Reflection groups as Coxeter groups
Euclidian and Hyperbolic reflection groups

Lectures by È. Vinberg

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Lecture 1

Preliminaries

Let X be an n-dimensional *space of constant curvature*, i.e. the Euclidean space E“, the n-dim sphere S^n or L^n – the Lobachevskij space.

1.1 Remark. “Space of constant curvature” $\equiv$ simply connected homogeneous space, the isotropy group (at each point) of which is the group of all orthogonal transformations with respect to some Euclidean metric.

1.2. Let $H \subset X$ be a hyperplane, i.e. H is the set of fixed points of an involution $\sigma \in \operatorname{Isom} X$, and $\operatorname{codim}_X H = 1$. Then $X \setminus H$ is a union of two connected (and convex) subsets of X. So, $X = H^+ \cup H^-$, $H^+ \cap H^- = \varnothing$ and the corresponding involution $\sigma \in \operatorname{Isom} X$ takes H^+ and H^- into each other. H^+ and H^- are called – half-spaces, determined by the hyperplane H. $r_H := \sigma$ is called the reflection with respect to H (or “in H”). So $r_H^2 = \mathrm{id}_X$. H is called the *mirror* of the reflection r_H.

1.3 Remark. More generally, a k-plane in X is a k-dimensional subspace, which is the set of fixed points of an involution in $\operatorname{Isom} X$. 1-planes are called “lines”. $(n-1)$-dimensional planes (the hyperplanes) define reflections (or are determined by reflections) in $\operatorname{Isom} X$.

1.4. Let H_1 and H_2 be two hyperplanes in X, such that $H_1 \cap H_2$ is a $(n-2)$-dimensional plane. Then $\Gamma_{H_1} \circ \Gamma_{H_2}$ is a rotation around $H_1 \cap H_2$ through an angle of 2α, where α is the angle between H_1 and H_2. (There are well defined normal subspaces to a plane at each its point in X; for a k-plane the normal subspace is $(n-k)$-dimensional. The angle between H_1 and H_2 is defined to be equal to π – (the angle between the corresponding 1-dim subspaces).)

It is obvious that for $g \in \operatorname{Isom} X$ and H – a hyperplane in X

$$g r_H g^{-1} = r_{gH}.$$

Discrete groups of motions of X.

1.5. $\Gamma \subset \operatorname{Isom} X$ is called a discrete group of motions, if for all compact subsets $K_1, K_2 \subset X$

$$\#\{\gamma \in \Gamma \mid \gamma K_1 \cap K_2 \neq \varnothing\} < \infty.$$

Properties:
1) any orbit of Γ is discrete in X
2) the stabilizer Γ_x of any $x \in X$ (in Γ) is a finite group

1.6. A subset $\mathcal{D} \subset X$ is called *a fundamental domain* of Γ, if
1) $\mathcal{D} = \overline{\mathcal{D}}^\circ$, where $\mathcal{D}^\circ$ is the interior of $\mathcal{D}$, and $\overline{\mathcal{D}}^\circ$ – its closure
2) $\{\gamma\mathcal{D} \mid \gamma \in \Gamma\}$ is a locally finite covering of X,
3) $\forall \gamma \in \Gamma$, $\gamma \neq e \Rightarrow \gamma\mathcal{D}^\circ \cap \mathcal{D}^\circ = \varnothing$.

1.7 Examples. 1. Let $\mathbf{Z}^n$ act on E^n by the formula

$$\vec{z} \in \mathbf{Z}^n, \quad x \in E^n \Rightarrow \vec{z}(x) := x + \vec{z},$$

then $\mathbf{Z}^n$ is a discrete group of transformation of E^n. A fundamental domain of $\mathbf{Z}^n$: $\mathcal{D} = \{x \in E^n \mid 0 \leqslant x_i \leqslant 1\}$.

2. Let S_n – the symmetric group in n elements – act on E^n by permuting the coordinates in some orthonormal basis of E^n. S_n is finite, hence a discrete group of motions of E^n. A fundamental domain of S_n: $\mathcal{D} = \{(x_1, \dots, x_n) \in E^n \mid x_1 \geqslant \dots \geqslant x_n\}$. Clearly $\{\gamma\mathcal{D} \mid \gamma \in S_n\}$ cover E^n. If furthermore $\mathcal{D}^\circ = \{(x_1, \dots, x_n) \mid x_1 > \dots > x_n\}$, then $\gamma\mathcal{D}^\circ \cap \mathcal{D}^\circ = \varnothing$ provided that $\gamma \neq e$.

1.8. In the sequel we will consider mainly groups which are discrete and generated by reflections ((**d**)iscrete (**r**)eflection (**g**)roups). Let $\Gamma \subset \operatorname{Isom} X$ be such a group. Consider the *mirrors* of all reflections of Γ

$$M = \{H \subset X \text{ – hyperplane } \mid r_H \in \Gamma\}.$$

Properties:

1) M is a locally finite family of hyperplanes. Indeed, let $B(x, a)$ be the ball with centre $x \in X$ and radius a; let $H \in M$. Note that

$$B(x, a) \cap H \neq \varnothing \Leftrightarrow r_H(x) \in B(x, 2a).$$

We may assume that x does not lie in any $H \in M$. Then H is uniquely determined by $r_H(x)$, and since. $B(x, 2a)$ is a compact set,

$$\{r_H(x) \mid H \in M, H \cap B(x, a) \neq \varnothing\} = \{r_H(x) \mid H \in M\} \cap B(x, 2a)$$

is a finite set.

2) M decomposes X into some convex polyhedra, called *chambers* of X.

3) Due to the relation $g r_H g^{-1} = r_{gH}$, we have that Γ permutes the chambers of X.

1.9 Remark. A *convex polyhedron* in X is defined to be an intersection of finitely many half-spaces, having a non-empty interior.

A convex polyhedron in X is called a *Coxeter polyhedron*, if all its dihedral angles are of the form $\frac{\pi}{k}$, $k = 2, 3, \ldots$

1.10 Remark. A dihedral angle of a convex polyhedron is the angle between the hyperplanes in X which contain *adjacent* $(n-1)$-dimensional faces of the polyhedron. Such hyperplanes are called walls of the polyhedron. And "adjacent" means that the $(n-2)$-dim intersection of those hyperplanes contains a $(n-2)$-dim face of the polyhedron.

1.11 Theorem. Any chamber of a discrete reflection group (d.r.g.) Γ is a Coxeter polyhedron.

Proof. Let P be a chamber of some d.r.g. Γ; let H_1 and H_2 be two adjacent walls of P, and set $r_i = r_{H_i}$ $(i = 1, 2)$, $\alpha = \angle H_1 H_2$ – the dihedral angle between H_1 and H_2. The composition $r_2 r_1 r_2 \in \Gamma$ is a reflection in the hyperplane $r_2 H_1$, which therefore contains the $(n-1)$-dimensional face of a chamber of Γ. Moreover it contains $H_1 \cap H_2$ and $\angle H_1(r_2 H_1) = 2\alpha$. Let $H_3 = r_2 H_1$, and $r_3 = r_{H_3}$. Then $H_4 = r_3(H_2)$ is a wall of some chamber, and $\angle H_4 H_1 \equiv 3\alpha \pmod{\pi}$. We can proceed in the same way finding walls $H_5, H_6, \ldots, H_k, \ldots$ such that

$$\angle H_k H_1 \equiv (k-1)\alpha \pmod{2\pi}.$$

If α does not have the type $\frac{\pi}{k}$ for some $k \in \mathbf{Z}$, then there will be a wall of a chamber of Γ which intersects P° – a contradiction. $\square$

1.12 Exercise. Find all Coxeter polygons in E^2.

1. E^2 is a Coxeter polygon in a trivial way, there are no sides at all.
2. Any half-plane is a Coxeter polygon (only one side).
3. Any angle π/k, $k \in \mathbf{N}$, $k \geqslant 2$, is a Coxeter polygon.
4. The strip between two parallel lines is a Coxeter polygon.
5. Any half-strip is a Coxeter polygon.
6. The triangular shapes: Let α, β, γ be the three angles, then

$$\alpha = \frac{\pi}{k}, \quad \beta = \frac{\pi}{\ell}, \quad \gamma = \frac{\pi}{m}, \qquad \frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} = 1 \quad \text{with integer } k, \ell, m \geqslant 2.$$

$$\Rightarrow \begin{array}{c|c|c} k & \ell & m \\ \hline 2 & 3 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 3 \end{array}$$

7. The quadrilateral shapes: Let $\alpha, \beta, \gamma, \delta$ be the four angles, then

$$\alpha = \frac{\pi}{k}, \quad \beta = \frac{\pi}{\ell}, \quad \gamma = \frac{\pi}{m}, \quad \delta = \frac{\pi}{n}, \quad \frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} = 2$$

with integers $k, \ell, m, n \geqslant 2$. Thus

$$2 = \frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} \leqslant 2, \Leftrightarrow \; k = \ell = m = n = 2,$$

and the only possibility is – a rectangular.

8. There are no other Coxeter polygons in E^2; the mean angle should be

$$\frac{n-2}{n}\pi > \frac{\pi}{2} \quad \text{for} \quad \frac{n-2}{n} > \frac{1}{2} \quad (\text{i.e.} \quad n > 4),$$

hence there should be an angle $> \frac{\pi}{2}$!.

