \subset \mathbb{P}^n$ which is locally Cohen–Macaulay and has equidimension $m$, then the* **Hartshorne–Rao modules** *of $X$,*

$$H^i_*\mathcal{J}_X \;, \quad 1 \leq i \leq m \;,$$

*are of finite length.* $\square$

Checking minimal free resolutions gives:

**Proposition 6.7.** *Let $M$ be a graded, finite length $S$–module, and let*

$$\mathcal{F}_i := \mathcal{S}yz_i(M) := \widetilde{\mathrm{Syz}_{i+1}(M)}, \quad 1 \leq i \leq n-1 \ .$$

*Then $\mathcal{F}_i$ is locally free with intermediate cohomology*

$$H^j_* \mathcal{F}_i = \begin{cases} M & j = i \\ 0 & j \neq i, \quad 1 \leq j \leq n-1 \ . \end{cases}$$

*Conversely, if $\mathcal{F}$ is any vector bundle on $\mathbb{P}^n$ with this intermediate cohomology, then*

$$\mathcal{F} \cong \mathcal{F}_i \oplus \mathcal{L}, \quad \mathcal{L} \text{ a direct sum of line bundles.}$$

*We call $\mathcal{F}_i$ the* **$i$th syzygy bundle** *of $M$.* □

**Example 6.8.** Think of $\mathbb{P}^n$ as the projective space of lines in the $\mathbb{C}$–vector space $V$. Consider $\mathbb{C}$ as a graded $S$–module sitting in degree 0. The minimal free resolution of $\mathbb{C}(i)$ is the **Koszul complex**

$$0 \longleftarrow \mathbb{C}(i) \longleftarrow \bigwedge^0 V^* \otimes S(i) \longleftarrow \ \dots \ \longleftarrow \bigwedge^{n+1} V^* \otimes S(i-n-1) \longleftarrow 0 \ .$$

The corresponding syzygy bundles are the twisted **bundles of differentials**,

$$\mathcal{S}yz_i(\mathbb{C}(i)) \cong \Omega^i_{\mathbb{P}^n}(i) = \bigwedge^i \mathcal{T}^*_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(i) \ , \quad 1 \leq i \leq n-1 \ .$$

We also have

$$\Omega^0_{\mathbb{P}^n}(0) \cong \mathcal{O}_{\mathbb{P}^n} \quad \text{and} \quad \Omega^n_{\mathbb{P}^n}(n) \cong \mathcal{O}_{\mathbb{P}^n}(-1) \ .$$

It follows from the sheafified Koszul complex that

$$\text{(1)} \quad H^j_* \Omega^i_{\mathbb{P}^n} = \begin{cases} \mathbb{C} & j = i, \\ 0 & j \neq i, \end{cases} \qquad 1 \leq i, j \leq n-1 \ ,$$

$$\text{(2)} \quad \mathrm{Hom}(\Omega^i_{\mathbb{P}^n}(i), \Omega^j_{\mathbb{P}^n}(j)) \cong \bigwedge^{i-j} V \ , \qquad 0 \leq i, j \leq n$$

(the isomorphisms in (2) are given by contraction). □

In what follows we write $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$ and $\Omega^i(i) = \Omega^i_{\mathbb{P}^n}(i)$.

**Example 6.9.** On $\mathbb{P}^3$ the Koszul complex reads

$$0 \longleftarrow \mathbb{C} \longleftarrow \mathcal{O} \xleftarrow{(x_0, \dots, x_3)} 4\mathcal{O}(-1) \longleftarrow 6\mathcal{O}(-2) \longleftarrow 4\mathcal{O}(-3) \longleftarrow \mathcal{O}(-4) \longleftarrow 0 \ .$$

Hence $\Omega^2(2)$ respectively $\Omega^1(1)$ have resolutions of type

$$0 \longleftarrow \Omega^2(2) \longleftarrow 4\mathcal{O}(-1) \longleftarrow \mathcal{O}(-2) \longleftarrow 0$$

respectively

$$0 \longleftarrow \Omega^1(1) \longleftarrow 6\mathcal{O}(-1) \longleftarrow 4\mathcal{O}(-2) \longleftarrow \mathcal{O}(-3) \longleftarrow 0 \ .$$

Any homomorphism $\varphi \in \mathrm{Hom}(\Omega^2(2), \Omega^1(1))$ lifts to a homomorphism between the resolutions:

$$\begin{tikzcd}
0 \arrow[d] & \vdots \arrow[d] \\
\mathcal{O}(-2) \arrow[r, dashed, "\varphi_1"] \arrow[d] & 4\mathcal{O}(-2) \arrow[d] \\
4\mathcal{O}(-1) \arrow[r, dashed, "\varphi_0"] \arrow[d] & 6\mathcal{O}(-1) \arrow[d] \\
\Omega^2(2) \arrow[r, "\varphi"] \arrow[d] & \Omega^1(1) \arrow[d] \\
0 & 0
\end{tikzcd}$$

For computational purposes we think of $\varphi_1$ and $\varphi_0$ as homomorphisms in the category of grades $S$–modules:

$$\begin{tikzcd}
0 \arrow[d] & \vdots \arrow[d] \\
S(-2) \arrow[r, dashed, "\varphi_1"] \arrow[d] & 4S(-2) \arrow[d] \\
4S(-1) \arrow[r, dashed, "\varphi_0"] \arrow[d] & 6S(-1) \arrow[d] \\
\mathrm{Syz}_3(\mathbb{C}(2)) \arrow[r, "\varphi"] \arrow[d] & \mathrm{Syz}_2(\mathbb{C}(1)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

So $\varphi$ is respresented by $\varphi_0$, that is by a $6 \times 4$–matrix with constant entries. In order to construct a morphism $\varphi$ we first pick $\varphi_1$, that is, we pick a $4 \times 1$–matrix

$\varphi_1$ with constant entries:

$$\begin{tikzcd}
0 \arrow[d] & \vdots \arrow[d] \\
S(-2) \arrow[r, "\varphi_1"] \arrow[d] \arrow[dr, dashed, "\tilde{\varphi}_1"] & 4S(-2) \arrow[d] \\
4S(-1) \arrow[r, dashed, "\varphi_0"] \arrow[d] & 6S(-1) \arrow[d] \\
\mathrm{Syz}_3(\mathbb{C}(2)) \arrow[d] & \mathrm{Syz}_2(\mathbb{C}(1)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

$\varphi_1$ gives rise to a map $\tilde{\varphi}_1$ as above which we need to lift to a map $\varphi_0$ (this can be done via a Gröbner basis computation, compare the exercises). $\square$

### 6.2. Structure theorems in codimension 2

For details and proofs in this section we refer to Eisenbud (1995), Hartshorne (1974 and 1978), Okonek, Schneider and Spindler (1980), and the references given there.

**Hilbert–Burch's Local Structure Theorem 6.10.** *Let $R$ be a regular local ring, and let $I \subset R$ be an ideal of codimension 2. Then the following are equivalent.*

(1) *$R/I$ is Cohen–Macaulay.*

(2) *$R/I$ has a resolution of type*

$$\begin{tikzcd}
0 & R/I \arrow[l] & R \arrow[l] & & R^{f+1} \arrow[ll, "\bigwedge^f \varphi"'] \arrow[dl] & R^f \arrow[l, "\varphi"'] & 0 \arrow[l] \\
& & & I \arrow[ul] \arrow[dl] & & & \\
& & 0 & & 0 \arrow[ul] & &
\end{tikzcd}$$

$\square$

Globally one has the following result.

**Theorem 6.11.** *Let $\mathcal{F}, \mathcal{G}$ be vector bundles on $\mathbb{P}^n$ with $\operatorname{rk} \mathcal{F} = f$, $\operatorname{rk} \mathcal{G} = f+1$, and let $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ be a morphism such that*

$$X = V(\varphi) = \{p \in \mathbb{P}^n \mid \operatorname{rk}(\varphi) < f\}$$

*has expected pure codimension 2. Then $X$ is locally Cohen–Macaulay with a resolution of type*

$$\begin{tikzcd}
 & & \bigwedge^f \mathcal{G}^* \otimes \bigwedge^{f+1} \mathcal{G} \arrow[d] & & & \\
0 & \mathcal{O}_X(m) \arrow[l] & \mathcal{O}_{\mathbb{P}^n}(m) = \bigwedge^f \mathcal{F}^* \otimes \bigwedge^{f+1} \mathcal{G} \arrow[l] & \mathcal{G} \arrow[l] \arrow[ul] \arrow[dl] & \mathcal{F} \arrow[l, "\varphi"'] & 0 \arrow[l] \\
 & & \mathcal{J}_X(m) \arrow[u] & & &
\end{tikzcd}$$

(**Eagon–Northcott complex**). *Conversely, every locally Cohen–Macaulay subscheme $X \subset \mathbb{P}^n$ of pure codimension 2 arises in this way.* $\square$

**Remark 6.12.** (1) How the maps above are defined can be best seen by dualizing:

$$\begin{tikzcd}
\bigwedge^f \mathcal{G} \otimes \bigwedge^{f+1} \mathcal{G}^* \arrow[dr] & \\
\bigwedge^f \mathcal{F} \otimes \bigwedge^{f+1} \mathcal{G}^* \arrow[u] \arrow[r] & \mathcal{G}^*
\end{tikzcd}$$

(2) Notice that $m = c_1 \mathcal{G} - c_1 \mathcal{F}$ .

(3) The cohomology modules of $\mathcal{F}$ and $\mathcal{G}$ reflect the structure of the Hartshorne–Rao modules of $X$.

