STRUCTURE OF OPERATOR ALGEBRAS ARISING FROM GROUPS AND DYNAMICS

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Mathematics has been crucial to the technological development of the modern society, and pure mathematics is the foundation of almost every science. In the future, there will probably be an even greater demand for more advanced mathematics, that will be used in new and unexpected ways. Any breakthrough usually comes as a consequence of a joint effort by researchers, working over a long time-span on improving the current status of knowledge. Operator algebras contribute to this development by studying methods that has its origin in quantum phenomena, such as noncommutativity, and is essential to progress in quantum information theory and quantum computing. The study of groups and dynamical systems in mathematics has applications and interactions across several fields of science and technology, and the research field of operator algebras, especially in connection with groups and dynamics, is currently very active both.

Introduction

The theory of C^*-algebras can be developed in two different ways, either as certain algebras of bounded operators on Hilbert spaces or as special cases of Banach algebras. The first approach has its origin in quantum mechanics and much of its motivation comes from unitary representations of locally compact groups. In the second, more abstract approach, a C^*-algebra is defined axiomatically as a Banach algebra A together with an involution $A \to A$, $x \mapsto x^*$ such that $\|x^*x\| = \|x\|^2$. In this way, concrete operators on Hilbert spaces are not necessarily involved. Since the basic example of a unital commutative abstract C^*-algebra is $C(X)$, the space of continuous complex-valued functions on some compact Hausdorff space X, the study of C^*-algebras is sometimes referred to as noncommutative topology in the modern language, an aspect of noncommutative geometry.

When X is a compact metric space and φ a homeomorphism $X \to X$, then φ induces an action of the group of integers $\mathbb{Z}$ on $C(X)$ by $n \cdot f(x) = f(\varphi^{-n}(x))$. This concept generalizes to actions of locally compact groups on C^*-algebras. That is, a C^*-dynamical system is a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G, and a continuous homomorphism $\alpha\colon G \to \operatorname{Aut} A$. Associated to a C^*-dynamical system there is a crossed product C^*-algebra $A \rtimes_\alpha G$, and its properties can often be studied in terms of properties of the action α. In fact, there are many crossed product C^*-algebras associated with a C^*-dynamical system, in particular there is a minimal (reduced) and a maximal (full) crossed product. In general, it is not known when the full and reduced crossed products coincide, and recently, there has also been increased interest for "exotic" crossed products lying in between these two.

If the C^*-algebra in question is just $\mathbb{C}$, the associated crossed products are group C^*-algebras. The reduced group C^*-algebra of a discrete group G can be defined as the C^*-subalgebra of $B(\ell^2(G))$ generated by the image of the left regular representation of G on the Hilbert space $\ell^2(G)$, and the full group C^*-algebra can be defined as the universal C^*-algebra generated by a set of unitaries satisfying all the relations coming from the group. In this situation it is known that the full and reduced versions coincide precisely when G is amenable, a notion that was introduced

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by von Neumann, one of the founders of operator algebras. Several of the important, long-standing unsolved problems in the field are related to group theory, e.g., the Baum-Connes and Kadison-Kaplansky conjectures and Connes' embedding problem (the latter also has connections to quantum information theory).

By now, the theory of crossed products of C^*-algebras by groups of automorphisms is a well-developed area. Crossed products are important to provide examples in operator algebras, in particular, this class of algebras is one of the main sources of simple C^*-algebras. Given the importance and success of that theory, it is natural to generalize the situation by allowing a twisted action, which is also required in decomposition results. Many aspects of the extended theory developed are analogous to results of the original classical theory. Nevertheless, there are significant differences. For instance, in the untwisted case, a group C^*-algebra can never be both nuclear and simple (unless the group is trivial), but this can happen in the twisted case, e.g., for noncommutative tori.

Moreover, investigation of semigroups (or usually monoids), often acting by endomorphisms, has proved to be a fruitful direction of study, especially in connection with number theory. Representations of semigroups are by isometries, and one associates various C^*-algebras to a semigroup. These algebras turn out to have very rich structures. Defining and analyzing full semigroup C^*-algebras is a complicated task in general, so often well-behaved situations are considered. In many cases of interest, there is a strong connection between semigroup and group dynamics, where one can form a group action from a semigroup action via a dilation process, and go back using full corners.