1.13 Exercise. Find all Coxeter polygons on S^2.

There are no Coxeter n-gons for $n \geqslant 4$ on S^2: The general fact is that "If P is an n-gon on S^2, $n \geqslant 3$, then the anglesum of P is $> (n-2)\pi$". So if the angles of P are $\alpha_i = \frac{\pi}{m_i}$, $m_i \geqslant 2$ – integer, $i = 1, \ldots, n$, then

$$\frac{n}{2} \geqslant \frac{1}{m_1} + \cdots + \frac{1}{m_n} > n-2,$$

whence $n < 4$. It is easy to see that the only possible Coxeter triangles on S^2 are of the form: $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{m}$, $\gamma = \frac{\pi}{n}$.

	ℓ	m	n	
	2	2	N	where $N \geqslant 2$ – an integer.
$(*)$	2	3	3	(Indeed, $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} > 1$, so at least one of ℓ, m, or n
	2	3	4	is equal to 2, and so on)
	2	3	5	

□

1.14. The cases $(*)$ above are connected with the regular polytopes in E^3. A convex polytope P in E^3 is called *regular* if its isometry group $\Gamma = \mathrm{Isom}(P)$ acts transitively on the flags $\{\text{vertex}\subset\text{edge}\subset\text{face}\}$ of P. Here are some facts about $\mathrm{Isom}(P)$:

- $\mathrm{Isom}(P)$ is generated by the reflections

$$\begin{aligned}
\{\mathbf{v} \subset \mathbf{e} \subset \mathbf{f}\} &\mapsto \{\mathbf{v}' \subset \mathbf{e} \subset \mathbf{f}\},\\
\{\mathbf{v} \subset \mathbf{e} \subset \mathbf{f}\} &\mapsto \{\mathbf{v} \subset \mathbf{e}' \subset \mathbf{f}\},\\
\{\mathbf{v} \subset \mathbf{e} \subset \mathbf{f}\} &\mapsto \{\mathbf{v} \subset \mathbf{e} \subset \mathbf{f}'\}.
\end{aligned}$$

- A fundamental domain of Γ is a cone with vertex – the center of the polytope (this is the only point of P, invariant under $\mathrm{Isom}\,P$), and rays passing through a vertex $\mathbf{v}$, the middle of an edge $\mathbf{e}$, and the center of a face $\mathbf{f}$, such that $\{\mathbf{v} \subset \mathbf{e} \subset \mathbf{f}\}$ is a flag of P.

1.15 Remark. It is easy to see that any face $\mathbf{f}$ of P is a regular polygon, and hence possesses a center.

1.16. Each regular polytope in E^3 is characterized by its Schläfli symbol $\{p, q\}$, where
$q = \#\{\text{edges through a vertex of } P\}$
$p = \#\{\text{edges adjacent to a face}\}$
It is a classical result that all the possibilities for (p, q) are

p	q		
3	3	– tetrahedron	– self dual
4	3	– cube	dual pair
3	4	– octahedron	
5	3	– dodecahedron	dual pair
3	5	– icosahedron	

To each regular polytope P on E^3 one can attach its dual P' with

vertices	–	the centres of the faces of P
edges	–	the segments connecting centres of adjacent faces of P
faces	–	the convex hulls of the centres of faces of P passing through a fixed vertex of P.

The Schläfli symbol of P' is that of P read in the inverse order $\{p, q\}_{P'} = \{q, p\}_P$.

The five regular polytopes in E^3 are unique up to a similarity in E^3. Suppose they are inscribed in $S^2 \subset E^3$. Then the projection (from the centre of S^2) of a fundamental domain of $\Gamma = \mathrm{Isom}(P)$ onto S^2 gives the relation between the regular polytopes and the cases $(*)$ from the previous example:

$(p, q) = (3, 3) \to$ the projection gives $(\ell, m, n) = (2, 3, 3)$
$(p, q) = (4, 3)$ or $(3, 4) \to (\ell, m, n) = (2, 3, 4)$
$(p, q) = (5, 3)$ or $(3, 5) \to (\ell, m, n) = (2, 3, 5)$

1.17 Remark. On S^2 the Coxeter triangles are unique up to an isometry.

1.18 Remark. The notion of a regular polytope in E^3 is easily generalized to the case of E^n (by using flags, etc.)

In E^2 the regular polytopes are the regular n-gons.

In E^3 there are 5 regular polytopes – two dual pairs and one selfdual polytope.

In E^4 there are 6 regular polytopes – two dual pairs and two selfdual polytopes.

In E^n, $n \geqslant 5$, there are three regular polytopes – the simplex, the cube, and the cocube (the dual of the cube).

1.19 Exercise. Find all Coxeter polygons on L^2.

A convex k-gon with angles $\alpha_1, \dots, \alpha_k$ exists in L^2 iff

$$\alpha_1 + \cdots + \alpha_k < \pi(k-2),$$

and depends (up to a motion) on $k-3$ parameters.

This can be easily seen as follows: Let $A_1, \dots, A_k$ be that k-gon. We can choose any point of L^2 to be A_1, and any point A_2 on a line through A_1; then α_2 determines the ray on which we can choose A_3 (arbitrarily), etc. When we choose $A_1, A_2, \dots, A_k, A_{k+1}$ it is needed that $A_{k+1} = A_1$ (two conditions), and $\angle A_1 = \alpha_1$ (one condition). So we have k parameters (when choosing $A_2, \dots, A_{k+1}$ on lines) and 3 conditions ($A_{k+1} = A_1$ and $\angle A_1 = \alpha_1$). □

Note that a triangle on L^2 is uniquely determined by its angles $\alpha_1, \alpha_2, \alpha_3$ provided that $\alpha_1 + \alpha_2 + \alpha_3 < \pi$.

There are infinitely many Coxeter triangles on L^2 – given by the solutions of $\frac{1}{k}+\frac{1}{\ell}+\frac{1}{m} < 1$, $k, \ell, m \geqslant 2$. Here one of α, β, or γ may be 0, which corresponds to k, ℓ, or $m = \infty$.

One famous Coxeter triangle in L^2 is the case

$$(k, \ell, m) = (2, 3, \infty).$$

To this Coxeter polygon corresponds an index two extension of the Klein group $PSL_2(\mathbf{Z})$, which in turn is related to the theory of elliptic curves. In the Poincare model this triangle consists of the points

$$P = \{z \in L^2 \quad | \quad 0 \leqslant \mathrm{Re} z \leqslant \frac{1}{2}, \quad |z| \geqslant 1\}.$$

(In the Poincare model the points of L^2 are the points z of $\mathbf{C}$ with $\mathrm{Im} z > 0$. Lines of L^2 are the half-circles, i.e. the intersections of circles of $\mathbf{C}$ with $\{\mathrm{Im} z > 0\}$, orthogonal to the line $\{\mathrm{Im} z = 0\}$, and the rays, i.e. the intersections of lines in $\mathbf{C}$ with $\{\mathrm{Im} z > 0\}$, orthognal to $\{\mathrm{Im} z = 0\}$. The angles between lines in L^2 are the Euclidean ones.)

So P above is a fundamental domain of a discrete reflection group, which is an index-two extension of $PSL_2(\mathbf{Z})$. And P is a half of a fundamental domain of $PSL_2(\mathbf{Z})$, which gives rise to the moduli space of elliptic curves.

Lecture 2

Generators and defining relations for discrete reflection groups

Let $\Gamma < \mathrm{Isom}\, X$, $X = E^n, S^n$, or L^n be a discrete reflection group. The mirrors of Γ decomposes X into chambers. We have the following

2.1 Theorem. 1. Γ acts simply-transitively on the set of chambers (i.e., for any two chambers P_1, P_2 there exists unique $\gamma \in \Gamma\colon \gamma P_1 = P_2$)

2. Γ is generated by the set S of reflections in the walls of a fixed chamber P of Γ.

3. Defining relations for Γ are $s^2 = e$ $(s \in S)$, and $(s_1 s_2)^{m(s_1,s_2)} = e$ $(s_1, s_2 \in S$, $s_1 \neq s_2)$, where the dihedral angle between the mirrors of s_1 and s_2 is $\pi/m(s_1, s_2)$.

Proof. Let P be a fixed chamber of Γ.

A chain of chambers $P = P_0, P_1, \ldots, P_k$ is a sequence of chambers of Γ such that $\forall i$ P_{i-1} and P_i are adjacent (i.e. they have a common face of codim 1).

To any word $w = s_1 \ldots s_k$, $s_i \in S$, one can associate a chain of chambers: $P_0 = P$, $P_1 = s_1 P, \ldots, P_k = s_1 s_2 \ldots s_k P$. Since P and $s_i P$ are adjacent, then $P_{i-1} = (s_1 \ldots s_{i-1})P$ and $P_i = (s_1, \ldots, s_{i-1}) s_i P$ are adjacents, too.

Conversely, any chain of chambers is obtained in this way – easy.

On the other hand, any chamber Q may be connected with P by a chain. (We connect points $q \in Q^\circ$ and $p \in P^\circ$ by a segment, then modify it in such a way that it do not intersect faces of dimension less than $n-1$: then it's easy).