(4) The graded version of the theorem of Hilbert–Burch implies by sheafifying that a locally Cohen–Macaulay subvariety $X \subset \mathbb{P}^n$ of pure codimension 2 is projectively Cohen–Macaulay iff $\mathcal{J}_X$ has a resolution of type

$$0 \longleftarrow \mathcal{J}_X \longleftarrow \bigoplus \mathcal{O}_{\mathbb{P}^n}(-d_i) \xleftarrow{\ \varphi\ } \bigoplus \mathcal{O}_{\mathbb{P}^n}(-e_j) \longleftarrow 0.$$

In other words, the homogeneous coordinate ring of $X$ is Cohen–Macaulay iff the ideal of $X$ is defined by the maximal minors of a graded $(f+1) \times f$–matrix of homogeneous polynomials. Notice that in this case the Hartshorne–Rao modules of $X$ are all zero. $\square$

**Example 6.13.** The twisted cubic curve $X \subset \mathbb{P}^3$ has the syzygies

$$0 \longleftarrow S/I_X \longleftarrow S \xleftarrow{\ \bigwedge^2 \varphi\ } 3S(-2) \xleftarrow{\ \varphi\ } 2S(-3) \longleftarrow 0$$

with

$$\varphi = \begin{pmatrix} y & z \\ x & y \\ w & x \end{pmatrix}$$

and

$$\bigwedge^2 \varphi = (wy - x^2, xy - wz, xz - y^2). \quad \square$$

**Remark 6.14.** If free resolutions of $\mathcal{F}$ and $\mathcal{G}$ and a complex homomorphism covering $\varphi$ are explicitly given, then one can compute a system of explicit equations defining $X$ via a mapping cone:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & & \\
\mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[d] & & \\
\mathcal{F}_1 \arrow[r] \arrow[d] \arrow[ur, phantom, "\oplus"] & \mathcal{G}_1 \arrow[d] & & \\
\mathcal{F}_0 \arrow[r] \arrow[d] \arrow[ur, phantom, "\oplus"] & \mathcal{G}_0 \arrow[d] \arrow[dr] & & \\
\mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{J}_X(m) \arrow[dr] & \\
0 & 0 & & 0
\end{tikzcd}$$

In fact, such a computation can be done in the category of graded $S$–modules. $\square$

**Example 6.15.** On $\mathbb{P}^4$ a generic morphism

$$\varphi : \mathcal{F} = 3\mathcal{O}(-1) \to \mathcal{G} = \Omega^1(1)$$

vanishes along a smooth surface $X$ (compare Kleiman's result below). $\varphi$ gives rise to a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & \mathcal{O}(-4) \arrow[d] \\
 & 5\mathcal{O}(-3) \arrow[d] \\
 & 10\mathcal{O}(-2) \arrow[d] \arrow[dl, dash, "\oplus"'] \\
3\mathcal{O}(-1) \arrow[r] \arrow[d, equal] & 10\mathcal{O}(-1) \arrow[d] \\
3\mathcal{O}(-1) \arrow[r] & \Omega^1(1) \arrow[d] \\
 & 0
\end{tikzcd} \quad .$$

Thus $X$ has syzygies of type

$$0 \leftarrow \mathcal{J}_X \leftarrow \mathcal{O} \leftarrow 7\mathcal{O}(-3) \leftarrow 10\mathcal{O}(-4) \leftarrow 5\mathcal{O}(-5) \leftarrow \mathcal{O}(-6) \leftarrow 0 \ .$$

In particular, $X$ is cut out by seven cubics, it has degree 4 and sectional genus 0 (compute the Hilbert polynomial). It easily follows that $X$ is a **Veronese surface** in $\mathbb{P}^4$. Classically, such a surface is constructed as follows. Consider the projective plane $\mathbb{P}^2$ with coordinates $x, y, z$. Then the morphism

$$p \to (x^2(p) : y^2(p) : z^2(p) : xy(p) : xz(p) : yz(p))$$

defined by all quadrics on $\mathbb{P}^2$ is an embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$. Its image is a **Veronese surface** $X$ in $\mathbb{P}^5$. It can be shown that the secant variety $\operatorname{Sec}(X)$ does not fill up $\mathbb{P}^5$. Thus a projection

$$X \subset \mathbb{P}^5 \setminus \{p\} \to \mathbb{P}^4$$

from a general point $p \in \mathbb{P}^5$ embeds $X$ as a smooth surface into $\mathbb{P}^4$ which is a Veronese surface in $\mathbb{P}^4$. Directly, such a surface can be obtained by embedding $\mathbb{P}^2$ into $\mathbb{P}^4$ via

$$p \to (x^2(p) : y^2(p) : z^2(p) : (xy - yz)(p) : (xz - yz)(p)). \qquad \square$$

Why does the generic morphism $\varphi$ above vanish along a smooth surface ?

**Kleiman's Bertini–Type Theorem 6.16.** (Kleiman, 1969) *Let $n \leq 5$, let $\mathcal{F}$ and $\mathcal{G}$ be vector bundles on $\mathbb{P}^n$, and let $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ be a generic morphism. Suppose that $\mathcal{H}om\,(\mathcal{F}, \mathcal{G})$ is globally generated. Then $V(\varphi)$ is smooth of pure codimension 2 or empty.* $\square$

**Remark 6.17.** (1) In the example of the Veronese surface the assumptions of Kleiman's theorem are fulfilled. Indeed,

$$\mathcal{H}om\,(3\mathcal{O}(-1), \Omega^1(1)) \cong 3\Omega^1(2)$$

is globally generated since it has syzygies of type

$$0 \leftarrow 3\Omega^1(2) \leftarrow 30\mathcal{O} \leftarrow \cdots$$

(2) In the context of non–general type surfaces in $\mathbb{P}^4$ Kleiman's theorem only applies in rather trivial examples. □

In order to construct a smooth variety in this way we need answers to the following questions.

**Problem 6.18.**

(1) How to choose $\mathcal{F}$ and $\mathcal{G}$ ?

(2) How to check smoothness ? □

An explicit construction of $X$ as a degeneracy locus as above leads to a system of explicit equations for $X$. Thus smoothness can be checked via the Jacobian criterion. We refer to Decker and Schreyer (2000) for a speedier method. An answer to (1) above will be given in Section 8 below. Now we recall some results around Hartshorne's conjecture in the case of codimension 2.

**Hartshorne's Conjecture (Codimension 2 Case) 6.19.** Let $n \geq 6$, and let $X \subset \mathbb{P}^n$ be a smooth subvariety of codimension 2. Then $X$ is a complete intersection, that is, $\mathcal{J}_X$ has a resolution of type

$$0 \leftarrow \mathcal{J}_X \leftarrow \mathcal{O}(-a) \oplus \mathcal{O}(-b) \leftarrow \mathcal{O}(-a-b) \leftarrow 0\ .$$

In particular,

$$\omega_X \cong \mathcal{O}_X(n+1-a-b)\ . \quad \square$$

Some support for this conjecture comes from the Lefshetz–type theorems of Barth and Larsen (Barth, 1970; Barth and Larsen, 1972; Larsen, 1973) which tell us that smooth subvarieties $X \subset \mathbb{P}^n$ of low codimension behave like complete intersections from a topological point of view. In particular, we have the following result.

**Theorem 6.20.** *Let $n \geq 6$, and let $X \subset \mathbb{P}^n$ be a smooth subvariety of codimension 2. Then $X$ is* **subcanonical**, *that is,*

$$\omega_X \cong \mathcal{O}_X(m) \quad \textit{for some} \quad m. \quad \square$$

Serre–correspondence allows to reformulate Hartshorne's conjecture in terms of vector bundles.

**Serre's Local Structure Theorem 6.21.** *Let $R$ be a regular local ring, and let $I \subset R$ be an ideal of codimension 2. Then the following are equivalent.*

(1) *$R/I$ is Gorenstein.*

(2) *$I$ is a complete intersection, that is, there exists a resolution of type*

$$0 \leftarrow R/I \leftarrow R \leftarrow R^2 \leftarrow R \leftarrow 0. \quad \square$$

Globally one has the following result.

**Theorem (Serre–Correspondence) 6.22.** *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a locally complete intersection. Then the following are equivalent:*

(1) *$X$ is subcanonical.*

(2) *There exists a resolution*

$$0 \longleftarrow \mathcal{O}_X(m) \longleftarrow \mathcal{O}(m) \longleftarrow \mathcal{E} \xleftarrow{s} \mathcal{O} \longleftarrow 0 \ ,$$

*where $\mathcal{E}$ is a rank–2 vector bundle on $\mathbb{P}^n$. That is, $X$ is the zero–scheme of a section $s$ of a rank–2 vector bundle $\mathcal{E}$ on $\mathbb{P}^n$.* $\square$

**Hartshorne's Conjecture (Codimension 2 case) 6.23.** Let $n \geq 6$, and let $\mathcal{E}$ be a rank–2 vector bundle on $\mathbb{P}^n$. Then $\mathcal{E}$ splits as a direct sum of two line bundles,

$$\mathcal{E} \cong \mathcal{O}(a) \oplus \mathcal{O}(b). \qquad \square$$

**Remark 6.24.** (1) On $\mathbb{P}^4$ the Horrocks–Mumford bundle (see Horrocks and Mumford, 1973) is essentially the only non–splitting rank–2 vector bundle known.

(2) On $\mathbb{P}^5$ no such bundle is known. $\square$

### 6.3. Beilinson's theorem

Beilinson's theorem tells us that the derived category of coherent sheaves on $\mathbb{P}^n$ is generated by the vector bundles $\Omega^i(i)$, $0 \leq i \leq n$ (see Beilinson, 1978).

**Beilinson's Theorem (Monad Version) 6.25.** *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ there is a complex $\mathcal{K}^\bullet$ with*

$$\mathcal{K}^\bullet \cong \bigoplus_j H^{i-j}(\mathbb{P}^n,\ \mathcal{F}(j)) \otimes \Omega^{-j}(-j)$$

*such that*

$$H^i(\mathcal{K}^\bullet) = \begin{cases} \mathcal{F} & \text{if } \ i = 0 \\ 0 & \text{if } \ i \neq 0\ . \end{cases} \qquad \square$$

**Definition 6.26.** $\mathcal{K}^\bullet$ above is called **Beilinson's monad** for $\mathcal{F}$. $\square$

For some applications it is useful to keep the following slightly weaker version of Beilinson's theorem in mind.

**Beilinson's Theorem (Spectral Sequence Version) 6.27.** *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ there is a spectral sequence with $E_1$–terms*

$$E_1^{ji} = H^i(\mathbb{P}^n, \mathcal{F}(j)) \otimes \Omega^{-j}(-j)$$

*converging to $\mathcal{F}$, that is, $E_\infty^{ji} = 0$ for $j + i \neq 0$ and $E_\infty^{-j,j}$ is the associated graded sheaf of a suitable filtration of $\mathcal{F}$.* $\square$

**Definition 6.28.** The spectral sequence above is called **Beilinson's spectral sequence** for $\mathcal{F}$. $\square$

**Remark 6.29.** (1) The differentials

$$\begin{aligned} d_1^{ji} &\in \operatorname{Hom}(H^i(\mathbb{P}^n, \mathcal{F}(j)) \otimes \Omega^{-j}(-j),\ H^i(\mathbb{P}^n, \mathcal{F}(j+1)) \otimes \Omega^{-j-1}(-j-1)) \\ &\cong \operatorname{Hom}(V^* \otimes H^i(\mathbb{P}^n, \mathcal{F}(j)),\ H^i(\mathbb{P}^n, \mathcal{F}(j+1))) \end{aligned}$$

of the monad (spectral sequence) coincide with the natural multiplication maps.

(2) The shape of the monad (spectral sequence) for $\mathcal{F}$ is determined by the dimensions

$$h^i\mathcal{F}(j) := H^i(\mathbb{P}^n, \mathcal{F}(j)), \quad 0 \leq i \leq n\,,$$

in the range $-n \leq j \leq 0$. In order to construct a sheaf $\mathcal{F}$ with given cohomology via Beilinson's theorem one has to determine the differentials of the monad (spectral sequence) first.