Finally, if (A, G, α) is a C^*-dynamical system and G is abelian, then there is a dual action $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes_\alpha G$. Motivated by the goal of extending this result to nonabelian groups, one introduces coactions, so that if G is abelian, then a coaction of G on a C^*-algebra B corresponds to an action of $\widehat{G}$ on B. An action β of $\widehat{G}$ on B may be identified with the map

$$\widetilde{\beta}\colon B \to C_b(\widehat{G}, M(B)) \quad \text{given by} \quad \widetilde{\beta}(b)(\gamma) = \beta_\gamma(b) \quad \text{for} \quad b \in B, \gamma \in \widehat{G}.$$

Since $C_b(\widehat{G}, M(B)) \cong M(B \otimes C_0(\widehat{G})) \cong M(B \otimes C^*(G))$, one defines a coaction of G on B as an injective nondegenerate homomorphism $\delta\colon B \to M(B \otimes C^*(G))$ satisfying certain natural properties. The associated (co-)crossed product of (B, G, δ) is a C^*-algebra $B \rtimes_\delta G$ whose representations are in one-to-one correspondence with the covariant representations of (B, G, δ), which is analogous to the ordinary crossed product coming from an action.

The main research challenges in the field of operator algebras include classification programs for C^*-algebras and making bridges between the theory of operator algebras and other branches of mathematics, for example geometry, topology, and number theory. In this regard, the methods used to study C^*-algebras in connection with dynamical systems and group theory are crucial to shed light upon these problems.

1. Group C^*-algebras

As is common, we consider ideals of C^*-algebras to be closed and two-sided. A C^*-algebra is simple if it contains no proper nontrivial ideals, and prime if any pair of nonzero ideals has nonzero intersection. A C^*-algebra with a faithful irreducible representation is called primitive. Every simple C^*-algebra is primitive, and every primitive C^*-algebra is prime. Conversely, every prime and separable C^*-algebra is primitive by a result of Dixmier, so that the notions of primeness and primitivity are equivalent for separable C^*-algebras. Moreover, it is also known that every prime C^*-algebra has trivial center.

In our study of group C^*-algebras corresponding to discrete group, some times twisted by a two-cocycle, the most essential problems are to find conditions so that these algebras are simple, primitive, prime, have trivial center, have a unique tracial state, or have stable rank one (i.e., the invertible elements are dense).

The full group C^*-algebra $C^*(G)$ is never simple, unless G is trivial (in fact, the group algebra $\mathbb{C}[G]$ is never algebraically simple), and therefore, primitive and prime group C^*-algebras may be considered as the building blocks for the class of full group C^*-algebras.

The problem of determining whether a full group C^*-algebra is primitive seems hard in general (some nontrivial cases are treated in [24, 2, 3]). For example, if $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$ fails to be primitive, then this would solve Connes' embedding problem negatively (see [2, Remark 2.2]; in [24], the author says he suspects that $\mathbb{F}_2 \times \mathbb{F}_2$ does not have a primitive full group C^*-algebra). Therefore, a natural problem to study is the primitivity of $C^*(\mathbb{F}_2 \times \mathbb{F}_2)$.

It is possible for an icc group (i.e., a group with infinite conjugacy classes) to have a full group C^*-algebra with nontrivial center (e.g. icc property (T) groups, cf. [33]), but the following problems are open: Does there exist a group with a full group C^*-algebra that has trivial center, but is not primitive? Is there an operator-theoretic characterization of property (T) that involves the center of $C^*(G)$?

For reduced group C^*-algebras the situation is quite different, and it is well-known [19, 24] that the following four properties are all equivalent: G is icc, $C^*_r(G)$ is prime, $C^*_r(G)$ has trivial center, the group von Neumann algebra $W^*(G)$ is a factor.

Moreover, the reduced group C^*-algebra $C^*_r(G)$ can be simple for nontrivial G (which is unrelated to primitivity of $C^*(G)$), and the question of deciding when the reduced group C^*-algebra $C^*_r(G)$ is simple is intriguing, and has attracted interest from many authors recently.

In [6] it is shown that simplicity is stronger than uniqueness of the trace for $C^*_r(G)$, in fact strictly stronger by [22]. One of the main results of [6] states that uniqueness of the trace for $C^*_r(G)$ is equivalent to G having a trivial amenable radical. Even though certain characterizations are now known [18], it is still generally not easy to check whether a group is C^*-simple. In joint work with Ivanov [13], we explore the theory of radical and residual classes of groups, and show that the class of C^*-simple groups is in fact residual, meaning that it has a "dual" radical class (analogously to amenability for the uniqueness of trace). Moreover, in [13, 7] we study groups acting on trees, and construct concrete examples of amalgamated free products and HNN-extensions that have a reduced group C^*-algebra that is not simple.

