Lecture notes for MAT3440 – Dynamical systems
Part I: Dynamical systems

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Contents

Note to the reader

Parts marked with * are considered optional, and are not essential for understanding the remaining parts of the material.

TODO: Dependency graph

Chapter 1

Introduction

A *continuous dynamical system*, or simply *dynamical system*, is an ordinary differential equation of the form

$$\dot{u}(t) = F(u(t)) \tag{1}$$

where $u\colon \mathbb{R} \to \mathbb{R}^n$ is the unknown and $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a given function.

1.1 Motivation — the SIR model

1.2 Overview of the notes

Well-posedness. Linear systems. Stability of fixed points. Periodic orbits.

1.3 Preliminaries

Basics about separable first-order equations, linear second-order equations with constant coefficients, non-autonomous to autonomous, high-order to first-order.

Chapter 2

Linear systems

A *linear system* is an ODE which can be written as

$$\dot{u}(t) = A(t)u(t) + h(t) \tag{2.1}$$

for given functions $A\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ and $h\colon \mathbb{R} \to \mathbb{R}^n$. Any ODE which cannot be written in the above form is called *nonlinear*. The function h is the *inhomogeneous part* or *inhomogeneity* of (2.1). If $h \equiv 0$ then the system is *homogeneous*, while if not it is *non-homogeneous*. If $A(t) \equiv A \in \mathbb{R}^{n\times n}$ then (2.1) is said to have *constant coefficients*.

Linear systems are important both because they are simple enough that we can solve them exactly, and because they can be used as a tool for understanding the behaviour of nonlinear systems. The main source of linear systems is via *linearization*. Consider a nonlinear ODE $\dot{v} = F(v,t)$, and assume that v^* is a fixed point for the ODE — that is, a point such that if $v(0) = v^*$ then $v(t) = v^*$ for all t. (This is equivalent to stating that $F(v^*,t) = 0$ for all t.) Taylor expanding around v^* yields

$$\begin{aligned} \dot{v}(t) = F(v(t),t) &= \underbrace{F(v^*,t)}_{=0} + D_v F(v^*,t)(v(t)-v^*) + O\big(\|v(t)-v^*\|^2\big) \\ &= D_v F(v^*,t)(v(t)-v^*) + O\big(\|v(t)-v^*\|^2\big). \end{aligned}$$

If v_0 is close to v^* then we may expect $v(t)$ to stay close to v^*, at least for some time. Then $\|v(t)-v^*\|^2 \approx 0$, so letting $u(t) := v(t) - v^*$, we see that

$$\dot{u}(t) \approx A(t)u(t)$$

where $A(t) := F(v^*,t)$. Thus, u satisfies a homogeneous linear equation — at least approximately. We will return to linearization as a tool for understanding nonlinear systems in Chapter 6.

2.1 General linear system

Basic well-posedness result (refer to next chapter).

Superposition. The space of solutions and its dimension. Solving non-homogeneous equations (general+particular solution).

2.2 Linear systems with constant coefficients

$$\begin{cases} \dot{u} = Au \\ u(0) = u_0 \end{cases} \tag{2.2}$$

2.2.1 The matrix exponential

Definition 2.1. The *matrix norm* (also called *operator norm*) of a matrix $A \in \mathbb{R}^{n\times n}$ is

$$\|A\|_{\mathcal{L}} := \sup_{\substack{u\in\mathbb{R}^n \\ u\neq 0}} \frac{\|Au\|}{\|u\|}.$$

(Here and elsewhere, the norm of a vector is the Euclidean norm.)

Theorem 2.2. *The matrix norm has the following properties:*

(i) It is a norm on $\mathbb{R}^{n\times n}$, i.e., for all $A, B \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}$,

(a) $\|A\|_{\mathcal{L}} \geqslant 0$, with $\|A\|_{\mathcal{L}} = 0$ if and only if $A = 0_n$,

(b) $\|\alpha A\|_{\mathcal{L}} = |\alpha| \|A\|_{\mathcal{L}}$,

(c) $\|A + B\|_{\mathcal{L}} \leqslant \|A\|_{\mathcal{L}} + \|B\|_{\mathcal{L}}$.

(ii) It is compatible with the norm $\|\cdot\|$, in the sense

$$\|Au\| \leqslant \|A\|_{\mathcal{L}}\|u\| \qquad \forall\ A \in \mathbb{R}^{n\times n},\ u \in \mathbb{R}^n.$$

(iii) It is sub-multiplicative:

$$\|AB\|_{\mathcal{L}} \leqslant \|A\|_{\mathcal{L}}\|B\|_{\mathcal{L}} \qquad \forall\ A, B \in \mathbb{R}^{n\times n}.$$

Definition 2.3. The *matrix exponential* of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as

$$e^A := I_n + \sum_{k=1}^{\infty} \frac{1}{k!} A^k. \tag{2.3}$$

Theorem 2.4. *The matrix exponential has the following properties.*

(i) The series (2.3) *converges for every $A \in \mathbb{R}^{n\times n}$*

(ii) $\|e^A\|_{\mathcal{L}} \leqslant e^{\|A\|_{\mathcal{L}}}$

(iii) $e^{0_n} = I_n$

(iv) If $A, B \in \mathbb{R}^{n\times n}$ commute, i.e. $AB = BA$, then $e^A e^B = e^{A+B}$