(3) It is often convenient to pick a twist $m$ and apply Beilinson's theorem to $\mathcal{F}(m)$ instead of $\mathcal{F}$ itself. $\square$

**Example 6.30.** The dimensions $h^i\mathcal{E}(j)$ of the cohomology groups of the Horrocks–Mumford bundle $\mathcal{E}$ on $\mathbb{P}^4$ are as follows (see Horrocks and Mumford, 1973):

$i$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 4 | | | | | | | | | | | |
| | 2 | 10 | 10 | 5 | | | | | | | | |
| | | | | | | 2 | | | | | | |
| | | | | | | | | 5 | 10 | 10 | 2 | |
| | | | | | | | | | | | 4 | |

$j$

Hence $\mathcal{E}$ is the cohomology of a monad of type

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 5\Omega^4(4) & \longrightarrow & 2\Omega^2(2) & \longrightarrow & 5\Omega^0(0) & \longrightarrow & 0\ . \\ & & \| & & & & \| & & \\ & & 5\mathcal{O}(-1) & & & & 5\mathcal{O} & & \end{array}$$

Notice that Serre–duality is reflected by the symmetry of the cohomology table (monad). □

## 6.4. Liaison

We recall some basic facts on liaison (linkage). For details and proofs we refer to Peskine and Szpiro (1974).

**Definition 6.31.** Let $X, X' \subset \mathbb{P}^n$ be two locally complete intersections of pure codimension 2 with no irreducible components in common. Then $X$ and $X'$ are said to be **linked** of type $(a, b)$, $X \underset{(a,b)}{\sim} X'$, if there exist hypersurfaces $V_1, V_2$ of degrees $a, b$ respectively such that

$$X \cup X' = V_1 \cap V_2\ . \quad \square$$

**Remark 6.32.** (1) There are formulae relating the numerical invariants of linked varieties.
(2) If $V_1, V_2$, and $X$ are given by explicit equations, then equations for $X'$ can be found by an ideal quotient computation.
(3) If $X$ and $X'$ are linked of type $(a, b)$, and if

$$0 \longleftarrow \mathcal{J}_X \longleftarrow \mathcal{G} \longleftarrow \mathcal{F} \longleftarrow 0$$

is a resolution with vector bundles $\mathcal{F}$ and $\mathcal{G}$, then, under suitable assumptions (for example if $H^1(\mathbb{P}^n, \mathcal{F}(a)) = H^1(\mathbb{P}^n, \mathcal{F}(b)) = 0$), $\mathcal{J}_{X'}$ has a resolution of type

$$0 \longleftarrow \mathcal{J}_{X'} \longleftarrow \mathcal{F}^*(-a-b) \oplus \mathcal{O}(-a) \oplus \mathcal{O}(-b) \longleftarrow \mathcal{G}^*(-a-b) \longleftarrow 0 \ .$$

Moreover, the Hartshorne–Rao modules of $X$ are $\mathbb{C}$–dual to those of $X'$:

$$(H_*^{n-1-i}\mathcal{J}_X)^* \cong (H_*^i \mathcal{J}_{X'})(n+1-a-b), \quad i = 1, \ldots, n-2 \ . \qquad \square$$

**Theorem (Peskine–Szpiro) 6.33.** *Let $X \subset \mathbb{P}^n$, $n \leq 5$, be a locally complete intersection of codimension 2. Let $m$ be a twist such that $\mathcal{J}_X(m)$ is globally generated. Then for every pair $d_1, d_2 \geq m$ there exist forms $f_i \in H^0(\mathbb{P}^n, \mathcal{J}_X(d_i))$, $i = 1, 2$, such that the corresponding hypersurfaces $V_1$ and $V_2$ intersect properly, $V_1 \cap V_2 = X \cup X'$, where*

(1) *$X'$ is a locally complete intersection,*

(2) *$X$ and $X'$ have no common component,*

(3) *$X'$ is nonsingular outside a set of positive codimension in* Sing X . $\square$

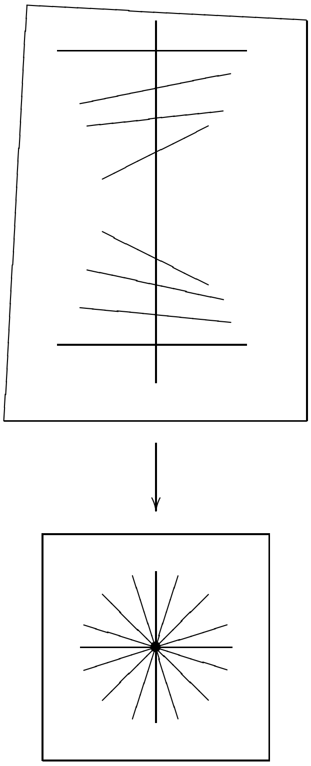
### 6.5. The Enriques–Kodaira classification

In this section a surface is a compact, connected complex manifold $X$ of dimension 2.

**Examples 6.35.** (1) The projective plane $\mathbb{P}^2$.

(2) Tori $\mathbb{C}^2/\Lambda$.

(3) Further examples are obtained by blowing up a given surface $X$ in a point:



contains exceptional curve $E$:

$$\tilde{X} \qquad E \cong \mathbb{P}^1, \quad E.E = -1$$

$$\begin{array}{c} \tilde{X} \\ \big\downarrow \sigma \\ X \end{array}$$

bimeromorphic

□

Surfaces are classified in two steps (see Barth, Peters and Van de Ven, 1984 for details and proofs).

- The first step consists of a coarse classification according to the numerical invariants of $X$, that is, mainly according to the **Kodaira dimension** $\mathrm{kod}(X)$. Other crucial invariants are the **geometric genus** $p_g$, the **arithmetic genus** $p_a$, the **irregularity** $q = p_g - p_a$, and the self–intersection number $K^2$ of the canonical class $K$ of $X$.
- The second step is the bimeromorphic classification: by successively blowing down exceptional curves one arrives at a **minimal model**:

$$\begin{array}{c} X \\ \downarrow \\ X_{\min} \end{array}$$

The classification of minimal surfaces is given by the following theorem.

**Theorem (Enriques–Kodaira classification) 6.35.** *Every surface has a minimal model in precisely one of the classes* (1)–(10) *below. This is uniquely determined (up to isomorphisms) except for the surfaces with minimal models* *in the classes* (1) *and* (3).

| | class of $X$ | kod $(X)$ | algebraic ? |
|---|---|---|---|
| (1) | minimal rational surfaces | $-\infty$ | yes |
| (2) | minimal surfaces of type VII | $-\infty$ | no |
| (3) | ruled surfaces with $q \geq 1$ | $-\infty$ | yes |
| (4) | Enriques surfaces | 0 | yes |
| (5) | bielliptic surfaces | 0 | yes |
| (6) | Kodaira surfaces | 0 | no |
| (7) | K3 surfaces | 0 | yes/no |
| (8) | Tori | 0 | yes/no |
| (9) | minimal elliptic surfaces with kod $(X) = 1$ | 1 | yes/no |
| (10) | minimal surfaces of general type | 2 | yes |

$\square$

It is a general philosophy that surfaces of general type are considered to be classified.

## 7. The theorem of Ellingsrud and Peskine

It is well–known that any smooth algebraic surface can be embedded in $\mathbb{P}^5$. Indeed, embed $X$ in some $\mathbb{P}^n$. As long as $n \geq 6$ the secant variety $\mathrm{Sec}(X)$ does not fill up $\mathbb{P}^n$, so we can project $X$ from a point off $\mathrm{Sec}(X)$ in order to embed $X$ into $\mathbb{P}^{n-1}$. For $n = 5$, however, the situation is different.

**Theorem (Severi, 1901) 7.1.** *Let $X \subset \mathbb{P}^5$ be a smooth, non–degenerate surface. Then* $\mathrm{Sec(X)} \neq \mathbb{P}^5$ *iff $X$ is a Veronese surface.* $\square$

This theorem tells us that if we project a smooth non–degenerate surface in $\mathbb{P}^5$ other then the Veronese surface from a general point off the surface then its image in $\mathbb{P}^4$ will have double points. The formula computing the number of these points yields constraints on the invariants of a smooth surface in $\mathbb{P}^4$.

**Double Point Formula 7.2.** *The numerical invariants of a smooth surface $X \subset \mathbb{P}^4$ satisfy*

$$d^2 - 10d - 5H.K - 2K^2 + 12\chi = 0 \ . \quad \square$$

Here and in what follows we denote by

- $H$ the hyperplane class of $X$,
- $K$ its canonical class,
- $d = H^2$ its degree,
- $\pi = \frac{1}{2}H.(K+H)+1$ its sectional genus, and
- $\chi = \chi\mathcal{O}_X = 1 - q + p_g$ its Euler–Poincaré characteristic.

**Theorem (Ellingsrud and Peskine, 1989) 7.3.** *There exists a number $d_0$ such that for every smooth surface $X \subset \mathbb{P}^4$ of degree $d$ the following holds:*

$$d > d_0 \;\Rightarrow\; X \quad \textit{is of general type.} \qquad \square$$

So there exist only finitely many families of smooth surfaces in $\mathbb{P}^4$ which are not of general type. Motivated by this result we pose the following problem.

**Problem 7.4.** Classify the smooth, non–general type surfaces in $\mathbb{P}^4$ . □

To solve this problem means to solve the following two problems.

**Problem 7.5.** What is the true $d_0$ ? □

We believe that $d_0$ is small. In fact, it is conjectured that $d_0 = 15$.

**Problem 7.6.** Classify the smooth, non–general type surfaces in $\mathbb{P}^4$ of low degree. □

In order to attack our problems we need

(I) bounds for the numerical invariants,

(II) an effective construction method, and

(III) a method to determine, where a given surface stands in the Enriques–Kodaira classification.

We give answers to (II) respectively (III) in sections 8 respectively 9. Let us now describe what is known with respect to (I).

We sketch an effective proof of the theorem of Ellingsrud and Peskine (see Decker and Schreyer, 2000 for further details and references). The proof is based on two lifting lemmas and a counting argument. Let us first recall the lifting lemmas.

**Theorem (Laudal's Lemma) 7.7.** (Laudal, 1977) *If the general hyperplane section $\Gamma \subset \mathbb{P}^2$ of an integral curve $C \subset \mathbb{P}^3$ of degree $d > s^2 + 1$ lies on a hypersurface of degree $s$ then the same holds for $C$.* □

**Theorem (Roth's Lemma) 7.8.** (Roth, 1937; Mezzetti and Raspanti, 1993) *If the general hyperplane section $C \subset \mathbb{P}^3$ of a smooth surface $X \subset \mathbb{P}^4$ of degree $d > s^2 - s + 2$ lies on a hypersurface of degree $s$ then the same holds for $X$.* □

A particularly transparent approach to the counting argument via ideas of Green (1998) on generic initial ideals is due to Braun and Floystad (1994). Let us explain the basic idea, that is, let us explain how to compute the Euler–Poincaré characteristic of a projective variety via generic initial ideals. We consider the polynomial ring $R = \mathbb{C}[x_0, \ldots, x_n]$ and fix the degree reverse lexicographical order $>_{\mathrm{dp}}$ on $R$. This order is particularly well–suited for our purposes.