Determining C^*-simplicity is very hard in general, as illustrated by the so-called Thompson's group F (it is known to be equivalent with non-amenability). In [13], we introduce the *amenablish radical* of a group. Our definition is rather abstract, and we will look for a more concrete description (maybe by using recurrent subgroups). In this way, we hope to be able to determine C^*-simplicity of groups by computing the amenablish radical.

We say that G is *amenablish* when it coincides with its amenablish radical, meaning that it does not have any nontrivial C^*-simple quotients. Recall that a group G is *inner amenable* if there exists a state m on $\ell^\infty(G)$ such that $m(\delta_e) = 0$ and $m(g \cdot \xi) = m(\xi)$ for all $\xi \in \ell^\infty(G)$ and $g \in G$, where $(g \cdot \xi)(h) = \xi(g^{-1}hg)$. It is known that if G is icc and $W^*(G)$ is not full (or equivalently, has property Γ), then G is inner amenable. Does there exist an icc group that is amenablish, but not inner amenable? Moreover, describing inner amenability for certain classes of groups, e.g., free groups with amalgamation or, more generally, groups acting on trees, would be an interesting task.

Henceforth, let $\sigma\colon G \times G \to \mathbb{T}$ denote a two-cocycle on G. The full twisted group C^*-algebra $C^*(G, \sigma)$ may be simple when G is amenable. For example, it is known that if G is abelian, then $\operatorname{Prim} C^*(G, \sigma)$ is homeomorphic to $\widehat{S_\sigma}$, where $S_\sigma = \{a \in G \mid \sigma(a, b) = \sigma(b, a) \text{ for all } b \in G\}$, and $C^*(G, \sigma)$ is simple if S_σ is trivial.

In [27], we give the following necessary and sufficient condition for primeness of $C^*_r(G, \sigma)$: If $g \in G \setminus \{e\}$ is such that $\sigma(g, h) = \sigma(h, g)$ whenever $gh = hg$, then the conjugacy class of g in G is infinite. In [19] it is shown that this property is also equivalent to the group von Neumann algebra $W^*(G,\sigma)$ being a factor, and therefore generalizes the icc property from the untwisted case. Motivated by [19] we call the property *Kleppner's condition*. In joint work with Bédos [4, 5] we begin the study of simplicity and uniqueness of the trace for reduced twisted group C^*-algebras of discrete groups, and introduce a notion of a *relative Kleppner condition* for a pair (G,N), where N is a normal subgroup of G, taking the normal subgroup structure of G into account. This approach is applied to certain Artin groups in [29]. Further, we aim to characterize simplicity and uniqueness of trace for $C^*(G,\sigma)$ when G is amenable.

A useful observation in this regard is the following: Murphy [25] defines a C^*-algebra A to have the *QTS property* if, for each proper ideal J of A, the quotient A/J admits a trace. It can be deduced from Murphy's work that if A has the QTS property and a faithful trace, then A is simple if and only if it has a unique trace. Moreover, if G is amenable, or if G is exact and $C_r^*(G,\sigma)$ has stable rank one, then $C_r^*(G,\sigma)$ has the QTS property. Also, the vector state associated with δ_e gives a faithful trace on $C_r^*(G,\sigma)$. Hence, if any of the two properties above hold, then (G,σ) is C^*-simple whenever it has the unique trace property. It is not known whether C^*-simplicity of (G,σ) is stronger than uniqueness of trace for any σ, and in contrast to the untwisted case, it is interesting to note that for C^*-algebras with the QTS property, then uniqueness of trace is stronger than simplicity.

If G is amenable, then $C_r^*(G,\sigma) \cong C^*(G,\sigma)$ (but to our knowledge, the converse is unknown for general σ). It is of particular interest to describe the class of amenable groups for which, for any two-cocycle σ, (G,σ) satisfies Kleppner's condition if and only if it is C^*-simple.

A question that has not been investigated so far, is (motivated by Murphy's QTS property) whether C^*-simplicity of (G,σ) implies that $C_r^*(G,\sigma)$ has stable rank one. In all cases where $C_r^*(G,\sigma)$ is known to be simple and the stable rank of $C_r^*(G,\sigma)$ has been computed, it is one. This topic is certainly of interest also when σ is trivial. In the first place, it would be natural to investigate this problem for classes we have good knowledge about, e.g., for groups acting on trees.

There is currently no example of a prime, nonprimitive group C^*-algebra. In fact, there are in general rather few examples of prime, nonprimitive C^*-algebras (the first were given by Weaver [34]), but it still seems likely that one would be able to produce examples coming from groups. A candidate for such an example could be an icc group, whose reduced group C^*-algebra does not have a faithful cyclic representation (which is also the case in [34]).