Let $\overline{\Gamma}$ be the group generated by the set S: $\overline{\Gamma} = \langle S \rangle$. Then $\overline{\Gamma}$ acts transitively on the set of chambers of Γ. We shall prove that $\Gamma \subset \overline{\Gamma}$, so $\Gamma = \overline{\Gamma}$. Let $r \in \Gamma$ be the reflection in a wall of a chamber P'. There exists $\gamma \in \overline{\Gamma}$ s.t. $P' = \gamma P$. Then $s = \gamma^{-1} r \gamma$ is a reflection in a wall of P, so $s \in \overline{\Gamma}$ (in fact, $s \in S$), and $r = \gamma s \gamma^{-1} \in \overline{\Gamma}$. Hence Γ is generated by the elements of S, $\Gamma = \langle S \rangle$, and acts transitively on the set of chambers of Γ.

We'll not prove 1) and 3) here, but give a sketch of a proof. Relations between elements in S correspond to cycles of chambers

$P = P_0, P_1, \ldots, P_k = P$ i.e. $s_1 \cdots s_k = e$ iff $s_1 \cdots s_k P = P$. Now, make a loop around the chambers and contract to a point. Then two kinds of relations occur: Contraction across a $(n-1)$-dimensional face, this corresponds to the relation $s^2 = e$, and contraction across the intersection of several $(n-1)$-dimensional faces, this corresponds to a relation $(s_i s_j)^{m(s_i s_j)} = e$. □

2.2 Corollary. Any chamber of Γ is a fundamental domain for Γ.

2.3 Examples. 1. S_n acts on E^n by permuting the basis vectors of some orthonormal basis $\{e_1, \ldots, e_n\}$: $\sigma(e_i) = e_{\sigma(i)}$, $i = 1, \ldots, n$, $\sigma \in S_n$. Any transposition (i, j) is a reflection in the subspace $\{x_i = x_j\}$. S_n is generated by transpositions, so it is a finite reflection group.

Any chamber of S_n is defined by a particular ordering of the coordinates (e.g., $P\colon x_1 \geqslant \cdots \geqslant x_n$). The walls of P are: $x_1 = x_2, \ldots, x_{n-1} = x_n$. It is well-known that S_n is generated by the transpositions $(1,2)(2,3)\cdots(n-1,n)$, i.e. $S_n = \langle \sigma_i = (i, i+1) \mid i = 1, \ldots, n-1\rangle$.

For the defining relations we have the following.

$\sigma_i^2 = \mathrm{id}$ (trivially); if $|i-j| > 1$, then $(\sigma_i\sigma_j)^2 = \mathrm{id}$ (this being equivalent to $\sigma_i\sigma_j = \sigma_j\sigma_i$ which is obvious); one can check that the angle between the walls $x_i = x_{i+1}$ and $x_{i+1} = x_{i+2}$ is $\pi/3$, hence $(\sigma_i\sigma_{i+1})^3 = \mathrm{id}$ is the defining relation in this case. By the theorem, these are the only relations between the elements $\{\sigma_1, \ldots, \sigma_{n+1}\}$.

2. The extended Klein modular group (cf. the hyperbolic triangle P of the previous lecture)

$$\Gamma = \langle s_1, s_2, s_3 \mid s_i^2 = \mathrm{id}, \quad (s_1s_2)^3 = \mathrm{id}, \quad (s_1s_3)^2 = \mathrm{id}\rangle$$

2.4 Exercise. Prove that the Klein modular group is a free product of $\mathbf{Z}_2$ and $\mathbf{Z}_3$ showing that

$$PSL_2(\mathbf{Z}) = \langle s_1s_3\rangle * \langle s_1s_2\rangle \cong \mathbf{Z}_2 * \mathbf{Z}_3.$$

2.5 Theorem. Any Coxeter polyhedron is a chamber of a discrete reflection group (generated by the reflections in its walls). $\square$

Finite reflection groups

Some problems concerning discrete reflection groups can be reduced to the case of finite reflection groups. The theory of finite reflection groups is very well developed and useful.

Let Γ be a finite reflection group of motions of E^n: $\Gamma \subset \operatorname{Isom} E^n$. Since Γ has a fixed point (the barycentre of any orbit is such a point), we can assume that $\Gamma \subset O_n := \operatorname{Isom}(S^{n-1} \subset E^n)$.

2.5 Remark. It is the same to study finite subgroups of $\operatorname{Isom}(S^{n-1})$ and finite subgroups of $\operatorname{Isom}(E^n)$.

Let P be a fixed chamber of Γ (in E^n). It is a convex polyhedral cone – all the walls are subspaces of E^n.

$$P = \bigcap_{i=1}^{m} H_i^+,$$

where H_i^+ is one of two half-spaces bounded by H_i. More precisely, let $e_i \perp H_i$,

$$(e_i, e_i) = 1 \quad \text{and} \quad H_i^+ = \{x \mid (e_i, x) \geqslant 0\},$$

then

$$P = \{x \in E^n \mid (e_i, x) \geqslant 0,\ i = 1, \ldots, m\}.$$

The dihedral angle $\angle H_iH_j$ is equal to $\pi - \angle e_ie_j$ and has the form

$$\angle H_iH_j = \pi - \angle e_ie_j = \frac{\pi}{m_{ij}}\ , \quad m_{ij} \geqslant 2 \quad \text{an integer}$$

Hence $(e_i, e_j) = -\cos\frac{\pi}{m_{ij}}$. The Gram matrix $G(e_1, \ldots, e_m)$ of the vectors $e_1, \ldots, e_m$ is called the Gram matrix of P and is denoted by $G(P)$.

Hence,

$$G(P) = \begin{pmatrix} 1 & & -\cos\frac{\pi}{m_{ij}} \\ & \ddots & \\ -\cos\frac{\pi}{m_{ij}} & & 1 \end{pmatrix}.$$

Without loss of generality we may assume that $\langle e_1, \ldots, e_m \rangle = E^n$. Otherwise we can reduce the investigation to a subspace of E^n.

We say that a system of vectors $e_1, \ldots, e_m$ in E^n is *decomposable* if it is a union of two orthogonal (non-empty) subsystems. It suffices to consider only the indecomposable systems of vectors.

2.6 Lemma (crucial). Any indecomposable system $e_1, \ldots, e_m \in E^n$ with $(e_i, e_j) \leqslant 0$ for $i \neq j$, is either linearly independent, or subject to a unique linear dependence with positive coefficients. (So, if $\langle e_1, \ldots, e_m \rangle = E^n$, then $m = n$, or $m = n + 1$.)

Proof. Any nontrivial linear relation between $e_1, \ldots, e_m$ can be represented in the form

$$\sum_{i \in I} \lambda_i e_i = \sum_{j \in J} \mu_j e_j,$$

where $I, J \subset \{1, \ldots, m\}$, $I \cap J = \varnothing$, and $\lambda_i, \mu_j > 0$.

Let $e = \sum\limits_{i \in I} \lambda_i e_i (= \sum\limits_{j \in J} \mu_j e_j)$. We have

$$0 \leqslant (e, e) = \sum_{\substack{i \in I \\ j \in J}} \lambda_i \mu_j (e_i, e_j) \leqslant 0 \quad (\text{since } I \cap J = \varnothing) \quad \text{hence} \quad e = 0,$$

$$\text{moreover} \quad (e_i, e_j) = 0 \quad (\text{since} \lambda_i, \mu_j > 0) \quad \forall i \in I,\ j \in J.$$

Let $k \notin I$, then $0 = (e, e_k) = \sum \lambda_i (e_i, e_k) \leqslant 0$ $(\lambda_i > 0,\ (e_1, e_k) \leqslant 0)$. Hence $(e_i, e_k) = 0 \quad \forall i \in I,\ k \notin I$ and $\{e_1, \ldots, e_m\}$ is decomposable unless $I = \{1, \ldots, m\}$.

Obviously, if there are nonproportional linear dependences between $\{e_1, \ldots, e_m\}$ then there is a relation

$$\sum_{i \in I} \lambda_i e_i = \sum_{j \in J} \mu_j e_j \qquad I \cap J = \varnothing,\ I, J \neq \{1, \ldots, m\}$$

Hence, up to proportionality, there may be at most one linear dependence (with positive coeffisients) between $\{e_1, \ldots, e_m\}$. □

2.7 Corollary. A chamber P of a finite reflection group Γ in E^n is a simplicial cone.

Proof. We know already that P is given by

$$\{x \in E^n \mid (e_i, x) \geqslant 0,\ i = 1, \ldots, m\}, \text{ and } (e_i, e_j) \leqslant 0 \text{ for } i \neq j.$$

Suppose $\sum\limits_{i=1}^{m} \lambda_i e_i = 0$, $\lambda_i > 0$, $i = 1, \ldots, m$. Then

$$\forall x \in P \Rightarrow 0 \leqslant \sum \lambda_i (e_i, x) = 0 \Rightarrow (e_i, x) = 0 \quad i = 1, \ldots, m.$$

But $P^\circ \neq \varnothing$, so $(\exists x \in P\colon (e_i, x) > 0\ \forall i)$, a contradiction. Thus $e_1, \ldots, e_m$ are linearly independent, and hence P is simplicial. □

2.8 Definition. A system of vectors $e_1, \ldots, e_n \in E^n$ is called an elliptic Coxeter system, if

1) $(e_i, e_i) = 1$, $(e_i, e_j) = -\cos\frac{\pi}{m_{ij}}$, $m_{ij} = 2, 3, \ldots$, $i \neq j$
2) $\langle e_1, \ldots, e_n \rangle = E^n$.

