A note on ℓ^p spaces

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In this note we define the ℓ^p spaces (pronounced "ell-pee", or sometimes "little ell-pee", to distinguish it from L^p) and list some of their properties. The starting point is real-valued sequences $\{a_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}$, which we can think of as "infinite-dimensional vectors" $(a_1, a_2, a_3, \dots) \in \mathbb{R}^\infty$. The space $\mathbb{R}^\infty$ is too big to have an interesting structure, so instead we study smaller (but still infinite-dimensional) subspaces, namely the ℓ^p spaces.

Notation. In this note we will think of a sequence $\{a_i\}_{i\in\mathbb{N}}$ in $\mathbb{K}$ (where $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$) as the function

$$a\colon \mathbb{N} \to \mathbb{K}, \qquad a(i) = a_i.$$

It should be clear that each sequence $\{a_i\}_{i\in\mathbb{N}}$ in $\mathbb{K}$ gives rise to one and only one such function $a\colon \mathbb{N} \to \mathbb{K}$, and *vice versa.* We will therefore refer to a function $a\colon \mathbb{N} \to \mathbb{K}$ as a *sequence (in $\mathbb{K}$).* The reason for viewing sequences as functions becomes apparent when we talk about sequences of sequences of numbers.

We first state the finite-dimensional variant of ℓ^p.

Definition 1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p \in [1,\infty]$. For $n \in \mathbb{N}$ we define the p-norm of a vector $u \in \mathbb{K}^n$ as

$$\|u\|_p = \begin{cases} \left(\sum_{i=1}^n |u_i|^p\right)^{1/p} & \text{if } p < \infty \\ \max_{i=1,\dots,n} |u_i| & \text{if } p = \infty. \end{cases}$$

Definition 2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $p \in [1,\infty]$. For a sequence $a\colon \mathbb{N} \to \mathbb{K}$ we define

$$\|a\|_{\ell^p} = \begin{cases} \left(\sum_{i\in\mathbb{N}} |a(i)|^p\right)^{1/p} & \text{if } p < \infty \\ \|a\|_{\ell^\infty} = \sup_{i\in\mathbb{N}} |a(i)| & \text{if } p = \infty \end{cases}$$

and we set

$$\ell^p(\mathbb{K}) = \left\{a\colon \mathbb{N} \to \mathbb{K} \ : \ \|a\|_{\ell^p} < \infty\right\}.$$

Example 3. If $a(i) = \frac{1}{i}$ then $\sum_{i\in\mathbb{N}} |a(i)|^p = \sum_{i\in\mathbb{N}} \frac{1}{i^p}$, which is finite only when $p > 1$. Clearly, $|a(i)| \leqslant 1$ for all i, so $\|a\|_{\ell^\infty} < \infty$. Hence, $a \in \ell^p(\mathbb{R})$ for all $p \in (1,\infty]$, but not for $p = 1$. Similarly, if $b(i) = \frac{1}{\sqrt{i}}$ then $\sum_{i\in\mathbb{N}} |b(i)|^p = \sum_{i\in\mathbb{N}} \frac{1}{i^{p/2}}$, which is finite only when $p > 2$.

Remark 4. It is not hard to see that $\ell^p(\mathbb{K}) \subset \ell^\infty(\mathbb{K})$ for every $p \in [1,\infty)$, but not *vice versa.* Indeed, if $a \in \ell^p(\mathbb{K})$ then $\sum_{i=1}^\infty |a(i)|^p < \infty$, so in particular $|a(i)|^p \to 0$ as $i \to \infty$, which is equivalent to $|a(i)| \to 0$ as $i \to \infty$. Thus, a is a sequence converging to 0, so it must be bounded. To see that the converse inclusion is not true it is enough to observe that $a = (1, 1, \dots) \in \ell^\infty(\mathbb{K})$ but $a \notin \ell^p(\mathbb{K})$ for every $p < \infty$.

In order to show that $\ell^p(\mathbb{K})$ is a normed vector space we first prove *Hölder's inequality* For some $p \in [1,\infty]$ we define its *conjugate exponent* $q \in [1,\infty]$ by

$$q = \begin{cases} \frac{p}{p-1} & \text{if } p < \infty \\ 1 & \text{if } p = \infty. \end{cases}$$

Note that

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{1}$$

and that q is the only element of $[1,\infty]$ satisfying this identity. (Here, we write $1/\infty = 0$.) Note also that $p = 2$ is the only exponent which is its own conjugate.