Moreover, group C^*-algebras, being somewhat concrete, are often good candidates in the search for counterexamples to conjectures. For examples, it is unknown whether every stably finite separable C^*-algebra is an MF (matricial field) algebra, and $C_r^*(G)$ for a property (T) group that is not RFD (residually finite dimensional), is a candidate for an algebra where this fails. Note that $C_r^*(G)$ is always stably finite since it has a faithful trace. Another topic that has not been studied much is when the full group C^*-algebra $C^*(G)$ has a faithful trace, which would mean it is stably finite.

Finally, let G be a group, and set $A = C^*(G)$ and $B = C_r^*(G)$. Some natural questions to ask are: How much does the group C^*-algebras remember of G, and what extra information is required to recover G from A or B? And moreover, up to what type of equivalence (weaker than group isomorphism) can G be recovered? A group G is said to be *C^*-superrigid* if for any other group H with $C_r^*(H) \cong C_r^*(G)$, we have $G \cong H$. Currently, only a few examples of C^*-superrigid groups are known, in particular, certain discrete torsion-free nilpotent groups [11, 30].

What characterizes the research in the area of group C^*-algebras, is that it often requires a wide range of techniques, so one needs to draw both on classical and modern results, both on general theory and on very specific computations. Irreducible representation theory of groups is a research field of its own, and progress in the questions of determining primitivity of $C^*(G)$ are of importance in geometric group theory. Moreover, twisted group C^*-algebras are strongly linked with projective representations of groups, which play various roles in mathematical physics (see [31]). Finally, simple twisted group C^*-algebras with a unique tracial state corresponding to countable discrete amenable groups are precisely of the type involved in the ongoing classification program for separable, simple, nuclear C^*-algebras. Therefore, advances in the twisted group C^*-algebra situation, for example by computation of nuclear dimension (cf. [35]), will lead to further insight in the classification program.

2. Recovering dynamical systems from crossed products

Given a C^*-dynamical system (A, G, α), one defines the crossed products $A \rtimes_\alpha G$ and $A \rtimes_\alpha^r G$. The questions are: How much do these C^*-algebras remember of the action? What extra information do we need in order to recover the action from the crossed product? And what do we mean by recover, that is, what are the various types of equivalences that we can we expect to recover the action with respect to? In general, if we only know that two crossed products are isomorphic, we cannot say much about how the corresponding C^*-dynamical system are related. Finally, we are interested in "rigidity" of actions, that is, situations where weaker and stronger forms of equivalences coincide.

Let us first mention the trivial situation: Let X be a locally compact Hausdorff space and set $A = C_0(X)$. Then we can always recover X up to homeomorphism from A via the famous Gelfand-Naimark theorem. The assignment $X \mapsto C_0(X)$ can also be made into a (contravariant) functor, giving an equivalence between the category of locally compact Hausdorff spaces and the nondegenerate category $\mathbf{C^*_{nd}}$ of C^*-algebras, in which a morphism $\phi\colon A \to B$ is a nondegenerate homomorphism $\phi\colon A \to M(B)$.

Returning to the original general approach, we will normally fix a group G and then just let (A, α) denote the C^*-dynamical system for the action α of G on A. "Crossed-product duality" refers to the problem of determining when a C^*-algebra is a crossed product (up to some equivalence), and then to recover the action from the crossed product together with the dual (co-)action, and sometimes other additional data. The first result in this direction is Takai duality, giving an isomorphism

$$A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} G \xrightarrow{\cong} A \otimes \mathcal{K}(L^2(G))$$

for abelian groups G, taking the double dual action $\widehat{\widehat{\alpha}}$ to $\alpha \otimes \operatorname{Ad}\rho$ (where ρ denotes the right regular representation), that is, recovers the action of an abelian group up to tensoring with the compact operators. This result was later generalized to reduced crossed products by arbitrary locally compact groups, where $\widehat{\alpha}$ then denotes the dual coaction.

Characterizing which C^*-algebras that are isomorphic to a (co-)crossed product by G, and recovery of the action up to conjugacy was first studied by Landstad [20] (thus often called "Landstad duality") for reduced crossed products, then for full crossed products, and then also dual versions for coactions were obtained. In the recent decades crossed-product duality has been put into a categorical framework.