The problem is to classify all Coxeter systems. This is equivalent to finding all symmetric matrices, which are positive definite and of the form

$$G = \begin{pmatrix} 1 & & -\cos\frac{\pi}{m_{ij}} \\ & \ddots & \\ -\cos\frac{\pi}{m_{ij}} & & 1 \end{pmatrix}.$$

2.9 Coxeter diagrams (elliptic ones). A Coxeter diagram is a graph with vertices $v_1, \ldots, v_n$ and edges with labels, determined as follows (here v_i corresponds to the vector e_i of a Coxeter system $e_1, \ldots, e_n$)

$\angle e_i e_j$	$v_i \quad v_j$	
$\frac{\pi}{2}$	○ ○	
$\frac{2\pi}{3}$	○—○	
$\frac{3\pi}{4}$	○═○	or ○ $\overset{4}{—}$ ○
$\frac{4\pi}{5}$	○≡○	or ○ $\overset{5}{—}$ ○
$\ldots$	$\ldots$	
$\frac{(m-1)\pi}{m}$	○ $\overset{m}{—}$ ○	
$\ldots$	$\ldots$	
π	○▬○	or ○ $\overset{\infty}{—}$ ○

2.10 Remark. A Coxeter system of vectors is indecomposable iff its Coxeter diagram is connected.

2.11 Examples.

<u>$n = 1$</u> ○

<u>$n = 2$</u> ○ $\overset{m}{—}$ ○, which has Gram matrix $\begin{pmatrix} 1 & -\cos\frac{\pi}{m} \\ -\cos\frac{\pi}{m} & 1 \end{pmatrix}$

<u>$n = 3$</u> The corresponding simplicial cone, intersected with the sphere $S^2 \subset E^3$ gives a Coxeter polygon on S^2. We already know all such polygons. They have the following diagrams:

○ ○ $\overset{m}{—}$ ○ – corresponding to $(2, 2, m)$, $m \geqslant 2$
○—○—○ – $(2, 3, 3)$
○—○═○ – $(2, 3, 4)$
○—○≡○ – $(2, 3, 5)$

□

Since any subdiagram of an elliptic Coxeter diagram is an elliptic Coxeter diagram, we get

2.12 Corollary. Any connected elliptic Coxeter diagram with $\geqslant 3$ vertices contains no edges with labels $\geqslant 6$.

2.13 Proposition. Any connected elliptic Coxeter diagram is a tree (i.e. does not contain cycles).

Proof. Suppose an elliptic Coxeter diagram contains a cycle. Consider a minimal such cycle. It is a subdiagram, hence an elliptic Coxeter diagram itself and the corresponding Gram matrix is positive definite. Let it have vertices $v_1, \dots, v_s$ and labels: m_i for $(v_i v_{i+1})$ (where $v_{s+1} = v_1$). Then

$$\sum_{i=1}^{s} x_i^2 - 2\sum_{i=1}^{s} x_i x_{i+1} \cos\frac{\pi}{m_i} > 0$$

is the corresponding quadratic form. Let $(x_1, \dots, x_s) = (1, \dots, 1)$, then $s - 2\sum\limits_{i=1}^{s} \cos\frac{\pi}{m_i} > 0$. But $\cos\frac{\pi}{m_i} \geqslant \frac{1}{2}$, so $s > 2\sum\limits_{i=1}^{x} \cos\frac{\pi}{m_i} \geqslant s$, a contradiction. $\square$

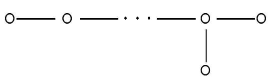
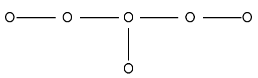
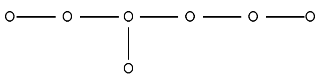
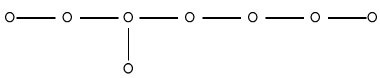
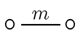
Lecture 3

3.1 Example. S_n acts on $E^n = \langle u_1, \dots, u_n \rangle$ via $\sigma(u_i) = u_{\sigma(i)}$, $(i = 1, \dots, n)$, where $\{u_1, \dots, u_n\}$ is an orthonormal basis of E^n. Our chamber for S_n is $P : \{x \in E^n \mid x_1 \geqslant \dots \geqslant x_n\}$; its walls are given by $x_1 = x_2$, $x_2 = x_3, \dots, x_{n-1} = x_n$. Normal unit vectors to the walls are: $e_1 = \frac{u_1 - u_2}{\sqrt{2}}, \dots, e_{n-1} = \frac{u_{n-1} - u_n}{\sqrt{2}}$ and $P = \{x \in E^n \mid (x, e_i) \geqslant 0,\ i = 1, \dots, n-1\}$.

Obviously, P contains the 1-dim subspace $\langle u_1 + \dots + u_n \rangle$. The (elliptic) Coxeter diagram, corresponding to $\{e_1, \dots, e_{n-1}\}$ is

$$\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\cdots-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ \qquad (n-1 \text{ nodes})$$

3.2 The list of all connected elliptic Coxeter diagrams

A_n		corresponds to S_{n+1} (acting on E^n)
$B_n(=C_n)$		corresponds to $S_n \leftthreetimes \mathbf{Z}_2^n$ acting on the coordinates of E^n by permutations and multiplication by (± 1).
D_n		corresponds to $S_n \leftthreetimes \mathbf{Z}_2^{n-1}$ $(n \geqslant 4)$
E_6		
E_7		
E_8		
F_4		corresponds to $\operatorname{Isom} P$, where P is a 4-dim selfdual regular polytope
$G_2^{(m)}$		corresponds to the dihedrial group D_m
H_3		corresponds to G = Isom (regular dodecahedron) $\simeq A_5 \times \mathbb{Z}_2$
H_4		corresponds to $\operatorname{Isom} P$, where P is a 4-dim regular polytope; the quotient $\operatorname{Isom} P/\{\pm\,\mathrm{id}\}$ is an extension of index 2 of $A_5 \times A_5$.

These Coxeter diagrams correspond to the finite reflection groups in E^n (they determine chambers, and the corresponding group is generated by the reflections in the walls of the corresponding chamber; all the chambers are simplicial cones).

Let now $\Gamma \subset \operatorname{Isom} E^n$ be a general discrete reflection group, and P – a fixed chamber. P is an intersection of half-spaces (for some number m). Let e_i for each $i = 1, \dots, m$ be the defining unit vector for H_i^+: e_i is orthogonal to H_i, and $H_i^+ = \{x \in E^n \mid (x, e_i) \geqslant c_i\}$, then

$$P = \{x \in E^n \mid (x, e_i) \geqslant c_i,\ i = 1, \dots, m\} = \bigcap_{i=1}^{m} H_i^+.$$

The system of vectors $\{e_1, \dots, e_m\}$ has the properties:

$$(e_i, e_i) = 1, \quad (e_i, e_j) = -\cos\frac{\pi}{m_{ij}}, \quad m_{ij} = 2, 3, \dots, \quad \text{where} \quad \angle H_i H_j = \frac{\pi}{m_{ij}}.$$

By the crucial lemma we know that $\{e_1, \dots, e_m\}$ decomposes into a union of indecomposable systems of vectors, each of them is either linearly independent, or subject to a unique linear dependence with positive coefficients.

In the first case the system of vectors defines a simplicial cone, while in the second one – a simplex (in the linear span of the system).

3.3 Direct product of convex polyhedra

Let $P_1 \subset E^{n_1}$, $P_2 \subset E^{n_2}$ be convex polyhedra
The direct product $P_1 \times P_2 = \{(x_1, x_2) \mid x_1 \in P_1, x_2 \in P_2\} \subset E^{n_1} \times E^{n_2} = E^{n_1+n_2}$: has faces $F = F_1 \times F_2$, where F_i is a face of P_i $(i = 1, 2)$.

$$\dim F = \dim F_1 + \dim F_2.$$

Hence, codimension 1 faces have the form $F = F_1 \times P_2$ or $F = P_1 \times F_2$, where F_i is a codimension 1 face of P_i $(i = 1, 2)$. The dihedral angles between the faces of the same type (i.e., either $F_1 \times P_2$ or $P_1 \times F_2$) are the same as in P_1 or P_2 (respectively). The faces of different types are adjacent and orthogonal.

3.4 Theorem. Any Coxeter polyhedron in E^n is a direct product of some number of simplices, a simplicial cone, and a Euclidean space of some dimension. In particular, bounded Coxeter polyhedra are direct products of simplices.

3.5 Parabolic Coxeter systems of vectors in E^n

A system of vectors $e_1, \ldots, e_m \in E^n$ is called a *parabolic Coxeter system* if

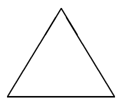
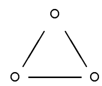
1. $(e_i, e_i) = 1 \quad \forall i,\ (e_i, e_j) = -\cos \frac{\pi}{m_{ij}} \quad \forall i \neq j,\ m_{ij} = 2, 3, \ldots$
2. $\{e_1, \ldots, e_m\}$ decomposes into orthogonal subsystems, each of which is subject to a unique linear dependence with positive coefficients

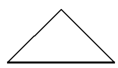
The diagram of a parabolic Coxeter system of vectors of E^n is called a *parabolic Coxeter diagram.*

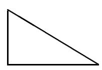
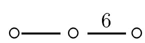
When a system of vectors cannot be decomposed into orthogonal subsystems, it is called *indecomposable* system of vectors.

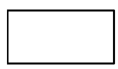
3.6 Examples (of parabolic Coxeter systems and their diagrams).