Theorem 5 (Hölder's inequality, finite-dimensional version)**.** *Let $p \in [1,\infty]$ and let q be its conjugate exponent. For any $n \in \mathbb{N}$ we have*

$$\|uv\|_1 \leqslant \|u\|_p \|v\|_q \qquad \forall\ u, v \in \mathbb{K}^n \tag{2}$$

where $uv = (u_1v_1, \dots, u_nv_n) \in \mathbb{K}^n$.

Proof. If $u = 0$ or $v = 0$ then (2) follows immediately, so we may assume $u, v \neq 0$. If $p = 1$ then $q = \infty$, and

$$\|uv\|_1 = \sum_{i=1}^n |u_iv_i| \leqslant \max_{j=1,\dots,n} |v_j| \sum_{i=1}^n |u_i| = \|u\|_1\|v\|_\infty.$$

The same argument applies to the case $p = \infty$, $q = 1$. Last, in the case $p, q \in (1,\infty)$ we first recall Young's inequality: If $s, t \geqslant 0$ then

$$st \leqslant \frac{s^p}{p} + \frac{t^q}{q}. \tag{3}$$

We get

$$\begin{aligned} \frac{\|uv\|_1}{\|u\|_p\|v\|_q} &= \frac{1}{\|u\|_p\|v\|_q} \sum_{i=1}^n |u_iv_i| = \sum_{i=1}^n \frac{|u_i|}{\|u\|_p} \frac{|v_i|}{\|v\|_q} \\ &\leqslant \sum_{i=1}^n \frac{|u_i|^p}{p\|u\|_p^p} + \frac{|v_i|^q}{q\|v\|_q^q} && (\textit{by Young's inequality}) \\ &= \frac{\sum_{i=1}^n |u_i|^p}{p\|u\|_p^p} + \frac{\sum_{i=1}^n |v_i|^q}{q\|v\|_q^q} \\ &= \frac{\|u\|_p^p}{p\|u\|_p^p} + \frac{\|v\|_q^q}{q\|v\|_q^q} = \frac{1}{p} + \frac{1}{q} \\ &= 1 && (\textit{by } (1)). \end{aligned}$$

Now multiply the above by $\|u\|_p\|v\|_q$ to get (2). □

Theorem 6 (Hölder's inequality, infinite-dimensional version)**.** *Let $p \in [1,\infty]$ and let q be its conjugate exponent. Then*

$$\|ab\|_{\ell^1} \leqslant \|a\|_{\ell^p}\|b\|_{\ell^q} \qquad \forall\ a \in \ell^p(\mathbb{K}),\ b \in \ell^q(\mathbb{K}), \tag{4}$$

where ab is the sequence $(ab)(i) = a(i)b(i)$.

Proof. For an arbitrary $n \in \mathbb{N}$, apply (2) to the vectors $u = (a(1), \ldots, a(n))$ and $v = (b(1), \ldots, b(n))$ to get

$$\sum_{i=1}^{n} |a(i)b(i)| \leqslant \|u\|_p \|v\|_q.$$

From the fact that $\|u\|_p = \big(\sum_{i=1}^{n} |a(i)|^p\big)^{1/p} \leqslant \big(\sum_{i=1}^{\infty} |a(i)|^p\big)^{1/p} = \|a\|_{\ell^p}$, and likewise for v, we get

$$\sum_{i=1}^{n} |a(i)b(i)| \leqslant \|a\|_{\ell^p} \|b\|_{\ell^q}.$$

Taking the limit $n \to \infty$ now yields (4). □

Remark 7. Hölder's inequality is useful in many applications. For instance, Hölder's inequality implies that if a sequence a lies in both ℓ^{p_1} and ℓ^{p_2} for some $p_1, p_2 \in [1, \infty]$, then it also lies in every space "in between". (Such a result is sometimes called an *interpolation result.*) Indeed, assume that, say, $p_1 < p_2 < \infty$, and let $p \in (p_1, p_2)$. Then there is some $\alpha \in (0, 1)$ such that $p = \alpha p_1 + (1-\alpha)p_2$. Apply Hölder's inequality with exponent $\frac{1}{\alpha}$ (whose conjugate exponent is $\frac{1}{1-\alpha}$) to get