Let us first explain the example based upon the nondegenerate category $\mathbf{C^*_{nd}}$ of C^*-algebras. The nondegenerate category $\mathbf{Ac_{nd}}$ of actions has actions (A, α) of G as objects, and when we say $\psi\colon (A, \alpha) \to (B, \beta)$ is a morphism in the category we mean $\psi\colon A \to B$ is a morphism in $\mathbf{C^*_{nd}}$ that is $\alpha - \beta$ equivariant. Isomorphisms in the category are equivariant C^*-isomorphisms.

Let δ be a coaction of G on a C^*-algebra C, and let $V\colon C^*(G) \to M(C)$ be a $\delta_G - \delta$ equivariant nondegenerate homomorphism, where δ_G denotes the canonical comultiplication on $C^*(G)$. Then we call the triple (C, δ, V) an equivariant coaction. If δ is normal or maximal, we denote the set of elements of $M(C)$ satisfying the so-called "Landstad's conditions" by $C^{\delta,V}$, and we call this the generalized fixed-point algebra of the equivariant coaction (C, δ, V). Further, we write α^V for the action $\operatorname{ad} V$ on $C^{\delta,V}$.

The nondegenerate equivariant category δ_G-$\mathbf{Co_{nd}}$ of coactions has equivariant maximal coactions (C, δ, V) of G as objects, and when we say $\psi\colon (C, \delta, V) \to (D, \epsilon, W)$ is a morphism in the category we mean $\psi\colon C \to D$ is a morphism in $\mathbf{C^*_{nd}}$ that is $\delta - \epsilon$ equivariant and satisfies $W = \psi \circ V$.

The nondegenerate equivariant crossed-product functor $\widetilde{\mathrm{CP}}_{\mathbf{nd}}$ is given on objects by

$$(A, \alpha) \mapsto (A \rtimes_\alpha G, \widehat{\alpha}, i_G),$$

where i_G denotes the canonical embedding of G into $M(A \rtimes_\alpha G)$, and on morphisms by

$$\big(\phi\colon (A, \alpha) \to (B, \beta)\big) \mapsto \big(\phi \rtimes G\colon (A \rtimes_\alpha G, \widehat{\alpha}, i_G) \to (B \rtimes_\beta G, \widehat{\beta}, i_G\big),$$

where we must keep in mind that $\phi \rtimes G$ is to be interpreted as a morphism in the nondegenerate category of C^*-algebras, and we recall that $\phi \rtimes G$ is $\widehat{\alpha} - \widehat{\beta}$ equivariant and takes i_G^α to i_G^β.

The functor $\widetilde{\mathrm{CP}}_{\mathbf{nd}}$ is an equivalence, and it follows that a quasi-inverse of the nondegenerate equivariant crossed-product functor is given by the nondegenerate fixed-point functor $\mathrm{Fix}_{\mathbf{nd}}$, defined on objects by $(C, \delta, V) \mapsto (C^{\delta,V}, \alpha^V)$ and on morphisms so that if $\psi\colon (C, \delta, V) \to (D, \epsilon, W)$ is a morphism in δ_G-$\mathbf{Co_{nd}}$, then $\mathrm{Fix}_{\mathbf{nd}}(\psi)\colon (C^{\delta,V}, \alpha^V) \to (D^{\epsilon,W}, \alpha^W)$ is the unique morphism in $\mathbf{Ac_{nd}}$ such that a certain diagram commutes in δ_G-$\mathbf{Co_{nd}}$. Since we have chosen the object map of $\mathrm{Fix}_{\mathbf{nd}}$ to take an equivariant maximal coaction (C, δ, V) to the C^*-subalgebra $C^{\delta,V}$ of $M(C)$, in our setting the nondegenerate homomorphism $\mathrm{Fix}_{\mathbf{nd}}(\psi)\colon C^{\delta,V} \to M(D^{\epsilon,W})$ is the restriction of (the canonical extension to $M(C)$ of) ψ. Thus, the additional data required to recover the action from the full crossed product $A \rtimes_\alpha G$ consists of the dual coaction $\widehat{\alpha}$ and the canonical homomorphism i_G. This procedure is sometimes called categorical nondegenerate crossed-product duality.

A similar result holds for the "enchilada categories" (introduced and treated thoroughly in [10]) of C^*-algebras with morphisms from A to B being G-equivariant C^*-correspondences, so that isomorphisms become G-equivariant Morita equivalences (i.e., actions and coactions are recovered up to Morita equivalence). Moreover, in both the nondegenerate and enchilada case, we may dualize the results, starting with categories of coactions instead of actions.