<u>$n = 1$</u> $\{e_1, e_2\}, (e_1, e_1) = (e_2, e_2) = 1, (e_1, e_2) = -1$ (with $e_1 + e_2 = 0$) $\leftrightarrow$

<u>$n = 2$</u> $\{e_1, e_2, e_3\}$, $m_{ij} = 3 \quad \forall i, j$ $\leftrightarrow$  $\leftrightarrow$ 

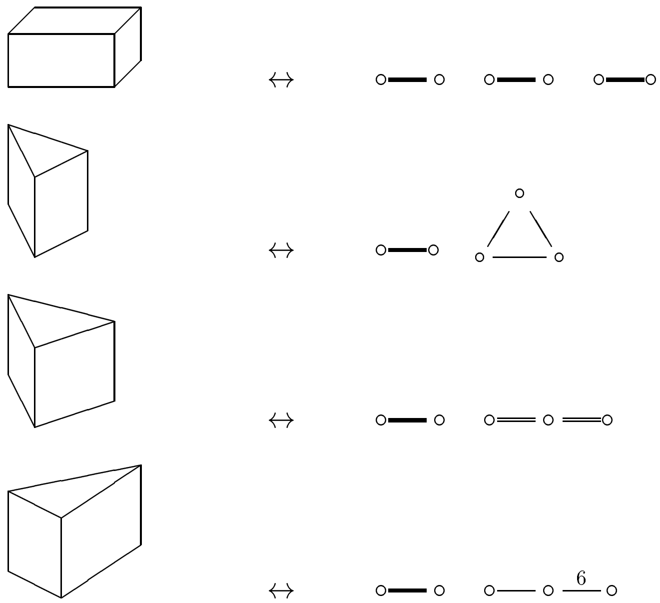
$\{e_1, e_2, e_3\}$, $m_{12} = 2$, $m_{13} = m_{23} = 4$ $\leftrightarrow$  $\leftrightarrow$

$\{e_1, e_2, e_3\}$, $m_{12} = 2$, $m_{13} = 3$, $m_{23} = 6$ $\leftrightarrow$  $\leftrightarrow$ 

$\{e_1, e_2, e_3, e_4\}$, $m_{12} = m_{23} = m_{34} = m_{41} = 2$, $m_{13} = m_{24} = \infty$ $\leftrightarrow$  $\leftrightarrow$

We already know that these are the only Coxeter polygons in E^2. Hence, these are the only parabolic Coxeter systems and diagrams.

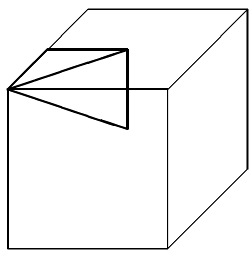
$\underline{n=3}$ We begin with the decomposable ones



An indecomposable one

(see the full list below).

3.7 Exercise. Show that the last diagram is a parabolic one and corresponds to the simplex P – the convex hull of the center of the cube, the center of a face of the cube, the middle of an edge adjacent to the face and the vertex adjacent to the edge.



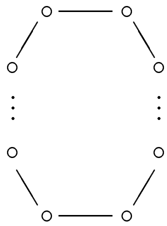
3.8 Classification of indecomposable parabolic Coxeter systems of vectors in E^n

Let $\{e_1, \dots, e_m\}$ be an indecomposable parabolic Coxeter system in E^n. Without loss of generality we may assume $m = n + 1$. By the crucial lemma any subsystem is linearly independent, hence elliptic. From this it follows that the quadratic form (in $\mathbf{R}^m$)

$$\sum_{i,j=1}^{n+1} x_i x_j (e_i, e_j)$$

is positive semidefinite of rank n.

3.9 Proposition. Any parabolic connected Coxeter diagram contains no cycles, except for one case – where it is just one cycles with simple edges

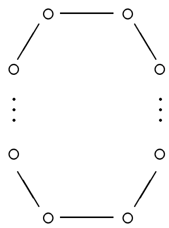


Proof. A proper subdiagram of a parabolic Coxeter diagram is elliptic one. We know all connected elliptic Coxeter diagrams. None of them is a cycle. So, a parabolic Coxeter diagram does not contain any proper cycles. Let the diagram itself be a cycle. Choose the vector $(1,\ldots,1) \in E^{n+1}$ Then $0 \leqslant n+1-2\sum\limits_{i=1}^{n+1} \cos\frac{\pi}{m_i} \leqslant 0$. Since $m_i \geqslant 3$ and $\cos\frac{\pi}{m_i} \geqslant \frac{1}{2}$, all $m_i = 3$ and the edges are simple ones. $\square$

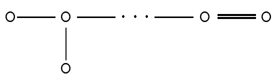
3.10 The list of all connected parabolic Coxeter diagrams

The index n is the dimension of the space spanned by the Coxeter system, each diagram $\widetilde{L}_n$ has $(n+1)$ vertices.

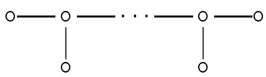
$\widetilde{A}_1$

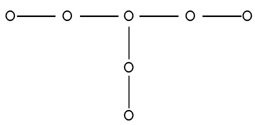


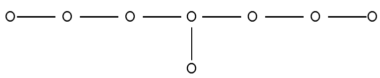
$\widetilde{A}_n$ $n \geqslant 2$

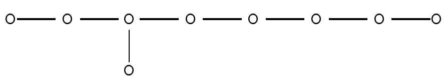
$\widetilde{B}_n$ $(n \geqslant 3)$ 

$\widetilde{C}_n$ $(n \geqslant 2)$

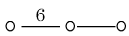
$\widetilde{D}_n$ $(n \geqslant 4)$ 

$\widetilde{E}_6$ 

$\widetilde{E}_7$ 

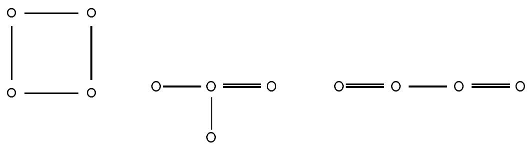
$\widetilde{E}_8$ 

$\widetilde{F}_4$

$\widetilde{G}_2$ 

3.11 Remark. The group Γ of motions of E^n corresponding to the diagram $\widetilde{L}_n$ is a semidirect product of the finite group $\Gamma_0 \subset \mathrm{Isom}(E^n)$ corresponding to the elliptic diagram L_n, and the commutative group generated by parallel translations along vectors orthogonal to the walls of a chamber of the group Γ_0.

3.12 Example. The indecomposable parabolic Coxeter diagrams for $n = 3$ are the following ones



$\widetilde{A}_3$ $\widetilde{B}_3$ $\widetilde{C}_3$ □

3.13 Remark. What we have done so far is to classify all d.r.g. $\Gamma \subset \operatorname{Isom} E^n$ by using elliptic (for finite d.r.g.) and parabolic Coxeter diagrams.

Hyperbolic discrete reflection groups

By definition a convex polyhedron P in L^n is an intersection of finitely many half-spaces, having non-empty interior

$$P = \bigcap_{i=1}^{m} H_i^+, \qquad P^\circ \neq \varnothing$$

A convex polyhedron P is called acute-angled if all its dihedral angles are $\leqslant \frac{\pi}{2}$. The definition of a Coxeter polyhedron in L^n is the same as in the case of E^n.

Still the case of Lobachevskij plane L^2 shows that there are several possibilities for a convex polyhedron: bounded polygons (they have finite area), unbounded polygons of finite area and polygons of infinite area.

The most interesting cases are of polygons with finite area. We already know that the (hyperbolic) polygons with k vertices and given angles depend on $(k-3)$ parameters. So, for $k \geqslant 4$ they are not uniquely defined by their angles.

In L^n, $n \geqslant 3$, there is a very nice theory of acute angled polyhedra. This theory and the theory of Coxeter polyhedra in L^n is much more interesting than that in E^n and S^n, where we know all the Coxeter polyhedra.

3.14 Theorem (Andreev). An (bounded) acute-angled polyhedron in L^n $(n \geqslant 3)$ is uniquely defined (up to a motion in L^n) by its combinatorial structure and dihedral angles. □

A polyhedron (of dimension n) in L^n is called simple, if any vertex belongs to exactly n faces of codimension 1.

3.15 Proposition. Any bounded acute-angled (Coxeter) polyhedron P in L^n is simple.

For any vertex v of P consider the intersection

$$Q := P \cap S(v, \varepsilon)$$

where ε is small enough positive integer. Then Q is an acute-angled polyhedron on $S^{n-1} = S(v, \varepsilon)$, and corresponds to a simplicial cone in E^n; hence is a simplex. □

3.16 Theorem (Andreev). A bounded acute-angled polyhedron in L^3 of a prescribed combinatorial structure, distinct from a triangular prism and tetrahedron, and with prescribed dihedral angles exists if and only if

1. when three faces have a common vertex, then the sum of the angles between them is greater than π,
2. when three faces are pairwise adjacent but don't have a common vertex, then the sum of the angles between them is less than π,
3. when four faces are "cyclically" adjacent, then the sum of the angles between them is less than 2π.

For a triangular prism an extra condition is necessary – at least one of the angles formed by the lateral faces with the top and the bottom of the prism must be different from $\frac{\pi}{2}$. And for a tetrahedron the extra condition is that the determinant of the Gram matrix of the normal vectors of the faces should be negative. □

There is a theorem concerning the acute-angled polyhedra in L^3 with finite volume (due to Andreev too).