$$\begin{aligned}
\sum_{i=1}^{\infty} |a(i)|^p &= \sum_{i=1}^{\infty} |a(i)|^{\alpha p_1} |a(i)|^{(1-\alpha)p_2} \\
&\leqslant \left(\sum_{i=1}^{\infty} \big(|a(i)|^{\alpha p_1}\big)^{1/\alpha}\right)^{1/(1/\alpha)} \left(\sum_{i=1}^{\infty} \big(|a(i)|^{(1-\alpha)p_2}\big)^{1/(1-\alpha)}\right)^{1/(1/(1-\alpha))} \\
&= \left(\sum_{i=1}^{\infty} |a(i)|^{p_1}\right)^{\alpha} \left(\sum_{i=1}^{\infty} |a(i)|^{p_2}\right)^{1-\alpha} \\
&= \|a\|_{\ell^{p_1}}^{\alpha p_1} \|a\|_{\ell^{p_2}}^{(1-\alpha)p_2} < \infty.
\end{aligned}$$

The case $p_2 = \infty$ is easier:

$$\sum_{i=1}^{\infty} |a(i)|^p = \sum_{i=1}^{\infty} |a(i)|^{p_1} |a(i)|^{p-p_1} \leqslant \|a\|_{\ell^\infty}^{p-p_1} \sum_{i=1}^{\infty} |a(i)|^{p_1} < \infty.$$

Remark 8. Combining Remarks 4 and 7, we see that if $a \in \ell^p(\mathbb{K})$ then also $a \in \ell^r(\mathbb{K})$ for all $r \in [p, \infty]$. (Exercise: Show by example that $a \in \ell^p(\mathbb{K})$ does not necessarily imply $a \in \ell^r(\mathbb{K})$ for $r < p$.)

In order to show that $\|\cdot\|_{\ell^p}$ is a norm, we first show that its finite-dimensional version $\|\cdot\|_p$ is a norm.

Theorem 9. *For every $p \in [1, \infty]$ and $n \in \mathbb{N}$, the function $\|\cdot\|_p$ is a norm on $\mathbb{K}^n$.*

Proof. It is clear that $\|u\|_p > 0$ for all $u \neq 0$, and that $\|u\|_p = 0$ implies $u = 0$. Let $u \in \mathbb{K}^n$ and $\alpha \in \mathbb{K}$. If $p < \infty$ then $\|\alpha u\|_p = \big(\sum_{i=1}^{n} |\alpha u_i|^p\big)^{1/p} = |\alpha|\big(\sum_{i=1}^{n} |u_i|^p\big)^{1/p} = |\alpha|\|u\|_p$. For $p = \infty$ we have $\|\alpha u\|_\infty = \max_{i=1,\ldots,n} |\alpha u_i| = |\alpha| \max_{i=1,\ldots,n} |u_i| = |\alpha|\|u\|_\infty$.

Last, we show the triangle inequality. If $u, v \in \mathbb{K}^n$ and $p = 1$ then

$$\|u+v\|_1 = \sum_{i=1}^n |u_i + v_i| \leqslant \sum_{i=1}^n |u_i| + |v_i| = \|u\|_1 + \|v\|_1.$$

If $p = \infty$ then

$$\begin{aligned} \|u+v\|_\infty &= \max_{i=1,\dots,n} |u_i + v_i| \leqslant \max_{i=1,\dots,n} \big(|u_i| + |v_i|\big) \\ &\leqslant \max_{i=1,\dots,n} |u_i| + \max_{i=1,\dots,n} |v_i| = \|u\|_\infty + \|v\|_\infty. \end{aligned}$$

Finally, if $p \in (1, \infty)$, then

$$\begin{aligned} \|u+v\|_p &= \sum_{i=1}^n |u_i + v_i|^p = \sum_{i=1}^n |u_i + v_i| \cdot |u_i + v_i|^{p-1} \\ &\leqslant \sum_{i=1}^n |u_i| \cdot |u_i + v_i|^{p-1} + \sum_{i=1}^n |v_i| \cdot |u_i + v_i|^{p-1} \\ &\leqslant \left(\sum_{i=1}^n |u_i|^p\right)^{1/p} \left(\sum_{i=1}^n |u_i + v_i|^{(p-1)q}\right)^{1/q} \\ &\quad + \left(\sum_{i=1}^n |v_i|^p\right)^{1/p} \left(\sum_{i=1}^n |u_i + v_i|^{(p-1)q}\right)^{1/q} \end{aligned}$$

where we have applied Hölder's inequality (2). Since $q = \frac{p}{p-1}$ we get $(p-1)q = p$. Divide both sides by $\big(\sum_{i=1}^n |u_i + v_i|^{(p-1)q}\big)^{1/q}$ to get

$$\left(\sum_{i=1}^n |u_i + v_i|^p\right)^{1-1/q} \leqslant \|u\|_p + \|v\|_p.$$

Since $1 - 1/q = 1/p$, the left-hand side equals $\|u+v\|_p$, so we are done. □

Theorem 10. *$(\ell^p(K), \|\cdot\|_{\ell^p})$ is a Banach space (a complete normed vector space) for every $p \in [1, \infty]$.*

Proof. The proof consist of three parts: ℓ^p is a vector space, $\|\cdot\|_{\ell^p}$ is a norm, and this space is complete.