In joint work [17] with Kaliszewski and Quigg, the "outer categories" of actions and coactions are introduced, lying in some sense between the nondegenerate and the enchilada, so that two actions (A, α) and (B, β) are isomorphic in the category if they are outer conjugate, i.e., if (B, β) is conjugate to some $(A, \widetilde{\alpha})$ that is exterior equivalent with (A, α).

We briefly recall the terminology. If (B, β) is an action of G, then a β-cocycle is a strictly continuous unitary map $u\colon G \to M(B)$ such that $u_{st} = u_s\beta_s(u_t)$ for all $s, t \in G$. Given a β-cocycle u, the map $s \mapsto \mathrm{Ad}\, u_s \circ \beta_s$ gives an action $\mathrm{Ad}\, u \circ \beta$ on B, which is said to be exterior equivalent to β. An action (A, α) is *outer conjugate* to (B, β) if it is conjugate to $\mathrm{ad}\, u \circ \beta$ for some β-cocycle u. Moreover, given a β-cocycle u, there is an associated isomorphism of $A \rtimes_{\mathrm{Ad}\, u \circ \beta} G$ onto $A \rtimes_\beta G$, denoted by Φ_u.

Now, the outer category $\mathbf{Ac_{ou}}$ of actions has the same objects as $\mathbf{Ac_{nd}}$, but when we say $(\phi, u)\colon (A, \alpha) \to (B, \beta)$ is a morphism in the category we mean u is a β-cocycle and $\phi\colon A \to B$ is a morphism in $\mathbf{C^*_{nd}}$ that is $\alpha - \mathrm{ad}\, u \circ \beta$ equivariant.

The fixed-point equivariant category δ_G-$\mathbf{Co_{ou}}$ of coactions has the same objects as δ_G-$\mathbf{Co_{nd}}$, and in which a morphism $\psi\colon (C, \delta, V) \to (D, \epsilon, W)$ is a morphism $\psi\colon C \to D$ in $\mathbf{C^*_{nd}}$ that is $\delta - \epsilon$ equivariant and satisfies $D^{\epsilon,W} = D^{\epsilon,\psi\circ V}$.

One of the main theorems from [17, Theorem 5.9] states, with the above notation, that the following assignments give a functor $\widetilde{\mathrm{CP}}_{\mathbf{ou}}\colon \mathbf{Ac_{ou}} \to \delta_G$-$\mathbf{Co_{ou}}$ which is a category equivalence:

$$(A, \alpha) \mapsto (A \rtimes_\alpha G, \widehat{\alpha}, i_G^\alpha) \quad \text{and} \quad (\phi, u) \mapsto \Phi_u \circ (\phi \rtimes G)$$

In particular, the crossed-product duality for outer categories gives the following: Let (A, α) and (B, β) be two C^*-dynamical systems (over the same group G), assume that there exists an isomorphism $\Phi\colon A\rtimes_\alpha G\to B\rtimes_\beta G$, and consider the following conditions:

$$\Phi \text{ is } \widehat{\alpha}-\widehat{\beta} \text{ equivariant} \tag{1}$$

$$\Phi(i_A(A))=i_B(B). \tag{2}$$

We show in [17, Section 3] that (A,α) and (B,β) are outer conjugate if and only if there is an isomorphism Φ satisfying both (1) and (2). An important part of our construction is inspired by a result of Pedersen [32, Theorem 35] for abelian groups. We are currently only able to generalize Pedersen's theorem to actions of arbitrary locally compact groups, and not to coactions, meaning that we only get "one-sided" results, see [17, Section 3] (which is an issue we hope to overcome).

Question (Pedersen's rigidity problem, cf. [15, 16])**.** Do there exist non-outer-conjugate actions (A,α) and (B,β) and an isomorphism $\Phi\colon A\rtimes_\alpha G\to B\rtimes_\beta G$ satisfying (1)? In other words, is the condition (2) redundant or not for outer duality?

In light of the above, we call an action (A,α) *Pedersen rigid* (with respect to G) if for every other action (B,β), if the dual coactions $(A\rtimes_\alpha G,\widehat{\alpha})$ and $(B\rtimes_\beta G,\widehat{\beta})$ are conjugate, then (A,α) and (B,β) are outer conjugate. If G is discrete, then every action is Pedersen rigid (with respect to G). However, even when G is abelian and non-discrete, the problem seems delicate.

Let (C,δ) be a coaction and $V\colon C^*(G)\to M(C)$ an equivariant homomorphism. A related question is whether the generalized fixed-point algebra of (C,δ,V) *only* depends on C and δ? There is currently no examples of V,W such that $C^{\gamma,V}\neq C^{\gamma,W}$. In other words, it is unclear whether V carries any information, although it is necessary for the construction. If (A,α) is an action such that $(A\rtimes_\alpha G,\widehat{\alpha})$ has a unique generalized fixed-point algebra, then (A,α) is Pedersen rigid.