Lecture 4

4.1 Examples of acute-angled polyhedra in L^3

1. A rectangular dodecahedron in L^3 does exist: all the conditions of Andreev's theorem are satisfied. It is a Coxeter polyhedron.
2. Consider the polygon (in E^3) obtained by truncating four vertices of the cube chosen so that no two of them have a common edge. The resulting polytope P has four triangular faces and six hexagonal faces. There is no acute-angled polyhedron in L^3 with the combinatiorial structure of P. Indeed, the first condition of Andreev's theorem gives

   $4\pi <$ the total sum of dihedral angles between faces meeting at "old" vertices (i.e. not renewed) of P.

   while the second condition of that theorem gives

   $4\pi >$ the total sum of dihedral angles between faces adjacent to a triangular face of P.

   Both total sums, obviously, coincide, which gives $4\pi < 4\pi$, a contradiction. □

4.2 A few words about the models of S^n and L^n

Fix a scalar product in E^{n+1} given by the formula

$$(x, x) = x_0^2 + \cdots + x_n^2.$$

S^n is given by the equation $(x, x) = 1$; $\operatorname{Isom} S^n = O_{n+1}$.

By definition, k-dimensional planes of S^n are intersections of S^n with $(k+1)$-dimensional subspaces of E^{n+1}. The distance on S^n is defined by the formula

$$x, y \in S^n \Rightarrow \rho(x, y) > 0 \quad \text{such that} \quad \cos(\rho(x, y)) = (x, y).$$

Following the pattern of S^n one can define a model of L^n as follows.

Let $E^{n,1}$ denote the $(n+1)$-dimensional real vector space with a scalar product

$$(x,x) = -x_0^2 + x_1^2 + \cdots + x_n^2.$$

L^n is given by the condition

$$(x,x) = -1, \qquad x_0 > 0$$

4.3 Remark. $\{(x,x) = -1\}$ has two components lying inside the cones

$$\{(x,x) = 0, x_0 > 0\} \quad \text{and} \quad \{(x,x) = 0, x_0 < 0\}.$$

The first cone is called "the cone of future" the second – "the cone of past".

Isom$\{(x,x) = -1\} =: O_{n,1}$ – the pseudoorthogonal group
Isom$(L^n) =: O'_{n,1}$ is a subgroup of index two in $O_{n,1}$.

By definition, k-dimensional subspaces of L^n are intersections of L^n with those $(k+1)$-dimensional subspaces of $E^{n,1}$ that intersect L^n.

4.4 Remark. There are 3 types of $(k+1)$ subspaces of $E^{n,1}$ – the first type don't intersect L^n and are not tangential to $\{(x,x) = 0\}$ (intrinsically, the restriction of $(\ ,\ )$ to them is positive definite); the second type – which are tangential to the cone $\{(x,x) = 0\}$ (intrinsically, the restriction of $(\ ,\ )$ to them is positive semidefinite, but not definite), and the third type – which do intersect L^n (the restriction of $(\ ,\ )$ to them is not definite). First-type subspaces are called – elliptic; second-type subspaces – parabolic, third-type subspaces – hyperbolic ones. □

The distance in L^n is defined by the formula

$$x, y \in L^n, \quad \rho(x,y) > 0 \qquad \text{such that} \quad \cosh(\rho(x,y)) = (x,y)$$

If H is a $(n-1)$-dimensional plane (a hyperplane) in L^n, then there exists naturally defined a normal space (1-dimensional) to H, and it coincides with the normal space to the defining H hyperspace in $E^{n,1}$.

To any convex polyhedron in L^n

$$P = \bigcap_{i=1}^{m} H_j^+$$

one can attach the corresponding unit normal vectors $e_1, \ldots, e_m$, such that $\{e_i\}$ are looking in the same direction, say outside of P (i.e. $H_j^+ = \{x \in L^n \mid (e_i, x) \leqslant 0\}$). So, the dihedral angles $\angle H_i H_j$ of P are equal to $\pi - \angle e_i e_j$.

Coxeter simplices in $X^n = S^n$, L^n, E^n

Let $e_0, e_1, \ldots, e_n$ be the unit normal vectors to the walls of a simplex, looking outside it. It is not hard to see that

1) $\forall i$ the Gram matrix $G(e_0, \ldots, \widehat{e_i}, \ldots, e_n)$ is positive definite
2) For the determinant of the Gram matrix of $e_0, \ldots, e_n$ we have

$$\det G(e_0, \ldots, e_n) \begin{cases} > 0 & \text{if } X^n = S^n \subset E^{n+1} \\ = 0 & \text{if } X^n = E^n \\ < 0 & \text{if } X^n = L^n \subset E^{n,1} \end{cases}$$

4.5 Theorem. Any symmetric matrix $G = (g_{ij})$ with $g_{ii} = 1$, $g_{ij} \leqslant 0$, $i \neq j$, satisfying 1) and 2) above, is the Gram matrix of the unit normal vectors of a simplex in X^n. This simplex is defined uniquely (modulo $\operatorname{Isom} X^n$) in the cases $X^n \in S^n, L^n$, and up to similarity for $X^n = E^n$, by the Gram matrix.

4.6 Bounded Coxeter simplices in L^n. The Coxeter diagrams of bounded Coxeter simplices in L^n are characterized by the properties

1) they are neither elliptic, nor parabolic
2) any proper subdiagram of them is elliptic

n	# bounded Coxeter simplices in L^n
3	9
4	5
$\geqslant 5$	0

4.7 Examples.
1) The diagram $\circ\!-\!\!-\!\circ\overset{5}{-\!\!-}\circ\!-\!\!-\!\circ$ is neither elliptic, nor parabolic, but any subdiagram is elliptic. It gives an example of a simplex in L^3;
2) The same for $\circ\overset{5}{-\!\!-}\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ$ in L^4.

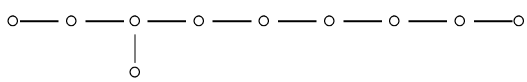
Unbounded Coxeter simplices of finite volume in L^n

The unbounded polyhedra contain vertices "at infinity" – they correspond to rays on the light cone $\{(x, x) = 0, x_0 > 0\}$. The characterizing conditions for the unbounded Coxeter simplexes of finite volume should be modified into the following:

1') their Coxeter diagrams are neither elliptic, nor parabolic;
2') any proper subdiagram is either elliptic, or parabolic.

There are more examples of such simplices (up to dim = 9):

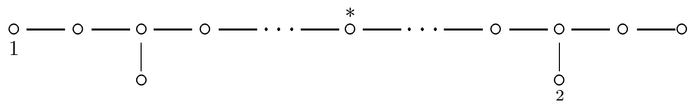
n	# unbounded Coxeter simplices of finite volume
3	23
4	9
5	12
6	3
7	4
8	4
9	3
> 9	0

4.8 Examples.
1) The diagram $\circ\!-\!\!-\!\circ\overset{6}{-\!\!-}\circ\!-\!\!-\!\circ$ in L^3
2)  in L^9 (it might be called – E_{10}).

The simplices (of finite volume) in L^n do not exhaust all the Coxeter polyhedra in L^n.

4.9 Examples.

1) Let



be a graph with 19 vertices corresponding to the vectors $e_1, \ldots, e_{19}$. If we delete the central vertex $*$, we'll get a disjoint union of two parabolic subdiagrams of type $\widetilde{E}_8$. The corresponding Gram matrix has the form,

$$G = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

each of the nonzero blocks of which has rank 8, so the rank of this matrix is 16.

The rank of the Gram matrix $G(e_1, \ldots, e_{19})$ obviously satisfies the inequality

$$\operatorname{rk} G(e_1, \ldots, e_{19}) \leqslant 16 + 2 = 18$$

(since we add to the above matrix one row and one column) therefore

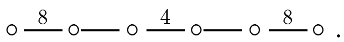
$$\det G(e_1, \ldots, e_{19}) = 0.$$

If we remove the vertices 1 and 2, we'll get the elliptic diagram D_{17}. On the other hand, the whole diagram is not parabolic, so the signature of $G(e_1, \ldots, e_{19})$ is

$$(17,\ 1,\ 1) \quad (+\ -\ 0).$$

By the theorem above, we can realize $e_1, \ldots, e_{19}$ in $E^{17,1}$ and define the cone $C = \{x \in E^{17,1} \mid (x, e_i) \geqslant 0, i = 1, \ldots, 19\}$. One can prove that the convex polyhedron $P = L^{17} \cap C$ is of finite volume, and of combinatorial type of a pyramid over the direct product of two 8-dimensional simplices. Some of its vertices are at infinity (so it is unbounded), for example the apex of P lies there.

2) Consider the diagram



The corresponding quadratic form is neither elliptic, nor parabolic; hence it is indefinite. Surprisingly, the determinant of that form is 0.

(One can easily check that

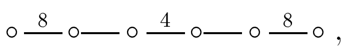
$$\det \begin{pmatrix} 1 & -\cos\frac{\pi}{8} & & & & 0 \\ -\cos\frac{\pi}{8} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} & & \\ & & -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & \\ & 0 & & -\frac{1}{2} & 1 & -\cos\frac{\pi}{8} \\ & & & & -\cos\frac{\pi}{8} & 1 \end{pmatrix} = 0\ )$$

Hence the signature of the quadratic form is

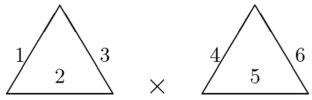
$$(4,\ 1,\ 1) \quad (+\ -\ 0).$$

By the theorem above, there exist vectors $e_1, \ldots, e_6 \in E^{4,1}$ with such a Gram matrix.