Claim: $\|\cdot\|_{\ell^p}$ is a norm. It is clear that $\|u\|_{\ell^p} > 0$ for all $u \neq 0$, and that $\|u\|_{\ell^p} = 0$ implies $u = 0$. Let $a \in \ell^p(\mathbb{K})$ and $\alpha \in \mathbb{K}$. If $p < \infty$ then $\|\alpha a\|_{\ell^p} = \big(\sum_{i=1}^\infty |\alpha a(i)|^p\big)^{1/p} = |\alpha|\big(\sum_{i=1}^\infty |a(i)|^p\big)^{1/p} = |\alpha|\|a\|_{\ell^p}$. For $p = \infty$ we have $\|\alpha a\|_{\ell^\infty} = \sup_{i\in\mathbb{N}} |\alpha a(i)| = |\alpha| \sup_{i\in\mathbb{N}} |a(i)| = |\alpha|\|a\|_{\ell^\infty}$. Last, we show the triangle inequality. If $a, b \in \ell^p(\mathbb{K})$ and $p = \infty$ then

$$\|a+b\|_{\ell^\infty} = \sup_{i\in\mathbb{N}} |a(i)+b(i)| \leqslant \sup_{i\in\mathbb{N}} |a(i)|+|b(i)| \leqslant \sup_{i\in\mathbb{N}} |a(i)| + \sup_{i\in\mathbb{N}} |b(i)| = \|a\|_{\ell^\infty} + \|b\|_{\ell^\infty}.$$

Finally, if $p \in (1, \infty)$, let $n \in \mathbb{N}$ be an arbitrary integer and define $u, v \in \mathbb{K}^n$ by $u = (a(1), \dots, a(n))$ and $v = (b(1), \dots, b(n))$. Then

$$\left(\sum_{i=1}^n |a(i) + b(i)|^p\right)^{1/p} = \|u+v\|_p \leqslant \|u\|_p + \|v\|_p \leqslant \|a\|_{\ell^p} + \|b\|_{\ell^p}.$$

Taking the limit $n \to \infty$ yields $\|a+b\|_{\ell^p}$ on the left-hand side.

Claim: $\ell^p(\mathbb{K})$ is a vector space. Most of the axioms follow immediately; we only show that $\ell^p(\mathbb{K})$ is closed under addition and multiplication by scalars. Indeed, from the fact that $\|\cdot\|_{\ell^p}$ is a norm on $\ell^p(\mathbb{K})$, we find that $\|\alpha a\|_{\ell^p} = |\alpha|\|a\|_{\ell^p} < \infty$ whenever $\alpha \in \mathbb{K}$ and $a \in \ell^p(\mathbb{K})$, implying that also $\alpha a \in \ell^p(\mathbb{K})$. If $a, b \in \ell^p(\mathbb{K})$ then $\|a+b\|_{\ell^p} \leqslant \|a\|_{\ell^p} + \|b\|_{\ell^p} < \infty$, so also $a+b \in \ell^p(\mathbb{K})$.

Claim: $\big(\ell^p(\mathbb{K}), \|\cdot\|_{\ell^p}\big)$ is complete. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell^p(\mathbb{K})$. Then for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $\|a_n - a_m\|_{\ell^p} < \varepsilon$ when $n, m \geqslant N$, so in particular,

$$|a_n(i) - a_m(i)| \leqslant \|a_n - a_m\|_{\ell^p} < \varepsilon \qquad \forall\, i \in \mathbb{N}.$$

It follows that for each $i \in \mathbb{N}$, the sequence $\{a_n(i)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete, $\{a_n(i)\}_{n\in\mathbb{N}}$ is convergent, converging to some $a(i) \in \mathbb{K}$. We claim that the sequence $a = (a(1), a(2), \dots)$ lies in $\ell^p(\mathbb{K})$ and that $\{a_n\}_{n\in\mathbb{N}}$ converges to a. We split the proof into the cases $p = \infty$ and $p < \infty$.