Moreover, we say that a class of actions is rigid if whenever (A,α) and (B,β) are any two actions belonging to this class such that $(A\rtimes_\alpha G,\widehat{\alpha})$ and $(B\rtimes_\beta G,\widehat{\beta})$ are conjugate, then (A,α) and (B,β) are outer conjugate. For example, we show in [15, 16] that for any locally compact group, the class of all actions on stable C^*-algebras is a rigid class.

On a different note, a result by Xin Li [23, Theorem 1.2] from topological dynamics (for topologically free systems of discrete groups) states that (X,G,α) and (Y,H,β) are continuous orbit equivalent if and only if there exists an isomorphism $\Phi\colon C_0(X)\rtimes^r_\alpha G\to C_0(Y)\rtimes^r_\beta H$ such that $\Phi(C_0(X))=C_0(Y)$, if and only if the transformation groupoids $X\rtimes G$ and $Y\rtimes H$ are isomorphic as topological groupoids. Inspired by this, given a discrete group G, we might say that two actions (A,α) and (B,β) are "noncommutative orbit equivalent" if there is an isomorphism Φ satisfying (2). Then, we wish to come up with a type of equivalence $\sim$ between (A,α) and (B,β) that does not use the crossed product construction, such that $(A,\alpha)\sim(B,\beta)$ if and only if they are "noncommutative orbit equivalent". Li's result depends on a notion of Cartan pairs studied by Renault, and our plan is to use a noncommutative version of Cartan subalgebras, probably similar to the one construction introduced by Exel [12, Section 12], in our investigation.

The Pedersen rigidity problem has an analog for quantum groups, and also in this setting, we study the various types of equivalences of (B,Δ)-actions, in particular conjugacy, outer equivalence, and Morita equivalence. For a compact quantum group (B,Δ), one defines the right action of (B,Δ) on a C^*-algebra C via a map $\gamma\colon C\to M(C\otimes B)$, satisfying certain natural (coaction) properties, and we say that (C,γ) is a (B,Δ)-action. Note that if (C,γ) and (D,δ) are right actions of (B,Δ), a nondegenerate homomorphism $\varphi\colon C\to M(B)$ is $\gamma-\delta$ equivariant if $(\varphi\otimes\mathrm{id})\circ\gamma=\delta\circ\varphi$, and such a map induces a nondegenerate homomorphism between appropriate crossed products. Moreover, let (C,γ) be an action of (B,Δ). A γ-cocycle is a unitary element $U\in M(C\otimes B)$ such that $(\mathrm{id}\otimes\Delta)(U)=(U\otimes 1)(\gamma\otimes\mathrm{id})(U)$. Given a γ-cocycle, $\mathrm{Ad}\,U\circ\gamma$ is an action on B which is said to be exterior equivalent to γ. A coaction (D,δ) is outer conjugate to (C,γ). if it is conjugate to $\mathrm{Ad}\,U\circ\gamma$ for some γ-cocycle U.

For the case of Morita equivalence, one would work with C^*-correspondences, and in all three cases, the goal is to find what extra information is needed, in addition to the crossed product, so that the action can be recovered up to the corresponding equivalence.

Moreover, we intend to formulate inverting the formation of crossed products within a category-theoretical framework, using techniques from [17]. In fact, when searching for weaker forms of equivalences between actions, we often rather search for categories of actions, so that isomorphism in the category serves as our equivalence.

Classification of group actions on topological spaces, or more generally on operator algebras, up to various types of equivalences, is one of the fundamental challenges in the field of dynamical systems. Thus, advances in this direction would hopefully give insight in rigidity theory for C^*-dynamical systems, a topic which is not yet much studied in the noncommutative case. This would again have applications in the already existing theories for topological and measurable dynamics.

Application: C^*-algebras arising from number theory

In this final section, related to both of the above topics, we give an example on how knowledge of group theory and dynamical systems can be applied. Recently, there has been much work in operator algebras in connection with algebraic number theory, in particular of Kirchberg algebras arising from number fields, or even more general rings. The origin is [8], where Cuntz introduces the C^*-algebra $\mathcal{Q}_\mathbb{N}$ associated with the $ax+b$-semigroup over the natural numbers, that is $\mathbb{Z} \rtimes \mathbb{N}^\times$, where $\mathbb{N}^\times$ acts on $\mathbb{Z}$ by multiplication. In [21] Larsen and Li define the 2-adic ring algebra of the integers $\mathcal{Q}_2$, attached to the semigroup $\mathbb{Z} \rtimes \langle 2 \rangle^+$, where $\langle 2 \rangle^+ = \{2^i : i \geq 0\} \subset \mathbb{N}^\times$ acts on $\mathbb{Z}$ by multiplication.