Define the cone C to be $\{x \in E^{4,1} \mid (x, e_i) \geqslant 0, i = 1, \ldots, 6\}$ and the convex polyhedron $P = C \cap L^4$. Investigating the proper subdiagrams of



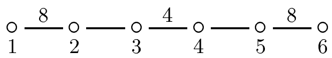
one can prove that P is a bounded polyhedron of combinatorial type of the direct product of two triangles



and the correspondence between the codimension one faces and vertices of the diagram are as follows

$$\{1\} \times \triangle, \{2\} \times \triangle, \{3\} \times \triangle, \triangle \times \{4\}, \triangle \times \{5\}, \triangle \times \{6\}$$

and



,

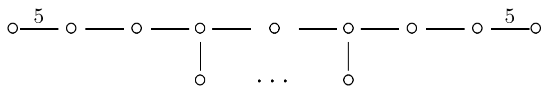
(In particular, P is not a simplex.) $\square$

Some results about the classification of Coxeter polyhedra in L^n

4.10 Theorem. There may exist bounded Coxeter polyhedra in L^n only for $n < 30$.

The proof of this theorem is very complicated and the estimate seem to be rough, there are known such polyhedra only for $n \leqslant 8$.

4.11 Example. The bounded Coxeter polyhedron of the highest known dimension in L^n has Coxeter diagram



(and $n = 8$), where the dotted line means that the corresponding faces are not adjacent.

4.12 Theorem. There may exist Coxeter polyhedra of finite volume in L^n only for $n < 996$.

There are known examples of such polyhedra only for $n \leqslant 21$.

Some words about the application of the theory of d.r.g. to integral quadratic forms

Consider the form $f(x) = -x_0^2 + x_1^2 + \cdots + x_n^2$

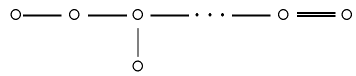
$$\begin{array}{rccccc} & O(f,\mathbf{R}) & \equiv & O(n,1) & \supset & O(f,\mathbf{Z}) \\ & \cup & & \cup & & \cup \\ \operatorname{Isom} L^n = & O'(f,\mathbf{R}) & \equiv & O'(n,1) & \supset & O'(f,\mathbf{Z}) \end{array}$$

and $O'(f,\mathbf{Z})$ is a discrete subgroup of transformations in $\operatorname{Isom} L^n$.

4.13 Remark. The importance of $O'(f,\mathbf{Z})$ stems from the fact that the equation $f(x) = m < 0$ has finitely many solutions in a fundamental domain of $O'(f,\mathbf{Z})$; hence all other solutions can be obtained by the action of $O'(f,\mathbf{Z})$ on the solutions in this domain.

4.14 Theorem. $O'(f, \mathbf{Z})$ contains a reflection subgroup of finite index if and only if $n \leqslant 19$, and the explicit description of the corresponding chambers is obtained.

4.15 Example. For $4 \leqslant n \leqslant 9$ the group $O'(f, \mathbf{Z})$ is a reflection group itself, and has a chamber with Coxeter diagram



Lecture 5

Abstract Coxeter groups

A group Γ with a fixed generating set S is called a Coxeter group, if the defining relations of Γ have the form

$$s^2 = e \quad (\forall s \in S), \quad (s_1 s_2)^{m(s_1,s_2)} = e \quad (s_1, s_2 \in S,\ s_1 \neq s_2)$$

where $m(s_1, s_2) = 2, 3, \dots, \infty$. ("$\infty$" mean that there is no relation between s_1 and s_2.) The numbers $\{m(s_1, s_2) \mid s_i \in S, s_1 \neq s_2\}$ are called the *exponents* of Γ.

5.1 Example. Any d.r.g. Γ is a Coxeter group. The corresponding generating set may be the set of reflections in the walls of some chamber of Γ.

For simplicity we'll assume that S is a finite set:

$$S = \{s_1, \dots, s_n\}, \qquad m_{ij} = m(s_i, s_j) \quad i \neq j.$$

We also assume that $m_{ij} < \infty$.

5.2 Remark. Actually, almost any problem on Coxeter groups can be reduced to the case of Coxeter groups with $|S| < \infty$ and $m_{ij} < \infty$.

The Coxeter group (Γ, S) is called *decomposable*, if $S = S_1 \coprod S_2$ (a disjoint union), $S_1, S_2 \neq \varnothing$, and

$$\forall s_1 \in S_1, s_2 \in S_2 \Rightarrow m(s_1, s_2) = 2.$$
$$(m_{12} = 2 \Leftrightarrow (s_1 s_2)^2 = e \Leftrightarrow s_1 s_2 = s_2 s_1)$$

If (Γ, S) is decomposable with decomposition $S = S_1 \coprod S_2$, then $\Gamma = \Gamma_1 \times \Gamma_2$, where S_i is a generating set for Γ_i, and Γ_i is a Coxeter group $(i = 1, 2)$.

Therefore any Coxeter group can be decomposed into a product of indecomposable groups.

5.3 Theorem. Any Coxeter group (Γ, S) admits a linear representation $\rho\colon \Gamma \to GL(V)$ over $\mathbf{R}$, such that $\forall s_i \in S$, $\rho(s_i) =: R_i$ is a linear reflection w.r.t. some subspace of codim 1 in V, and

$$\operatorname{ord}(R_i \circ R_j) = m_{ij} \qquad \text{for } i \neq j.$$

5.4 Corollary. $s_i \neq e \quad \forall i$ and $\operatorname{ord}(s_i s_j) = m_{ij} \quad \forall i \neq j$.

Proof of the Theorem. $S = \{s_1, \ldots, s_n\}$. Let $V = \langle e_1, \ldots, e_n \rangle$ be n-dimensional real vector space. Define a bilinear form in V:

$$(e_i, e_j) = 1, \quad i = 1, \ldots, n; \qquad (e_i, e_j) = -\cos\frac{\pi}{m_{ij}} \quad (i \neq j).$$

This bilinear form is symmetric, but need not be definite. It can be even degenerate! Nevertheless, for each i there exists a codimension 1 subspace $H_i = e_i^{\perp}$.

Let R_i be the reflection in H_i parallel to e_i: analytically

$$R_i x = x - 2(e_i, x)e_i$$

(indeed, on H_i the reflection R_i acts as the identity map; $R_i : e_i \mapsto -e_i$). Obviously, $R_i^2 = E$, and

$$R_i R_j = \begin{cases} E \text{ on } (e_i, e_j)^{\perp} \\ \text{a rotation through an angle of } \frac{2\pi}{m_{ij}} \text{ in } (e_i, e_j) \end{cases}$$

Note that (e_i, e_j) is the usual Euclidean plane (with a basis of unit vectors $\{e_i, e_j\}$), and R_i, R_j are the usual Euclidean reflections there. Hence $R_i \circ R_j$ is a rotation through an angle of $2(\pi - \angle e_i e_j) = \frac{2\pi}{m_{ij}}$ ($(e_i, e_j) = \cos\angle e_i e_j = -\cos\frac{\pi}{m_{ij}}$) i.e. $\operatorname{ord}(R_i \circ R_j) = m_{ij}$.

The mapping $S_i \mapsto R_i$ defines a representation $\rho\colon \Gamma \to GL(V)$ as required. $\square$

5.5 Theorem (Tits). In the notation of the above theorem consider the cone $P \subset V$, $P = \{x \in V \mid (e_i, x) \geqslant 0,\ i = 1, \ldots, n\}$. ($\Rightarrow \{R_i\}$ are the reflections in the walls of the cone P). Define $P_{\text{fin}} \subset P$ to be the open subset of P for which $x \in P_{\text{fin}} \Leftrightarrow$ the group $\langle R_i \mid (e_i, x) = 0\rangle$ is finite. Then we have the following:

The (Tits) cone $T := \bigcup_{\gamma \in \Gamma} \rho(\gamma) P_{\text{fin}}$ is convex and open in V, and

1) $\rho(\Gamma)$ is a discrete group of transformations of T
2) P_{fin} is a fundamental domain of Γ
3) the representation ρ is faithful.

Moreover, let $X = \mathbf{P}(T) \subset \mathbf{P}(V)$ be the projectivization of T (in $\mathbf{P}(V)$). In some cases X carries a structure of a space of constant curvature, and Γ is realized as a discrete reflection group of X. $\square$

5.6 Remark. All reflection groups we have seen so far can be realized in this way.

5.7 Theorem (Coxeter). A Coxeter group Γ is finite, iff the matrix (the Gram matrix of the vectors $\{e_1, \ldots, e_n\}$ above)

$$G = \begin{pmatrix} 1 & & -\cos\frac{\pi}{m_{ij}} \\ & \ddots & \\ -\cos\frac{\pi}{m_{ij}} & & 1 \end{pmatrix} \qquad \text{is positive definite.}$$

Proof. 1) Let the bilinear form defined by G in V be positive definite. Then the cone P is simplicial (in the Euclidean space V) with dihedral angles $\{\pi/m_{ij}\}$. Hence, the group generated by the reflections in its walls (noted by $\{R_i\}$) is finite and has defining relations (we know this from the previous lectures).

$$R_i^2 = E, \qquad (R_iR_j)^{m_{ij}} = E, \quad i \neq j$$

It follows that the representation ρ from above is an isomorpism of Γ onto this finite group; hence, Γ is finite.