**Definition 7.9.** A monomial ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is said to be **Borel–fixed** if $x_i m \in I$, $m$ a monomial in $\mathbb{C}[x_0, \ldots, x_n]$, implies $x_j m \in I$ for all $j \leq i$. □

Let now $I \subset \mathbb{C}[x_0, \ldots, x_n]$ be an arbitrary homogeneous ideal and denote by $X \subset \mathbb{P}^n$ the closed subscheme defined by $I$. Then the homogeneous ideal $I_X$ of $X$ is obtained by saturation:

$$I_X = I^{\mathrm{sat}} := (I : \langle x_0, \ldots, x_n \rangle^\infty).$$

**Theorem (Galligo, 1974) 7.10.** *There is a non–empty Zariski open subset $U \subset \mathrm{GL}\,(n+1, \mathbb{C})$ such that the initial ideal $\mathrm{L}(g(I))$ is constant and Borel–fixed for $g \in U$.* □

**Definition 7.11.** $\mathrm{gin}(I) := \mathrm{L}(g(I))$, $g \in U$, is called the **generic initial ideal** of $I$. □

**Theorem (Properties of Generic Initial Ideals) 7.12.** (Bayer and Stillman, 1987)

(1) *The homogenous ideal $I_X$ of $X$ has the generic initial ideal*

$$\mathrm{gin}(I^{\mathrm{sat}}) = (\mathrm{gin}(I) : x_n^\infty).$$

*In particular, $I$ is saturated iff no minimal generator of* $\mathrm{gin}(I)$ *involves $x_n$.*

(2) *The homogeneous ideal of a general hyperplane section $X \cap H$ of $X$ has the generic initial ideal*

$$\operatorname{gin}(I_{X\cap H}) = (\operatorname{gin}(I) : x_{n-1}^{\infty}) \cap \mathbb{C}[x_0, \dots , x_{n-1}].$$

(3) *We have*

$$(\operatorname{gin}(I) : x_{n-\alpha}^{\infty}) = (\operatorname{gin}(I) : \langle x_{n-\alpha} \cdot \ldots \cdot x_n\rangle^{\infty}), \quad 0 \le \alpha \le n.$$

*In particular,*

$$\operatorname{depth} R/I = n - \max\{\alpha \mid \operatorname{gin}(I) = (\operatorname{gin}(I) : x_{n-\alpha}^{\infty})\}.$$

(4) *The Castelnuovo–Mumford regularity of $I$ is equal to the highest degree of a minimal generator of* $\operatorname{gin}(I)$. □

In order to compute the Hilbert function of $R/I$ or equivalently of $R/\operatorname{gin}(I)$ we stratify the set of monomials not in $\operatorname{gin}(I)$ by studying the generic initial ideal of successive hyperplane sections of $X$. For $0 \le \alpha \le \dim X$ consider the monomial ideal

$$J_\alpha := J_\alpha(I) := (\operatorname{gin}(I) : x_{n-\alpha}^{\infty}).$$

By the properties of generic initial ideals $J_\alpha = (\operatorname{gin}(I) : \langle x_{n-\alpha} \cdot \ldots \cdot x_n\rangle^{\infty})$, whence $J_\alpha \cap \mathbb{C}[x_0, \dots , x_{n-\alpha}] = \operatorname{gin}(I_{X\cap H_1 \cap \ldots \cap H_\alpha})$ is the generic initial ideal of the intersection of $X$ with $\alpha$ general hyperplanes $H_1, \dots , H_\alpha$. The $J_\alpha$ define a filtration

$$\operatorname{gin}(I) =: J_{-1} \subset J_0 \subset J_1 \subset \ldots \subset J_{\dim X} \subset \langle 1\rangle =: J_{\dim X+1}.$$

For $-1 \le \alpha \le \dim X$ let

$$M_\alpha := M_\alpha(I) := \{\text{monomials } m \in J_{\alpha+1} \cap \mathbb{C}[x_0, \dots , x_{n-\alpha-1}] \mid m \notin J_\alpha\}.$$

Borel–fixedness implies that each $M_\alpha$ is a finite set of monomials and that we have a decomposition

$$\mathbb{C}[x_0, \dots , x_n]/\operatorname{gin}(I) \cong \bigoplus_{\alpha=-1}^{\dim X} \bigoplus_{m\in M_\alpha} m\mathbb{C}[x_{n-\alpha}, \dots , x_n] \subset \mathbb{C}[x_0, \dots , x_n]$$

as a $\mathbb{C}$–vector space. For the Hilbert function $H_{R/I_X}$ of $X$ this gives

$$H_{R/I_X}(t) = \sum_{\alpha=0}^{\dim X} \sum_{m\in M_\alpha} \binom{t- \mid m \mid +\alpha}{\alpha}$$

for $t \gg 0$. Here $\mid m \mid$ denotes the degree of the monomial $m$. Hence we have the following result.

**Proposition (Formula for $\chi$) 7.13.** *With notations as above*

$$\chi(\mathcal{O}_X) = \sum_{\alpha=0}^{\dim X} (-1)^\alpha \sum_{m \in M_\alpha} \binom{\mid m \mid -1}{\alpha}. \qquad \square$$

**Definition 7.14.** For $0 \leq \alpha \leq \dim X$ let $M_\alpha(X) := M_\alpha(I_X)$. $\square$

**Remark 7.15.** Let $X \cap H$ be a general hyperplane section. Then

$$M_\alpha(X \cap H) = M_{\alpha+1}(X), \quad 0 \leq \alpha \leq \dim X - 1. \qquad \square$$

Consider for example a set $\Gamma$ of $d$ points in the plane $\mathbb{P}^2$. By property (1) of generic initial ideals and Borel–fixedness $\mathrm{gin}(I_\Gamma)$ is of the form

$$I_\Gamma = \langle x_0^s, x_0^{s-1} x_1^{\lambda_{s-1}}, \ldots, x_0 x_1^{\lambda_1}, x_1^{\lambda_0} \rangle$$

with

$$\lambda_0 > \cdots > \lambda_{s-1} \geq 1.$$

Then $s$ is the minimal degree of a hypersurface containing $\Gamma$ and

$$M_0(\Gamma) = \{x_0^a x_1^b \mid 0 \leq a \leq s-1,\ 0 \leq b \leq \lambda_i - 1\}.$$

The sequence $n_i := \lambda_i + i$, $0 \leq i \leq s-1$, is the **numerical character** of $\Gamma$ as introduced by Gruson and Peskine (1977).

**Definition 7.16.** $\lambda_0, \ldots, \lambda_{s-1}$ are called the **GP–invariants** of $\Gamma$. $\square$

The formula for $\chi(\mathcal{O}_\Gamma)$ reads

$$d = \chi(\mathcal{O}_\Gamma) = \sum_{i=0}^{s-1} \lambda_i.$$

In particular, two sets of points in the plane have the same GP–invariants iff they have the same Hilbert function.

**Definition 7.17.** $\Gamma$ is said to be in **uniform position** if every two subsets of $\Gamma$ containing the same number of points have the same Hilbert function. $\square$

**Theorem (Connectedness of the GP–invariants) 7.18.** (Gruson and Peaskine, 1977) *If $\Gamma$ is a set of points in the plane in uniform position, then the GP–invariants of $\Gamma$ are* **connected**, *that is,*

$$\lambda_i - 1 \geq \lambda_{i+1} \geq \lambda_i - 2, \quad 0 \leq i \leq s-2. \qquad \square$$

By Harris' uniform position principle (see Harris, 1980) the above theorem applies in particular to a general plane section $\Gamma$ of an integral space curve $C$.

Following Green (1998) we visualize the monomials in $\mathrm{gin}(I_C)$ by the **diagram** of $\mathrm{gin}(I_C)$ as follows. By property (1) of generic initial ideals $\mathrm{gin}(I_C)$ is generated by monomials of type $x_0^a x_1^b x_2^c$. Consider integers $a \geq b \geq 0$. The $b$th entry in the $a$th row of the diagram represents the monomials $x_0^a x_1^b x_2^c$, $c \geq 0$. There are three possible types of entries. We put an empty circle iff $x_0^a x_1^b x_2^c$ is not in $\mathrm{gin}(I_C)$ for every $c \geq 0$ (then $x_0^a x_1^b \in M_1(C)$). We put a circle containing an integer $c > 0$ iff $x_0^a x_1^b x_2^c \in \mathrm{gin}(I_C)$ but $x_0^a x_1^b x_2^{c-1} \notin \mathrm{gin}(I_C)$ (then $x_0^a x_1^b, \ldots, x_0^a x_1^b x_2^{c-1} \in M_0(C)$). Finally, we put an $X$ iff $x_0^a x_1^b$ is in $\mathrm{gin}(I_C)$. If the $a$th row of the diagram contains $X$'s only, all information on $\mathrm{gin}(I_C)$ can be read off from the first $a$ rows of the diagram (we always picture the first $a-1$ rows). For fixed $a \geq 0$, all entries representing monomials of type $x_0^a x_1^b$, $b \geq 0$, form the $a$th **diagonal** of the diagram. The diagram of $\mathrm{gin}(I_\Gamma)$ is defined in the same way. In this case the entries are empty circles and X's only. In order to obtain the diagram of $\mathrm{gin}(I_\Gamma)$ from that one of $\mathrm{gin}(I_C)$ we have to replace each circle containing a positive number by an $X$ (this follows from property (2) of generic initial ideals).

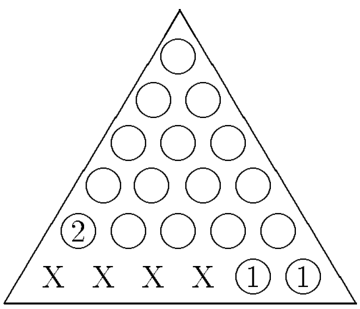
**Example 7.19.** Let $C$ be a general hyperplane section of a K3–surface $X \subset \mathbb{P}^4$ with degree $d = 14$ and sectional genus $\pi = 19$ as constructed by Popescu (1993). Then, as one can check,

$$\mathrm{gin}(I_C) = \langle x_0^5, x_0^4x_1, x_0^3x_1^2, x_0^2x_1^3, x_0x_1^5, x_1^6, x_0x_1^4x_2, x_1^5x_2, x_0^4x_2^2 \rangle$$

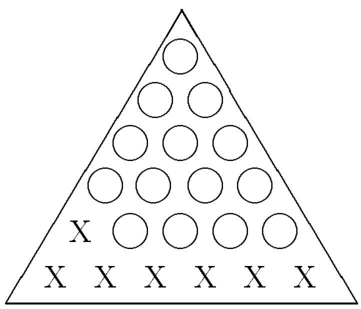
and

$$\mathrm{gin}(I_\Gamma) = \langle x_0^4, x_0^3x_1^2, x_0^2x_1^3, x_0x_1^4, x_1^5 \rangle.$$

The corresponding diagrams are:




$\mathrm{gin}(I_C)$




$\mathrm{gin}(I_\Gamma)$ $\square$

**Definition 7.20.** Every monomial contained in one of the $M_\alpha(C)$ is called a **zero** of $C$. The monomials in $M_1(C)$ are called the **ordinary zeros** of $C$, those in $M_0(C)$ the **sporadic zeros** of $C$. □

By the above discussion the numerical data of $\Gamma$ can be read off the diagram of $\mathrm{gin}(I_\Gamma)$. For example, $\lambda_i$ is the number number of circles in the $i$th diagonal, $0 \leq i \leq s-1$, and the degree of $\Gamma$ is obtained by counting the number of all circles. The arithmetic genus $\pi$ of $C$ can be read off the diagram of $\mathrm{gin}(I_C)$:

**Remark 7.21.** Let $\pi$ be the arithmetic genus of $C$, and let $\lambda_0 > \cdots > \lambda_{s-1}$ be the GP–invariants of $\Gamma$. Then the formula for $\chi(\mathcal{O}_C) = 1 - \pi$ gives

$$\pi = 1 + \sum_{m \in M_1(C)} (\mid m \mid -1) \quad - \sum_{m \in M_0(C)} 1$$

$$= 1 + \sum_{i=0}^{s-1} \left( \binom{\lambda_i}{2} + (i-1)\lambda_i \right) \quad - \sum_{m \in M_0(C)} 1. \qquad \square$$

By applying Laudal's lemma and by maximizing $1 + \sum_{m \in M_1(C)} (\mid m \mid -1)$ $- \sum_{m \in M_0(C)} 1$ among all curves $C$ as in the following assumption one can prove the following result.