(v) e^A is invertible for every $A \in \mathbb{R}^{n\times n}$, with $(e^A)^{-1} = e^{-A}$

(vi) If (λ, r) is an eigenpair for A, then (e^λ, r) is an eigenpair for e^A

(vii) Let $\lambda_{\max} = \max_{i=1,\dots,n} \mathrm{Re}(\lambda_i)$, the maximal real part of the eigenvalues of A. For any $\varepsilon > 0$ there is some $C > 0$ such that

$$\|e^{tA}\|_{\mathcal{L}} \leqslant C e^{(\lambda_{\max}+\varepsilon)t} \qquad \text{for all } t \geqslant 0.$$

Definition 2.5. A matrix $A \in \mathbb{R}^{n\times n}$ is *diagonalizable* if there is a diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$, where $\lambda_1,\dots,\lambda_n \in \mathbb{C}$, and an invertible matrix $R \in \mathbb{C}^{n\times n}$ such that

$$A = R\Lambda R^{-1}. \tag{2.4}$$

Letting $r_1,\dots,r_n \in \mathbb{R}^n$ denote the column vectors of R and multiplying (2.4) by R from the right, we see that

$$Ar_k = \lambda_k r_k, \qquad k = 1,\dots,n,$$

so (λ_k, r_k) are eigenpairs of A. Since invertibility of R is equivalent to the linear independence of its column vectors, we conclude that diagonalizability is equivalent to the existence of *n linearly independent* eigenvectors $r_1,\dots,r_n$.

If A itself is a diagonal matrix, $A = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$, then $A^k = \operatorname{diag}(\lambda_1^k,\dots,\lambda_n^k)$, so from (2.4) we see that

$$e^A = \operatorname{diag}\bigl(e^{\lambda_1},\dots,e^{\lambda_n}\bigr).$$

More generally, if A is merely diagonalizable, then

$$\begin{aligned}
e^A &= I_n + \sum_{k=1}^{\infty}\frac{1}{k!}(R\Lambda R^{-1})^k = I_n + \sum_{k=1}^{\infty}\frac{1}{k!}R\Lambda R^{-1}R\Lambda R^{-1}\cdots R\Lambda R^{-1} \\
&= I_n + \sum_{k=1}^{\infty}\frac{1}{k!}R\Lambda^k R^{-1} = I_n + R\left(\sum_{k=1}^{\infty}\frac{1}{k!}\Lambda^k\right)R^{-1} \\
&= R\left(I_n + \sum_{k=1}^{\infty}\frac{1}{k!}\Lambda^k\right)R^{-1} = Re^{\Lambda}R^{-1}.
\end{aligned}$$

Since Λ is diagonal, the above is very easy to compute. In particular, for any $u \in \mathbb{R}^n$,

$$e^A u = Re^{\Lambda}\underbrace{R^{-1}u}_{=:v} = Re^{\Lambda}v.$$

Writing out the product $e^{\Lambda}v$ and inserting $R = \bigl(r_1 \quad \dots \quad r_n\bigr)$, we see that

$$e^A u = v^{(1)}e^{\lambda_1}r_1 + \cdots + v^{(n)}e^{\lambda_n}r_n, \qquad \text{where } v := R^{-1}u.$$

2.2.2 The fundamental theorem

The fundamental theorem for linear systems (solution formula). Write out and interpret the solution for diagonalizable systems.

Theorem 2.6. *The function $M\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ defined by $M(t) := e^{tA}$ satisfies*

$$\begin{cases} \dot{M} = AM \\ M(0) = I_n. \end{cases} \tag{2.5}$$

In particular, the solution to the linear system (2.2) *is*

$$u(t) = e^{tA}u_0. \tag{2.6}$$

The function M is called the *fundamental solution* to the problem (2.2). It is "fundamental" in the sense that it can be used to construct the solution (2.6) to the general initial value problem.

2.2.3 The solution for diagonalizable systems

If A is diagonalizable then we can compute the solution rather easily. If $A = R\Lambda R^{-1}$ then $tA = R(t\Lambda)R^{-1}$, so tA is also diagonalizable. Thus,

$$u(t) = e^{tA}u_0 = Re^{t\Lambda}R^{-1}u_0.$$

Denote $v_0 := R^{-1}u_0$, and note that $u_0 = v_0^{(1)}r_1 + \cdots + v_0^{(n)}r_n$, that is, the components $v_0^{(1)}, \ldots, v_0^{(n)}$ are the coefficients u_0 in the basis $\{r_1, \ldots, r_n\}$. From the above we see that

$$u(t) = v_0^{(1)}e^{\lambda_1 t}r_1 + \cdots + v_0^{(n)}e^{\lambda_n t}r_n. \tag{2.7}$$

Clearly, $u(0) = v_0^{(1)}r_1 + \cdots + v_0^{(n)}r_n = u_0$, and as time evolves, each eigenvector r_k is scaled by a factor $e^{\lambda_k t}$.