$p = \infty$: Let $\varepsilon > 0$ and let N be as above. Then $|a(i) - a_n(i)| = \lim_{m\to\infty} |a_m(i) - a_n(i)| \leqslant \varepsilon$ for all $n \geqslant N$ and $i \in \mathbb{N}$, so

$$|a(i)| \leqslant |a(i) - a_n(i)| + |a_n(i)| \leqslant \varepsilon + \|a_n\|_{\ell^p}.$$

Since this holds for every $i \in \mathbb{N}$ we get $\|a\|_{\ell^\infty} \leqslant \varepsilon + \|a_n\|_{\ell^p} < \infty$, so $a \in \ell^\infty(\mathbb{K})$. Moreover, $\|a - a_n\|_{\ell^\infty} = \sup_{i\in\mathbb{N}} |a(i) - a_n(i)| \leqslant \varepsilon$, so we conclude that $a_n \to a$ as $n \to \infty$.

$p < \infty$: Let $\varepsilon > 0$ and let N be as above. For every $n, I \in \mathbb{N}$ we have

$$\left(\sum_{i=1}^{I} |a(i) - a_n(i)|^p\right)^{1/p} = \lim_{m\to\infty} \underbrace{\left(\sum_{i=1}^{I} |a_m(i) - a_n(i)|^p\right)^{1/p}}_{\leqslant \|a_n - a_m\|_{\ell^p} < \varepsilon} \leqslant \varepsilon,$$

so $\|a - a_n\|_{\ell^p} \leqslant \varepsilon$. Hence, $\|a - a_n\|_{\ell^p} \to 0$ as $n \to \infty$. Last, from the inverse triangle inequality,

$$\|a\|_{\ell^p} \leqslant \|a_n\|_{\ell^p} + \|a - a_n\|_{\ell^p} \leqslant \|a_n\|_{\ell^p} + \varepsilon < \infty,$$

so $a \in \ell^p(\mathbb{K})$.

□

We complete this note by showing that ℓ^p has a (Schauder) basis whenever $p < \infty$.

Proposition 11. *$\ell^p(\mathbb{K})$ is infinite-dimensional for all $p \in [1, \infty]$. If $e_n \in \ell^p(\mathbb{K})$ is given by*

$$e_n = (0, \dots, 0, 1, 0, \dots)$$

(the 1 occuring in the nth position), then $\{e_n\}_{n\in\mathbb{N}}$ is a Schauder basis for $\ell^p(\mathbb{K})$ for every $p \in [1, \infty)$, but not for $\ell^\infty(\mathbb{K})$.

Proof. The set $\{e_n\}_{n\in\mathbb{N}}$ is infinite and linearly independent, so $\ell^p(\mathbb{K})$ is infinite-dimensional. If $p < \infty$ and $a \in \ell^p(\mathbb{K})$, let $\alpha_i = a(i)$ for each $i \in \mathbb{N}$. Then the partial sum $s_n = \sum_{i=1}^n \alpha_i e_i = (a(1), \dots, a(n), 0, \dots)$ satisfies

$$\|a - s_n\|_{\ell^p} = \left(\sum_{i=n+1}^{\infty} |a(i)|^p \right)^{1/p}.$$

From the fact that $\sum_{i=1}^{\infty} |a(i)|^p < \infty$, the above sum must converge to 0 as $n \to \infty$. It follows that $a = \sum_{i=1}^{\infty} \alpha_i e_i$. This proves that $\{e_n\}_{n\in\mathbb{N}}$ is a Schauder basis for $\ell^p(\mathbb{K})$.

For $\ell^\infty(\mathbb{K})$, let $a = (1, 1, \dots) \in \ell^\infty(\mathbb{K})$. If α_i are such that $a = \sum_{i=1}^{\infty} \alpha_i e_i$ then necessarily $\alpha_i = 1$ for all i. But

$$\|a - s_n\|_{\ell^\infty} = \|(0, \dots, 0, 1, 1, \dots)\|_{\ell^\infty} = 1,$$

a contradiction. Hence, $\{e_n\}_{n\in\mathbb{N}}$ is not a Schauder basis for $\ell^\infty(\mathbb{K})$. $\square$

It can also be shown that ℓ^∞ does not possess *any* Schauder basis. In this sense, ℓ^∞ is "much bigger" than the other ℓ^p spaces.