**Theorem (Halphen's Bound) 7.22.** (Gruson and Peskine, 1977) *Let $C \subset \mathbb{P}^3$ be an integral space curve of degree $d$ and arithmetic genus $\pi$. Let $s > 0$ be an integer such that $d > s(s-1)$ and such that $I_C$ is not contained in a hypersurface of degree $< s$. Then*

$$\pi \leq 1 + \frac{d^2 + s(s-4)d}{2s} - \frac{r(s-r)(s-1)}{2s},$$

*where*

$$d + r \equiv 0 \ (\mathrm{mod}\ s) \quad \textit{with} \quad 0 \leq r < s. \qquad \square$$

In order to prove the theorem of Ellingsrud and Peskine we need in addition to the results presented so far the following three easy facts on smooth surfaces in $\mathbb{P}^4$. We denote by $X$ such a surface and by $C$ respectively $\Gamma$ a general hyperplane section of $X$ respectively $C$.

**Lemma 7.23.** *If $X$ is not of general type, then $d \leq 5$ or*

$$K^2 \leq 0 \quad \textit{and} \quad 6\chi(\mathcal{O}_S) - K^2 \geq 0.$$

*Proof.* This follows by Severi's theorem and classification. □

**Lemma 7.24.** *If* $6\chi(\mathcal{O}_S) - K^2 \geq 0$ *and* $d > 90$ *then* $X$ *is contained in a hypersurface of degree 5.*

*Proof.* Apply the double point formula, Roth's lemma and Halphen's bound. □

**Definition 7.25.** If $Z \subset \mathbb{P}^n$ is a closed subscheme, then

$$s_Z := \min \{k \in \mathbb{Z} \mid (I_Z)_k \neq 0\}$$

denotes the minimal degree of a hypersurface containing $Z$. □

**Lemma 7.26.** *If* $d > (s_X - 1)^2 + 1$ *then* $s := s_X = s_C = s_\Gamma$ *and*

$$\mu := \sum_{m \in M_0(C)} 1 \leq \frac{d(s-1)^2}{2s}.$$

*Proof.* Apply the lemmas of Roth and Laudal and consider $c_2(\mathcal{N}_X^*(s_X))$. □

We are now ready to show the key lemma in the proof of the theorem of Ellingsrud and Peskine.

**Proposition 7.27.** *For every integer* $s > 0$ *there is a cubic polynomial* $P_s(d)$ *with leading term* $\frac{d^3}{6s^2}$ *such that every smooth surface* $X \subset \mathbb{P}^4$ *with* $s = s_S$ *and degree* $d > (s-1)^2 + 1$ *satisfies*

$$\chi(\mathcal{O}_X) \geq P_s(d).$$

*Proof.* Let $X$ be a surface as in the assertion with $C$ and $\Gamma$ as above. Then

$$s := s_X = s_C = s_\Gamma$$

since $d > (s-1)^2 + 1$. Let $\lambda_0 > \cdots > \lambda_{s-1}$ be the GP–invariants of $\Gamma$ and write $M_i = M_i(X)$, $i = 0, 1, 2$. Then

$$M_2 = M_1(C) = M_0(\Gamma) \quad \text{and} \quad M_1 = M_0(C).$$

We want to bound

$$\chi(\mathcal{O}_X) = \sum_{m \in M_2} \binom{\mid m \mid -1}{2} \quad - \sum_{m \in M_1} (\mid m \mid -1) \quad + \sum_{m \in M_0} 1$$

from below. Let us consider each summand seperately.

(1) The first summand is smallest, if the $d$ monomials of $M_2$ have smallest possible degrees, that is, if at most one of the differences $\lambda_{i-1} - \lambda_i$ is 2. In that case $\lambda_0$ is the smallest integer $\geq \frac{d}{s} + \frac{s-1}{2}$. Hence

$$\sum_{m \in M_2} \binom{\mid m \mid -1}{2} = \sum_{i=0}^{s-1} (\binom{\lambda_i + i - 1}{3} - \binom{i-1}{3}) \geq s\binom{\frac{d}{s} + \frac{s-3}{2}}{3} + 1 - \binom{s-1}{4}.$$

(2) Next consider the second summand. The Castelnuovo–Mumford regularity of $I_C$ is at most $d-1$ (see Gruson, Lazarsfeld and Peskine, 1983). It follows from property (4) of generic initial ideals that the degree of each sporadic zero of $C$ is bounded by $d-2$. Together with the bound for the number of sporadic zeros of $C$ from Lemma 7.26 this gives

$$-\sum_{m \in M_1} (\mid m \mid -1) \geq -\mu(d-3) \geq -\frac{(s-1)^2}{2s} d(d-3).$$

(3) The third summand in the formula for $\chi(\mathcal{O}_X)$ can be neglected since it is non–negative.

By summing up we obtain the desired polynomial

$$P_s(d) = s\binom{\frac{d}{s} + \frac{s-3}{2}}{3} + 1 - \binom{s-1}{4} - \frac{(s-1)^2}{2s} d(d-3)$$

with leading term $\frac{d^3}{6s^2}$. □

**Proof of the Theorem of Ellingsrud and Peskine.** By Lemma 7.23 we may suppose that $K^2 \leq 0$ and that $6\chi(\mathcal{O}_S) - K^2 \geq 0$. By Lemma 7.24 $d \leq 90$ or $X$ is contained in a hypersurface of degree 5. We thus may and will assume that $s := s_X \leq 5$ and that $d > (s-1)^2 + 1$. Now

$$\begin{aligned} 0 \geq 2K^2 &= d^2 - 5d - 10(\pi - 1) + 12\chi(\mathcal{O}_X) \\ &\geq d^2 - 5d - 10(G(d,s) - 1) + 12P_s(d) \end{aligned}$$

by the double point formula, Halphen's bound and Proposition 7.27. The right hand side is a cubic polynomial in $d$ with positive leading coefficient $\frac{2}{s^2}$. So $d$ is bounded. □

**Remark 7.28.** Decker and Schreyer (2000) provide further information on the configuration of the sporadic zeros of a space curve arising as a general hyperplane section of a smooth surface in $\mathbb{P}^4$. As a consequence they show that $d_0 \leq 52$. □

## 8. Constructions

How to construct smooth surfaces in $\mathbb{P}^4$? Besides using general projections which does not work due to Severi's theorem there are two other classical construction methods.

- Consider a suitable **linear system** on an abstract surface and blow up the base points.

**Example (Bordiga surface) 8.1.** The blow–up

$$X = \mathbb{P}^2(p_1, \ldots, p_{10})$$

of the projective plane in ten general points can be embedded into $\mathbb{P}^4$ by the linear system

$$H = 4L - \sum_{i=1}^{10} E_i$$

(with $L$ the preimage of a line and $E_i$ the exceptional divisors). □

The problem with this approach is that typically the base points have to be in a very special position. Such configurations are hard to find.

- Apply **Liaison**: let $X \subset \mathbb{P}^4$ be a surface already known (presumably of small degree). Consider two hypersurfaces $V_1$ and $V_2$ containing $X$. Then, under assumptions as described in Section 6.4, the residual surface $X'$ in
$$V_1 \cap V_2 = X \cup X'$$
is smooth.

**Example 8.2.** Let $X$ be the Veronese surface in $\mathbb{P}^4$. Then $\mathcal{J}_X(3)$ is globally generated. Hence $X \underset{(3,3)}{\sim} X'$, where $X'$ is a smooth surface of degree 5 (by classification $X$ is an elliptic scroll). □

The problem with this approach is that typically the surfaces to be constructed tend to be minimal in their even liaison class.

Essentially, the classical construction methods failed to produce examples in degree $d \geq 11$. Let us discuss how to construct examples as the determinantal locus of a morphism between vector bundles $\mathcal{F}$ and $\mathcal{G}$ as in Section 6.2. We face the following basic question.

- How can we find $\mathcal{F}$ and $\mathcal{G}$ (depending on the numerical invariants of the surface to be constructed)?

An answer to this question is given in Decker, Ein and Schreyer (1993) (see Decker and Schreyer, 2000 for further details). The basic idea is to construct the Hartshorne–Rao modules

$$H^i_*\mathcal{J}_X = \bigoplus_{\nu\in\mathbb{Z}} H^i(\mathbb{P}^4, \mathcal{J}_X(\nu)), \quad i = 1, 2,$$

first. Then one involves syzygy bundles of these modules of finite length as direct summands in order to construct $\mathcal{F}$ and $\mathcal{G}$. Which syzygy bundles to choose comes out by analyzing Beilinson's monad or spectral sequence. For this analysis one needs a priori information on the dimensions $h^i\mathcal{J}_X(\nu)$ of the cohomology groups of the ideal sheaf of $X$ in some twists.

**Theorem (Riemann–Roch) 8.3.** *Let $X \subset \mathbb{P}^4$ be a smooth surface. Then*

$$\chi(\mathcal{J}_X(\nu)) = \chi(\mathcal{O}_{\mathbb{P}^4}(\nu)) - \binom{\nu+1}{2} d + \nu(\pi - 1) - 1 + q - p_g \ . \qquad \square$$

**Proposition 8.4.** *Let $X \subset \mathbb{P}^4$ be a smooth, non–general type surface which is not contained in any cubic hypersurface. Then we have the following table for the $h^i\mathcal{J}_X(\nu)$:*

$\uparrow i$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 0 | |
| | $n'+1$ | $p_g$ | 0 | 0 | 0 | 0 | |
| | 0 | $q$ | $*$ | $*$ | $*$ | $*$ | |
| | 0 | 0 | 0 | $*$ | $*$ | $*$ | |
| | 0 | 0 | 0 | 0 | 0 | $*$ | |

$\nu \rightarrow$

*where*

$$n' = \pi - q + p_g - 1 \ . \qquad \square$$

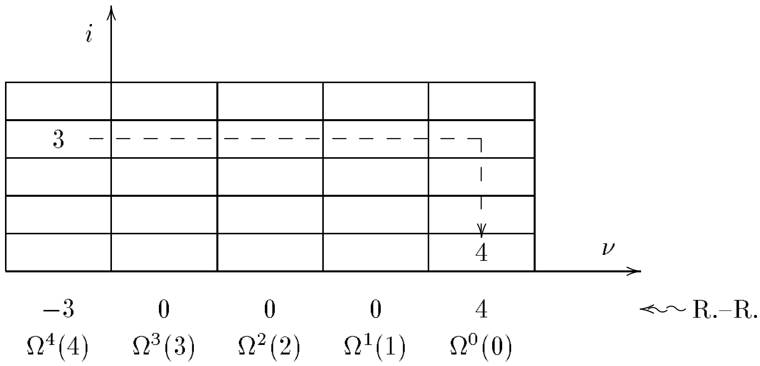
In what follows we represent a zero in a cohomology table by an empty box.