In joint works with Kaliszewski, Quigg [14] and Barlak, Stammeier [1] we consider algebras $\mathcal{Q}_S$ attached to the semigroup $\mathbb{Z} \rtimes \langle S \rangle^+$, where S denotes a nonempty set of mutually relatively prime natural numbers. This class of C^*-algebras generalizes the ones from Cuntz, Li, and Larsen. The algebras have several nice properties, in particular they are UCT Kirchberg algebras. In [1] we make a big step towards classifying the algebras by computing their K-theory. The strategy is the following: First, let $\Omega = \prod'_{p \in \mathcal{P}} \mathbb{Q}_p$, where $\mathcal{P}$ is the set of all primes that divide some element of S. The stabilization of $\mathcal{Q}_S$ is isomorphic to $C_0(\Omega) \rtimes N \rtimes \langle S \rangle$, where $N = \mathbb{Z}[\{\frac{1}{p} : p \in S\}] \subseteq \mathbb{Q}$, and $\langle S \rangle$ denotes the multiplicative subgroup of $\mathbb{Q}^\times$. Then we use a duality result from [14] to obtain a Morita equivalence between $C_0(\Omega) \rtimes N \rtimes \langle S \rangle$ and $C_0(\mathbb{R}) \rtimes N \rtimes \langle S \rangle$.

The advantage of working with the real numbers is that it makes homotopy arguments possible. Since $\langle S \rangle$ is the free abelian group on S, it is isomorphic to $\mathbb{Z}^{|S|}$. However, the K-theory of C^*-dynamical systems with actions of $\mathbb{Z}^n$ is in general a subtle topic, and standard techniques do not apply. In [1] we were able to compute the torsion-free part of the K-groups, but in many cases there is torsion in both K_0 and K_1. Interestingly, it seems like we can identify a subalgebra of $\mathcal{Q}_S$ that is responsible for this torsion subgroup. This "torsion subalgebra" looks intriguing itself, e.g., it always admits a description as a higher rank graph. In other words, there are more work to do in this direction before a complete classification is available, and interestingly, there are many of the fundamental algebras in the theory that plays a role in this construction.

When $\mathcal{P}$ consists of all primes, then Ω coincides with the finite adeles $\mathcal{A}_f$. The Bost-Connes algebra $\mathcal{C}_\mathbb{Q}$ is isomorphic to the semigroup crossed product $C(\prod_{p \in \mathcal{P}} \mathbb{Z}_p) \rtimes \mathbb{N}^\times$, which dilates to $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^\times$, and the latter has $\mathcal{C}_\mathbb{Q}$ as a full corner. The stabilization of $\mathcal{Q}_\mathbb{N}$ is isomorphic to $C_0(\mathcal{A}_f) \rtimes \mathbb{Q} \rtimes \mathbb{Q}_+^\times$ and $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^\times$ sits inside in a natural way. Similarly, for general S, the stabilization of $\mathcal{Q}_S$, $C_0(\Omega) \rtimes N \rtimes \langle S \rangle$ contains the "Bost-Connes type" algebra $C_0(\Omega) \rtimes \langle S \rangle$ as a subalgebra. In future work, we wish to study this class of algebras, and in particular, initiate a K-theory computation, using the approach of adding one generator of $\mathbb{Z}^{|S|}$ at a time, which has similarities to the method indicated above. However, this situation is much more complex, since we have no duality theorem that provide a passage to real dynamics.

There is another class of algebras, related to [9], arising from a multiplicative subgroup of $\mathbb{Q}^\times$ generated by elements that are non-integers. It is not yet clear how to analyze this situation, as it seems to require a new construction of a C^*-algebra from a "generalized" partial C^*-dynamical system, that differs from the partial crossed product.

Moreover, we also aim to study a more general setup, where so-called "rings of S-integers" in a global field (or an $\mathcal{A}$-field) are considered. When the class number is 1, things work out much like for $\mathbb{Q}$, but otherwise it gets more complicated.

Global fields and their completions, local fields, are precisely the objects of interest from the viewpoint of algebraic number theory, so this is another good opportunity to see the interplay between operator algebras, groups, dynamics, and another branch of mathematics.

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