2) Conversely, let $|\Gamma| < \infty$. Then there exists a positive definite scalar product in V which is Γ-invariant; denote it by $(\ ,\ )_0$. Hence, R_i is an orthogonal transformation with respect to $(\ ,\ )_0$. $(R_ix, R_iy)_0 = (x, y)_0$.

R_i is a reflection, hence

$$R_ie_j = e_j - 2\frac{(e_i, e_j)_0}{(e_i, e_i)_0}e_i = e_j - 2(e_i, e_j)e_i$$
$$\Rightarrow (e_i, e_j) = \frac{(e_i, e_j)_0}{(e_i, e_i)_0} \quad \forall i, j$$

We may suppose that $(e_1, e_1)_0 = 1$; $\Rightarrow (e_1, e_i) = (e_1, e_i)_0$, $i = 1, \dots, n$; further, $(e_1, e_i) = \frac{(e_1,e_i)_0}{(e_1,e_1)_0}$ and $(e_1, e_i) \neq 0 \Rightarrow (e_i, e_i)_0 = 1$.

We may assume that (Γ, S) is indecomposable. In this case each k may be joined to 1 with a chain $1 = i_0, i_1, \dots, i_s = k$ such that $(e_{i_{p-1}}, e_{i_p}) \neq 0$ $\forall p = 1, \dots, s$. From this it follows that $\forall i, j \Rightarrow (e_i, e_j) = (e_i, e_j)_0$.

$\Rightarrow (\ ,\ ) = (\ ,\ )_0$, and hence $(\ ,\ )$ is positive definite. $\square$ From the theorem of Coxeter

we get that the finite Coxeter groups are just the finite reflection groups.

A few words about the word problem

Let Γ be a group generated by the (finite) set S. Denote by $W(S)$ the semigroup of words in (the alphabet) S, i.e. the set of sequences (finite) of elements of S and their inverses (written usually as a product), the multiplication is just the union of the two words.

The word problem is how to understand whether two words in the alphabet S represent the same element in Γ? A famous result says that in general there is no algorithm for that. But for the Coxeter groups the problem is solved (due to Tits). Let (Γ, S) be a Coxeter group (with finite S and specific defining relations: $s_i^2 = e$ $\forall i$, $(s_is_j)^{m_{ij}} = e$ $\forall i, j$, $m_{ij} = 2, 3, \dots, \infty$).

The defining relations of Γ determine *elementary reductions of words in S*

1) $w = us_is_iv \mapsto w' = uv$
2) $w = u\underbrace{s_is_js_i\dots}_{h}v \mapsto w' = u\underbrace{s_js_is_j\dots}_{2m_{ij}-h}v$ where $2m_{ij} \geqslant h \geqslant m_{ij}$, $i \neq j$.

In both cases above $w \underset{\Gamma}{=} w'$ (means that $w = w'$ in Γ), and $\ell(w') \leqslant \ell(w)$ ($(=) \Leftrightarrow h = m_{ij}$). $\ell(w) := $ length of w. A word w is called reduced if its length cannot be diminished by means of several consecutive elementary reductions.

5.8 Remark. There are only finitely many words with given length ($|S| < \infty$). So, in finitely many steps we can check if it is possible to reduce the word to one with a smaller length.

Any word can be transformed to a reduced one by a restricted number of elementary reductions.

5.9 Theorem (Tits). Let w_1 and w_2 be two reduced words. Then, $w_1 \underset{\Gamma}{=} w_2 \Leftrightarrow \ell(w_1) = \ell(w_2)$, and each of them can be obtained from the other by means of elementary reductions that do not change the length. □

5.10 Example. We shall apply this last theorem proving that if the Coxeter diagram of (Γ, S) contains a cycle, then $|\Gamma| = \infty$. (We already know the Coxeter diagrams of finite Coxeter groups. None of them contains a cycle – hence the result. But here we'll use the theorem above).

Let $C = (s_1 s_2 \dots s_k)$ be a minimal cycle. We have

$$\begin{aligned} (s_i s_{i+1})^{m_i} &= e \qquad (m_i \geqslant 3) \\ s_i s_j &= s_j s_i \qquad |i-j| > 1. \end{aligned}$$

Consider the word $\gamma = s_1 \dots s_k$.

$$\gamma^m = \underbrace{\underbrace{s_1 \dots s_k}_{\gamma} \underbrace{s_1 \dots s_k}_{\gamma} \dots \underbrace{s_1 \dots s_k}_{\gamma}}_{m \text{ times}} .$$

- γ^m is reduced: any two consecutive letters differ by at least 1 in the indices; there are no blocks of the form $s_i s_j s_i \dots$. So, no elementary reduction can be performed.
- $\gamma^m \neq \gamma^n$ for $m \neq n$; $\ell(\gamma^m) = mk \neq nk = \ell(\gamma^n)$, hence Tits' theorem gives that $\gamma^m \underset{\Gamma}{\neq} \gamma^n$
  $\Rightarrow |\Gamma| = \infty$.

□

Exercises

1. Prove that if any orbit of a group $\Gamma \subset \operatorname{Isom} X$ is discrete then Γ is discrete in the sense of the definition given in the lecture. (For simplicity consider the case $X = E^2$.)

2. Let $\Gamma \subset \operatorname{Isom} X$ be a discrete group and $x_0 \in \Gamma$ a point with trivial stabilizer. Prove that the Dirichlet domain $\{x \in X : \rho(x, x_0) \leqslant \rho(x, \gamma x_0)\ \forall \gamma \in \Gamma\}$ is a fundamental domain for Γ. (For simplicity consider the case $X = E^2$.)

3. For $X = E^n$, prove that $r_{H_1} r_{H_2}$ is a rotation around $H_1 \cap H_2$ if $H_1 \cap H_2 \neq \varnothing$, and a parallel translation along a vector orthogonal to H_1 if $H_1 \| H_2$.

4. Prove the formula $g r_H g^{-1} = r_{gH}$ for any X.

5. In the Poincaré model of L^2, find an explicit formula for the reflection in a hyperbolic straight line $Re z = a$. What is the product of two such reflections?

6*. Prove that the extended Klein modular group is generated by the reflections in the sides of the triangle $0 \leqslant Re z \leqslant \frac{1}{2}$, $|z| \geqslant 1$ (with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{\infty} = 0$).

7. Is the regular tetrahedron in E^3 a Coxeter polyhedron?

8. Making use of generators and defining relations of the extended Klein modular group, prove that the very Klein modular group $PSL_2(\mathbf{Z})$ is the free product of cyclic groups of orders 2 and 3.

9. Let Γ be the group of permutations of basic vectors of E^n with possible multiplications of them by ± 1. Prove that Γ is generated by reflections, find a chamber and draw the Coxeter diagram.

10*. Making use of the fact that the Coxeter diagram of a finite reflection group contains no cycles, prove that the product of all the generating reflections (the so-called Coxeter element) up to conjugacy does not depend on the order of the factors.

11. Prove that the Coxeter diagram of a finite reflection group cannot contain vertices joined with 4 or more other vertices.

12. For which m $\circ\!-\!\!-\circ\overset{m}{-\!\!-}\circ\!-\!\!-\circ$ is the Coxeter diagram of a finite relection group?

13. Is $\begin{array}{c} \circ\!-\!\!-\circ\!-\!\!-\circ\!-\!\!-\circ\!-\!\!-\circ \\ \,|\, \\ \circ \\ \,|\, \\ \circ \end{array}$ the Coxeter diagram of a finite reflection group?

14. Prove that 24 hyperplanes $x_i = 0$, $\pm x_i \pm x_j = 0$ $(i \neq j)$, $\pm x_1 \pm x_2 \pm x_3 \pm x_4 = 0$ in E^4 are the mirrors of a finite reflection group of type F_4.

15. Prove that any two 3-faces of a direct product of two triangles are adjacent.

16*. Prove that the diagram $\circ\overset{8}{-\!\!-}\circ\!-\!\!-\circ\!=\!\!=\circ\!-\!\!-\circ\overset{8}{-\!\!-}\circ$ defines a Coxeter polyhedron in L^4 of combinatorial type of the direct product of two triangles.

17. Prove that in L^3 there are rectangular polyhedra with arbitrarily large number of faces. (Make use of Andreev's theorem.)

18. Find all bounded Coxeter simplices in L^3.

19. Prove that the diagram $\circ\!\boldsymbol{-\!\!-}\!\circ\!-\!\!-\circ\!-\!\!-\circ\!\boldsymbol{-\!\!-}\!\circ$ is the Coxeter diagram of an unbounded Coxeter polyhedron of finite volume in L^3 of combinatorial type of a quadrilateral pyramid.

20. Prove that in a finite Coxeter group, there is a unique element of the maximal length equal to the number of all reflections in the group. Which is the element of the maximal length in S_n?

21. Let Γ be a finite reflection group without non-zero fixed vectors, whose diagram is a connected Coxeter diagram distinct from A_n, D_n, E_6 and $G_2^{(m)}$. Prove that $\Gamma \ni -E$.

22. Prove that the subgroup of a Coxeter group $\Gamma = \langle S \rangle$ generated by a subset $S' \subset S$ is also a Coxeter group (with the same exponents).

23. Making use of the solution of the word problem in Coxeter groups, prove that any Coxeter group, whose diagram contains two or more ramifications, is infinite.