**Remark 8.5.** By work of Roth (1937), Aure (1987) and Koelblen (1992) surfaces on cubic hypersurfaces in $\mathbb{P}^4$ are classified. In particular, we have the following result:

$$d \geq 9,\ h^0\mathcal{J}_X(3) \neq 0 \ \Rightarrow\ X \quad \text{is of general type.} \qquad \square$$

In order to construct new surfaces via these ideas one first needs to find admissible numerical invariants. Here one can use again some of the results discussed in Section 7.

**Example 8.6.** Surfaces with $d = 6$, $\pi = 3$, $p_g = q = 0$.
A plausible cohomology table for the ideal sheaf of a surface with these invariants is

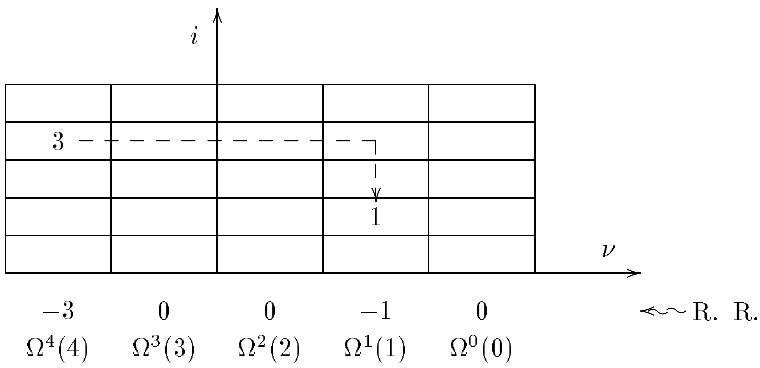



By Riemann–Roch these are the smallest possible values for a surface with the given invariants. If a surface with these values exists then the Beilinson–monad for $\mathcal{J}_X(3)$ is of type

$$0 \longleftarrow \mathcal{J}_X(3) \longleftarrow \mathcal{G} = 4\mathcal{O} \xleftarrow{\varphi} \mathcal{F} = 3\mathcal{O}(-1) \longleftarrow 0\ .$$

Kleiman's Bertini–type result shows that a generic $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ vanishes indeed along a smooth surface. $\square$

**Example 8.7.** Surfaces with $d = 4$, $\pi = 0$, $p_g = q = 0$.
A plausible cohomology table is




A corresponding Beilinson–monad for $\mathcal{J}_X(2)$ is of type

$$0 \longleftarrow \mathcal{G} = \Omega^1(1) \xleftarrow{\varphi} \mathcal{F} = 3\mathcal{O}(-1) \longleftarrow 0\ .$$

So such a diagram leads to a Veronese surface (see Example 6.15 ). $\square$

**Example 8.8.** Surfaces with $d = 10$, $\pi = 9$, $p_g = q = 0$
A plausible cohomology table is

$i$

| | | | | |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | 2 | | | |
| | | | 2 | |
| | | | | 1 |

$\nu$

A corresponding Beilinson–monad for $\mathcal{J}_X(4)$ is of type

$$0 \longleftarrow \mathcal{J}_X(4) \longleftarrow \mathcal{G} = 2\Omega^1(1) \oplus \mathcal{O} \xleftarrow{\varphi} \mathcal{F} = 2\Omega^3(3) \longleftarrow 0\ .$$

In this case Kleiman's theorem does not apply. That a generic $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ gives indeed rise to a smooth surface $X$ can be checked with the computer in an explicit example. By construction, $X$ has syzygies of type

$$0 \longleftarrow \mathcal{J}_X \longleftarrow \begin{matrix} \mathcal{O}(-4) \\ \oplus \\ 10\mathcal{O}(-5) \end{matrix} \longleftarrow 18\mathcal{O}(-6) \longleftarrow 10\mathcal{O}(-7) \longleftarrow 2\mathcal{O}(-8) \longleftarrow 0\ .$$

Let us also try the following cohomology table:

$i$

| | | | | |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | 2 | | | |
| | | | 2 | 1 |
| | | | | 2 |

$\nu$

A corresponding Beilinson–monad is of type

$$0 \longleftarrow \mathcal{O} \xleftarrow{0} 2\mathcal{O} \oplus 2\Omega^1(1) \longleftarrow 2\Omega^3(3) \longleftarrow 0$$

with $\mathcal{J}_X(4)$ being the cohomology in the middle. In other words, we are looking for an exact sequence

$$0 \longleftarrow \mathcal{J}_X(4) \longleftarrow \mathcal{G} = \mathcal{E} \oplus 2\mathcal{O} \longleftarrow \mathcal{F} = 2\Omega^3(3) \longleftarrow 0\ ,$$

where $\mathcal{E}$ is a kernel

$$0 \longleftarrow \mathcal{O} \longleftarrow 2\Omega^1(1) \longleftarrow \mathcal{E} \longleftarrow 0\ .$$

In particular, $\mathcal{E}$ has the intermediate cohomology

$$H^i_*(\mathcal{E}) = \begin{cases} H^1_*\mathcal{J}_X(4) & i = 1 \\ 0 & i = 2, 3\ . \end{cases}$$

So $\mathcal{E}$ is a vector bundle with the same intermediate cohomology as the first syzygy bundle of

$$M = H^1_*\mathcal{J}_X(4)\ .$$

$M$ is supposed to have Hilbert function (2,1) (here we use an obvious short hand notation). A generic module with this Hilbert function has syzygies of type

$$0 \leftarrow M \leftarrow 2S(1) \leftarrow 9S \leftarrow \begin{array}{c} 15S(-1) \\ \oplus \\ S(-2) \end{array} \leftarrow \begin{array}{c} 11S(-2) \\ \oplus \\ 3S(-3) \end{array} \leftarrow \begin{array}{c} 3S(-3) \\ \oplus \\ 3S(-4) \end{array} \leftarrow S(-5) \leftarrow 0\ .$$

In particular,

$$\operatorname{rk}(\mathcal{S}yz_1(M)) = 7 = \operatorname{rk}\mathcal{E}\ .$$

In view of Proposition 6.7 this dicussion suggests to pick $\mathcal{E}$ as the first syzygy bundle of a generic module as above, and to choose $\mathcal{F} = 2\Omega^3(3)$ and $\mathcal{G} = \mathcal{E} \oplus 2\mathcal{O}$. A check in an explicit example shows that a generic $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ vanishes along a smooth surface $X$ (see the Singular program in Example 2.2). $X$ has syzygies of type

$$0 \longleftarrow \mathcal{J}_X \longleftarrow \begin{array}{c} 2\mathcal{O}(-4) \\ \oplus \\ 5\mathcal{O}(-5) \\ \oplus \\ \mathcal{O}(-6) \end{array} \longleftarrow \begin{array}{c} 9\mathcal{O}(-6) \\ \oplus \\ 3\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 3\mathcal{O}(-7) \\ \oplus \\ 3\mathcal{O}(-8) \end{array} \longleftarrow \mathcal{O}(-9) \longleftarrow 0\ . \quad \square$$

**Example 8.9.** Surfaces with $d = 10$, $\pi = 8$, $q = p_g = 0$

A plausible cohomology table is

$i$

| | | | | |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | 1 | | | |
| | | 2 | 5 | 3 |
| | | | | |

$\nu$

Arguments as above suggest to construct $X$ by choosing

$$\mathcal{F} = \Omega^3(3), \quad \mathcal{G} = \mathcal{S}yz_1(M),$$

where $M$ has Hilbert function $(2,5,3)$. A short discussion shows that a generic $M$ with this Hilbert function has a presentation of type

$$0 \longleftarrow M \longleftarrow 2S(2) \xleftarrow{\psi=(\psi_1,\psi_2)} 5S(1) \oplus 2S\ ,$$

where the cokernel of $\psi_1$ is supported on five points. Without loss of generality we may therefore suppose that $\psi$ is of type

$$\psi = \left(\begin{array}{c|cc} \overbrace{\begin{matrix} x_0 \dots x_4 \\ a_0x_0 \dots a_4x_4 \end{matrix}}^{\psi_1} & \begin{matrix} 0 \\ q_1 \end{matrix} & \begin{matrix} 0 \\ q_2 \end{matrix} \end{array}\right)$$

with pairwise distinct constants $a_i$ and quadrics $q_1, q_2$. If $q_1, q_2$ are chosen generically, then $\mathcal{G}$ has a resolution of type

$$0 \longleftarrow \mathcal{G} \longleftarrow 15\mathcal{O}(-1) \longleftarrow \begin{matrix} 10\mathcal{O}(-2) \\ \oplus \\ 10\mathcal{O}(-3) \end{matrix} \longleftarrow \begin{matrix} 3\mathcal{O}(-3) \\ \oplus \\ 10\mathcal{O}(-4) \end{matrix} \longleftarrow 3\mathcal{O}(-5) \longleftarrow 0\ ,$$

and no $\varphi \in \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ is injective. Indeed, such a $\varphi$ is covered by a morphism of complexes

$$\begin{tikzcd} 0 & \mathcal{G} \arrow[l] & 15\mathcal{O}(-1) \arrow[l] & \begin{matrix} 10\mathcal{O}(-2) \\ \oplus \\ 10\mathcal{O}(-3) \end{matrix} \arrow[l] & \cdots \arrow[l] \\ 0 & \mathcal{F} \arrow[l] \arrow[u, "\varphi"] & 5\mathcal{O}(-1) \arrow[l] \arrow[u, "\varphi_0"] & \mathcal{O}(-2) \arrow[l] \arrow[u, "\varphi_1"] & 0 \arrow[l] \end{tikzcd}$$

which necessarily maps to the subcomplex

$$\begin{tikzcd} 0 & \underset{\text{rk}\,3}{\ker \psi_1} \arrow[l] & 10\mathcal{O}(-1) \arrow[l] & 10\mathcal{O}(-2) \arrow[l] & 3\mathcal{O}(-3) \arrow[l] & 0 \arrow[l] \\ 0 & \underset{\text{rk}\,4}{\Omega^3(3)} \arrow[l] \arrow[u, "\varphi"] & 5\mathcal{O}(-1) \arrow[l] \arrow[u, "\varphi_0"] & \mathcal{O}(-2) \arrow[l] \arrow[u, "\varphi_1"] & 0 \arrow[l] \end{tikzcd}$$

defined by the Eagon–Northcott complex of $\psi_1$ .

The trick is to choose $q_1, q_2$ special in order to obtain more syzygies. If we take

$$q_1 = b_0^{(1)}x_0^2 + b_1^{(1)}x_1^2 + b_2^{(1)}x_2^2, \qquad q_2 = b_2^{(2)}x_2^2 + b_3^{(2)}x_3^2 + b_4^{(2)}x_4^2,$$

with $b_i^{(j)} \in \mathbb{C}^*$, then we obtain a resolution of type

$$0 \leftarrow \mathcal{G}_a \leftarrow \begin{matrix} 15\mathcal{O}(-1) \\ \oplus \\ 2\mathcal{O}(-2) \end{matrix} \longleftarrow \begin{matrix} 12\mathcal{O}(-2) \\ \oplus \\ 10\mathcal{O}(-3) \end{matrix} \longleftarrow \begin{matrix} 3\mathcal{O}(-3) \\ \oplus \\ 10\mathcal{O}(-4) \end{matrix} \leftarrow 3\mathcal{O}(-5) \leftarrow 0 \ ,$$

whereas

$$q_1 = b_0^{(1)}x_0^2 + b_1^{(1)}x_1^2 + b_2^{(1)}x_2^2, \qquad q_2 = b_3^{(2)}x_3^2 + b_4^{(2)}x_4^2$$

with $b_i^{(j)} \in \mathbb{C}^*$ yields a resolution of type

$$0 \leftarrow \mathcal{G}_b \leftarrow \begin{matrix} 15\mathcal{O}(-1) \\ \oplus \\ 4\mathcal{O}(-2) \end{matrix} \longleftarrow \begin{matrix} 14\mathcal{O}(-2) \\ \oplus \\ 11\mathcal{O}(-3) \end{matrix} \longleftarrow \begin{matrix} 4\mathcal{O}(-3) \\ \oplus \\ 10\mathcal{O}(-4) \end{matrix} \leftarrow 3\mathcal{O}(-5) \leftarrow 0 \ .$$

In this way we obtain in both cases a family of smooth surfaces which are cut out by 10 quintics and 2 sextics respectively 10 quintics and 4 sextics. $\square$

## 9. Adjunction theory for surfaces

Adjunction theory provides a powerful tool for spotting a given surface in the Enriques–Kodaira classification. We recall the basic results and refer to Beltrametti and Sommese (1995) for details and proofs.

**Example 9.1.** Let $X \subset \mathbb{P}^4$ be a general and thus smooth surface with syzygies of type

$$0 \longleftarrow \mathcal{J}_X(3) \longleftarrow 4\mathcal{O} \xleftarrow{\ \varphi\ } 3\mathcal{O}(-1) \longleftarrow 0$$

as in Example 8.6 . Then the twisted dualizing sheaf

$$\omega_X(1) = \mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^4}(-5))(1)$$

has a resolution of type

$$0 \longleftarrow \omega_X(1) \longleftarrow 3\mathcal{O} \xleftarrow{\ \varphi^t\ } 4\mathcal{O}(-1) \longleftarrow \mathcal{O}(-4) \longleftarrow 0 \ .$$

It follows that the **adjoint linear system** $\mathbb{P}(H^0(\mathbb{P}^4, \omega_X(1))) = |K+H|$ is base point free of dimension $n' = \dim|K+H| = 2$, that is, we have a well–defined

**adjunction map** $\Phi = \Phi_{|K+H|} : X \to \mathbb{P}^2$ . Let $y_0, y_1, y_2$ be coordinates on $\mathbb{P}^2$. Then graph $\Phi \subset \mathbb{P}^4 \times \mathbb{P}^2$ is given by the equations

$$y_0\varphi_{0j}(x) + y_1\varphi_{1j}(x) + y_2\varphi_{2j}(x) = 0\,, \quad j = 0, \ldots, 3\ ,$$

where $\varphi = (\varphi_{ij})$. We may rewrite these equations as

$$x_0\psi_{j0}(y) + \cdots + x_4\psi_{j4}(y) = 0\,, \quad j = 0, \ldots, 3\ ,$$

with linear forms $\psi_{jk} \in \mathbb{C}[y_0, y_1, y_2]$. The general fibre of $\Phi$ is defined by four independent linear forms in $\mathbb{C}[x_0, \ldots, x_4]$. Hence $\Phi$ is birational with positive dimensional fibres precisely in the points where $\psi = (\psi_{jk})$ drops rank:

$$0 \longleftarrow \mathcal{J}_Z \longleftarrow 5\mathcal{O}_{\mathbb{P}^2}(-4) \xleftarrow{\ \psi\ } 4\mathcal{O}_{\mathbb{P}^2}(-5) \longleftarrow 0\ .$$

So $\Phi : X \to \mathbb{P}^2$ expresses $X$ as the blowing up of $\mathbb{P}^2$ in 10 points and $X$ is embedded by the quartics through these points, that is, by the $4 \times 4$–minors of $\psi$. In other words, $X$ is rational and in fact a Bordiga surface (see Example 8.1). □

In more complicated examples one has to apply the above idea repeatedly. The resulting **adjunction process** eventually yields a a minimal model $X_{\min}$.

**Theorem 9.2.** (Sommese, 1979; Van de Ven, 1979; Sommese and Van de Ven, 1987) *Let $X \subset \mathbb{P}^n$ be a smooth surface. Then the adjoint linear system $|K + H|$ is non–special of dimension $n' = \pi - q + p_g - 1$. It defines a birational morphism*

$$\Phi = \Phi_{|K+H|} : X \to X' \subset \mathbb{P}^{n'}$$

*onto a smooth surface $X'$ which blows down precisely all $(-1)$–curves on $X$ which are lines in the given embedding, unless*

(1) *$X$ is a plane, or the Veronese surface of degree 4, or $X$ is ruled by lines, or*

(2) *$X$ is a Del Pezzo surface, or a conic bundle, or*

(3) *$X$ belongs to one of the following four families (with obvious notations):*

   (a) $X = \mathbb{P}^2(p_1, \ldots, p_7)$ *embedded by* $H \equiv 6L - \sum_{i=1}^{7} 2E_i$,

   (b) $X = \mathbb{P}^2(p_1, \ldots, p_8)$ *embedded by* $H \equiv 6L - \sum_{i=1}^{7} 2E_i - E_8$,

   (c) $X = \mathbb{P}^2(p_1, \ldots, p_8)$ *embedded by* $H \equiv 9L - \sum_{i=1}^{8} 3E_i$,

   (d) *$X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is an indecomposable rank 2 bundle over an elliptic curve, and $H \equiv 3B$, where $B$ is a section with $B^2 = 1$ on $S$.* □

**Remark 9.3.** (1) In the exceptional case (1) $|K+H| = \emptyset$ and $(K+H)^2 < 0$. In the exceptional case (2) $(K+H)^2 = 0$ and $|K+H|$ maps to a point, or defines the conic fibration (unless $d = 8$, $q = 1$ and $\pi = 3$). In all other cases $(K+H)^2 > 0$, but in the exceptional case (3) $|K+H|$ is simply too small to define a birational map.

(2) For surfaces in $\mathbb{P}^4$ the exceptional cases occur in degrees $\leq 8$. This follows from the classification of scrolls and conic bundles in $\mathbb{P}^4$ (see Lanteri, 1980; Aure, 1987; Ellia and Sacchiero, 1998). □

The adjunction process yields finitely many surfaces $X \to X' \to X'' \to \dots$ which are called the **adjoint surfaces** of $X$. The last adjoint surface is either minimal or belongs to one of the exceptional cases. In order to avoid cumbersome case by case considerations it is crucial to control the numerical data of the first adjoint surface $X'$. We know

$$\deg X' = (K+H)^2 \quad \text{and} \quad 2\pi' - 2 = (K+H).(2K+H).$$

For

$$K'^2 = K^2 + a,$$

however, we have to compute the number $a$ of exceptional lines on $X$. Here Le Barz' 6–secant formula is useful. Let $X \subset \mathbb{P}^4$ be a smooth surface of degree $d$, sectional genus $\pi$ and Euler–Poincare characteristic $\chi$. Consider the double curve of a general projection of $X$ to $\mathbb{P}^3$ and denote by

$$\delta = \binom{d-1}{2} - \pi$$

its degree, by

$$t = \binom{d-1}{3} - \pi(d-3) + 2\chi - 2$$

its number of triple points and by

$$h = \frac{1}{2}(\delta(\delta - d + 2) - 3t)$$

its number of apparent double points. Let

$$\begin{aligned} N_6(d,\,\pi,\chi) = &-\frac{1}{144}d(d-4)(d-5)(d^3+30d^2-577d+786)\\ &+\delta(2\binom{d}{4} + d\binom{d}{3} - 45\binom{d}{2} + 148d - 317)\\ &-\frac{1}{2}\binom{\delta}{2}(d^2-27d+120) - 2\binom{\delta}{3}\\ &+h(\delta - 8d + 56) + t(9d - 3\delta - 28) + \binom{t}{2}. \end{aligned}$$

**Theorem (6–secant formula of Le Barz, 1981) 9.4.** *Suppose that there are no lines on $X$ with self–intersection $\geq 0$. Then the number of 6–secantes (if finite) plus the number of exceptional lines on $X$ is equal to $N_6(d, \pi, \chi)$.* □

**Remark 9.5.** (1) It is a plausible guess and in many cases true that the number of 6–secants to $X$ is precisely the number of sextic generators of the homogeneous ideal $I_X$ of $X$. For example, if $I_X$ is generated in degrees $\leq 5$, then there are no 6–secants at all.

(2) In any case, a construction of $X$ via syzygies as in Section 8 produces explicit equations of $X$. This allows to compute the number of 6–secants by computing the ideal quotient $((I_X)_{\leq 5} : I_X)$. □

Let us describe the adjunction process for the examples in Section 8.

**Notation 9.6.** $X_\pi^d \subset \mathbb{P}^n$ stands for a surface of degree d and sectional genus $\pi$ in $\mathbb{P}^n$.

$$X_\pi^d \overset{\alpha}{\to} X_{\pi'}^{d'}$$

denotes an adjunction map which simultaneously blows down $\alpha$ $(-1)$–lines. □

**Example 9.7.** Let $X$ belong to one of the two families constructed in Example 8.8 . Then

$$d = 10,\ \ \pi = 9,\ \ q = p_g = 0,$$

whence $\chi = 1$. So $K^2 = -9$ by the double point formula, the intersection matrix of $X$ is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & -9 \end{pmatrix} ,$$

and

$$N_6 = N_6(10, 9, 1) = 7.$$

(1) If $X$ is a member of the first family then $I_X$ is generated by quartics and quintics, so $N_6 = 7$ is the number of the exceptional lines on $X$. It follows that the intersection matrix of the first adjoint surface $X'$ is

$$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 13 & -3 \\ -3 & -2 \end{pmatrix} .$$

In particular, no multiple of $K'$ is effective. So $X$ is rational since $q = 0$. The second adjoint surface is a smooth surface $X'' \subset \mathbb{P}^5$ of degree 5. Hence $X''$ is a Del Pezzo surface with intersection matrix

$$\begin{pmatrix} (H'')^2 & H''.K'' \\ H''.K'' & (K'')^2 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} .$$

Altogether the adjunction process yields

$$\begin{array}{ccccc} \mathbb{P}^4 & & \mathbb{P}^8 & & \mathbb{P}^5 \\ \cup & & \cup & & \cup \\ X_9^{10} & \xrightarrow{7} & X_6^{13} & \xrightarrow{7} & X_1^5 \end{array},$$

and $X$ is the blow–up

$$X = \mathbb{P}^2(p_1, \ldots, p_{18})$$

of $\mathbb{P}^2$ in 18 possibly infinitesimal near points, embedded by

$$H = 4L - \sum_{i=1}^{4} 3E_i - \sum_{i=5}^{11} 2E_i - \sum_{i=12}^{18} E_i$$

(with $L$ the preimage of a line and $E_i$ the exceptional divisors).

(2) In the second case $I_X$ has one sextic generator and we guess that there is one 6–secant to $X$. Indeed, the subcomplex

$$\mathcal{O}(-6) \leftarrow 3\mathcal{O}(-7) \leftarrow 3\mathcal{O}(-8) \leftarrow \mathcal{O}(-9)$$

of the syzygies

$$\begin{array}{ccccccccccc} & & & & 2\mathcal{O}(-4) & & & & & & \\ & & & & \oplus & & & & & & \\ 0 & \longleftarrow & \mathcal{J}_X & \longleftarrow & 5\mathcal{O}(-5) & & 9\mathcal{O}(-6) & & 3\mathcal{O}(-7) & & \\ & & & & \oplus & \nwarrow & \oplus & \longleftarrow & \oplus & \nwarrow & \\ & & & & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & & \mathcal{O}(-9) \longleftarrow 0 \end{array}$$

of $\mathcal{J}_X$ is the Koszul resolution of a line $L$, and one can check in an explicit example that $L$ is the unique 6–secant to $X$ (see the Singular program in Example 2.2). So this time there are six exceptional lines and going through the adjunction process yields that the second adjoint surface is the Veronese surface in $\mathbb{P}^5$:

$$\begin{array}{ccccc} \mathbb{P}^4 & & \mathbb{P}^8 & & \mathbb{P}^5 \\ \cup & & \cup & & \cup \\ X_9^{10} & \xrightarrow{6} & X_6^{13} & \xrightarrow{12} & X_0^4 \end{array}.$$

So again $X$ is the blow–up of $\mathbb{P}^2$ in 18 possibly infinitesimal near points, but now $H$ is of type

$$H = 8L - \sum_{i=1}^{12} 2E_i - \sum_{i=13}^{18} E_i \ . \qquad \square$$

**Example 9.8.** Let $X$ belong to one of the two families constructed in Example 8.9 . Then

$$d = 10, \ \pi = 8, \ q = p_g = 0,$$

whence $\chi = 1$. So $K^2 = -4$ by the double point formula, the intersection matrix of $X$ is

$$\begin{pmatrix} H^2 & H.K \\ H.K & K^2 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & -4 \end{pmatrix} ,$$

and

$$N_6 = N_6(10, 8, 1) = 6.$$

From the number of sextic generators of the ideals we guess that there are two 6–secants and hence four exceptional lines in the first case, and four 6–secants and hence two exceptional lines in the second case. That this is indeed true can be checked in explicit examples. So the first adjoint surfaces have the intersection matrices

$$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} (H')^2 & H'.K' \\ H'.K' & (K')^2 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & -2 \end{pmatrix} .$$

respectively.

(1) In the first case $K'$ is numerically trivial by the Hodge index theorem (see Hartshorne, 1977). Hence $X_{\min} = X'$ is a minimal Enriques surface since $p_g = q = 0$, and $X$ is an Enriques surface with four points blown up:

$$X = X_{\min}(p_1, \ldots, p_4) \quad \text{and} \quad H = H_{\min} - \sum_{i=1}^{4} E_i \ .$$

(2) In the second case $X$ has negative Kodaira dimension, so $X$ is rational since $q = 0$. The adjunction process yields that $X$ is the blow–up

$$X = \mathbb{P}^2(p_1, \ldots, p_{12})$$

of $\mathbb{P}^2$ in 12 points, embedded by

$$H = 14L - 6E_0 - \sum_{i=1}^{9} 4E_i - 2E_{10} - E_{11} - E_{12} \ . \qquad \square$$

## Exercises

**Exercise 1.** Consider the ideal

$$I = \langle x_1^2 + x_1^3, x_2^2 \rangle \subset K[x_1, x_2] \ .$$

Compute a $K$–basis of $\check{R}/\langle I \rangle_{\check{R}}$ in the cases

(1) $\check{R} = K[x_1, x_2]$ and

(2) $\check{R} = K[x_1, x_2]_{\langle x_1, x_2 \rangle}$.

**Hint.** Use Macaulay's Theorem 1.15 for (1) and show that $\langle I \rangle_{\check{R}} = \langle x_1^2, x_2^2 \rangle_{\check{R}}$ in case (2).

**Exercise 2.** Suppose that there exists an algorithm for computing syzygies. Derive an algorithm for computing the intersection of two ideals $I, J \subset K[x]$.

**Hint.** If $I = \langle f_1, \dots, f_s \rangle$ and $J = \langle g_1, \dots, g_t \rangle$, then compute the first order syzygies of the module generated by the columns of a convenient matrix $M$ involving $f_1, \dots, f_s$ and $g_1, \dots, g_t$:

$$S^2 \xleftarrow{M} S^{s+t+1} .$$

**Exercise 3.** Verify the correctness of the solution to the **radical membership problem** given in Section 4.

**Exercise 4.** Solve the **lifting problem**: given free $K[x_1, \dots, x_n]$–modules $F, G, H$ with finite bases consider a diagram of type

$$\begin{tikzcd} & G \arrow[dl, dashed, "\psi"'] \arrow[d, "\gamma"] \\ H \arrow[r, "\varphi"] & F \end{tikzcd} .$$

Suppose that $\varphi$ and $\gamma$ are given with

$$\operatorname{im} \gamma \subset \operatorname{im} \varphi .$$

Explain how to construct a lift $\psi : G \to H$, that is, a map $\psi$ with $\varphi \circ \psi = \gamma$

**Exercise 5.** Find an algorithm for computing ideal quotients.

**Hint.** As for the intersection of ideals resolve a convenient matrix.

## Computer exercises

**Computer exercise 1.** Study the ring command in Singular. Find out how to create the monomial orders discussed in Section 3.

**Computer exercise 2.** Compute free resolutions of the ideals

$$I = \langle y - x^2, z - x^3 \rangle, \quad J = \langle x, y, z \rangle .$$

How do the syzygies look like ? What is the data type of the syzygies? Is it possible to convert this type into the type matrix ?

**Computer exercise 3.** Write a Singular procedure which checks for two quadratic matrices $A$ and $B$ whether $AB = BA$.

**Computer exercise 4.** Generate 10 homogeneous random polynomials in 5 variables of degree 5 over a finite (!) field. Compute a Gröbner basis of the ideal generated by the 10 polynomials. How many elements of which degree do you get ?

**Computer exercise 5.** Solve Exercise 1 with the help of Singular.

**Computer exercise 6.** Implement the algorithm from Exercise 2 as a Singular procedure.

**Computer exercise 7.** Construct the Veronese surface $X \subset \mathbb{P}^4$

(1) as an embedding

$$\mathbb{P}^2 \hookrightarrow \mathbb{P}^4, \quad p \to (x^2(p) : y^2(p) : z^2(p) : (xy - yz)(p) : (xz - yz(p)) \ .$$

(2) As the degeneracy locus of a generic morphism

$$3\mathcal{O}(-1) \to \Omega^1(1) \ .$$

Check that $X$ is smooth and compute its syzygies.

**Computer exercise 8.** The matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \text{Mat } (4 \times 4, \mathbb{F}_2)$$

generates a matrix group $G$ which acts on $\mathbb{F}_2[x_1, \dots, x_4]$ by linear substitution. Use the Singular library finvar.lib to answer the following questions.

(1) How many elements has $G$ ?

(2) What does a fundamental system of invariants look like ?

(3) Is the invariant ring Cohen–Macaulay ?

**Computer exercise 9.** Construct surfaces in $\mathbb{P}^4$ with resolutions as in Example 8.8:

$$0 \longleftarrow \mathcal{J}_{X_1}(4) \longleftarrow 2\Omega^1(1) \oplus \mathcal{O} \longleftarrow 2\Omega^3(3) \longleftarrow 0$$

and

$$0 \longleftarrow \mathcal{J}_{X_2}(4) \longleftarrow \mathcal{S}yz_1(M) \oplus 2\mathcal{O} \longleftarrow 2\Omega^3(3) \longleftarrow 0 \ .$$

where $M$ is a generic module with a presentation of type

$$0 \leftarrow M \leftarrow 2S(1) \leftarrow 9S \; .$$

**Computer exercise 10.** Recall the following result (see Cox, Little and O'Shea, 1997).

**Theorem.** Let

$$I \subset K[x,y] = K[x_0,\ldots,x_n,\, y_0,\ldots,y_m]$$

be an ideal generated by bihomogeneous polynomials. Define its **projective elimination ideal** by

$$\tilde{I} := (I :< x_0,\ldots,x_n >^{\infty}) \cap K[y] \; .$$

Consider the variety

$$V(I) \subset \mathbb{P}^n \times \mathbb{P}^m$$

and the projection

$$\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m \; .$$

Then

$$\pi(V(I)) = V(\tilde{I}) \; . \qquad \square$$

Construct the Bordiga surface $X$:

$$0 \longleftarrow \mathcal{J}_X(3) \longleftarrow 4\mathcal{O} \xleftarrow{\;\varphi\;} 3\mathcal{O}(-1) \longleftarrow 0 \; .$$

Then $\omega_X(1)$ has resolution of type

$$0 \longleftarrow \omega_X(1) \longleftarrow 3\mathcal{O} \xleftarrow{\;\varphi^t\;} 4\mathcal{O}(-1) \longleftarrow \mathcal{O}(-4) \longleftarrow 0 \; .$$

The sections of $\omega_X(1)$ define the adjunction map

$$\Phi : X \longrightarrow \mathbb{P}^2 \; .$$

graph $\Phi \subset \mathbb{P}^4 \times \mathbb{P}^2$ is given by the equations

$$y_0\varphi_{0j}(x) + y_1\varphi_{1j}(x) + y_2\varphi_{2j}(x) = 0, \qquad j = 0,\ldots,3 \; .$$

Compute three general hyperplane sections of $X$ and the intersection of their images in $\mathbb{P}^2$ under the adjunction map. What do you get ?

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