2.2.4 Dealing with complex eigenvalues

Let A be diagonalizable, as in the previous section. If an eigenvalue λ_k of A is complex, then the corresponding eigenvector r_k might also be complex. Thus, the terms in (2.7) are complex, but by some miracle, the solution $u(t)$ is real-valued.

2.3 *Non-diagonalizable systems

2.3.1 The Jordan normal form

Assume now that $A \in \mathbb{R}^{n\times n}$ is *not* diagonalizable. Every matrix has n eigenvalues $\lambda_1, \ldots, \lambda_n$, and if $\lambda_k \neq \lambda_l$ then corresponding eigenvectors r_k and r_l are linearly independent. Thus, the only way that a matrix can be non-diagonalizable, is that two or more eigenvalues are equal — its algebraic multiplicity $\mu_A(\lambda_k)$ is greater than 1 — and there are too few eigenvectors — $\mu_G(\lambda_k)$, the number of linearly independent eigenvectors corresponding to λ_k, is strictly smaller than $\mu_A(\lambda_k)$.

The following theorem, which we will not prove, shows that the idea of diagonalization can be generalized slightly so that it applies to *any* square matrix.

Theorem 2.7 (The Jordan normal form)**.** *Let $A \in \mathbb{R}^{n\times n}$ and let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ (where $m \leqslant n$) be the* distinct *eigenvalues of A. Then there is an invertible matrix $R \in \mathbb{C}^{n\times n}$ such that*

$$A = R\Lambda R^{-1}, \qquad \Lambda = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{pmatrix}$$

where J_k is the matrix

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix} \in \mathbb{C}^{q_k \times q_k}$$

and where $q_k = \mu_A(\lambda_k)$ for $k = 1, \ldots, m$ is the algebraic multiplicity of the eigenvalue λ_k. The decomposition $A = R\Lambda R^{-1}$ is the Jordan normal form *of A. Each* *"block" J_k is a* Jordan block. *If r is one of the column vectors of R, and the corresponding column in Λ lies in block J_k, then we say that r is a* generalized eigenvector *for λ_k.*

Remark 2.8. We can write $J_k = \lambda_k I_{q_k} + N_{q_k}$, where $I_q = \big(\delta_{i,j}\big)_{i,j=1,\dots,q}$ is the $q \times q$ identity matrix, and $N_q = \big(\delta_{i,j-1}\big)_{i,j=1,\dots,q}$. △

To get an idea of how to compute the Jordan normal form of A, let us concentrate on one Jordan block, say, J_k. Let $n_1 \leqslant n_2$ be such that J_k lies in columns $n_1, \dots, n_2$ in the matrix Λ. Let $r_1, \dots, r_q$ be the vectors lying in the corresponding column numbers $n_1, \dots, n_2$ in R. Right-multiplying the identity $A = R\Lambda R^{-1}$ by R gives $AR = R\Lambda$, and columns $n_1, \dots, n_2$ of this identity read

We have $n_1 = q_1 + \cdots + q_{k-1} + 1$ and $n_2 = q_1 + \cdots + q_k$.

$$Ar_1 = \lambda_k r_1, \qquad Ar_2 = r_1 + \lambda_k r_2, \qquad \dots \qquad Ar_q = r_{q-1} + \lambda_k r_q.$$

Reordering these equations yields

$$\big(A - \lambda_k I\big) r_1 = 0, \qquad \big(A - \lambda_k I\big) r_2 = r_1, \qquad \dots \qquad \big(A - \lambda_k I\big) r_q = r_{q-1}$$

From these identities, we can see that the generalized eigenvectors $r_1,\ r_2, \dots, r_q$ of λ_k satisfy

$$\big(A - \lambda_k I\big)^p r_j \begin{cases} \neq 0 & \text{for all } p = 0, 1, \dots, j-1 \\ = 0 & \text{for } p = j. \end{cases} \tag{2.8}$$

In fact, it can be shown that the above property uniquely determines the generalized eigenvectors, in the sense that if the columns of R have been chosen to satisfy (2.8), then R is invertible, and $A = R\Lambda R^{-1}$. This gives us an algorithm for computing the Jordan normal form:

Algorithm 2.9 (Computing the Jordan normal form)**.** Let $A \in \mathbb{R}^{n\times n}$. Then $A = R\Lambda R^{-1}$, where R and Λ are computed as follows:

1. Compute the (distinct) eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ of A.
2. For each $k = 1, \dots, m$:
   (a) Let $q = \mu_A(\lambda_k)$, the algebraic multiplicity of λ_k.
   (b) Let $J_k = \lambda_k I_q + N_q$ be the kth Jordan block.
   (c) Find nonzero vectors $r_1, \dots, r_q \in \mathbb{C}^n$ satisfying
   $$\big(A - \lambda_k I\big)^j r_j = 0, \qquad j = 1, \dots, q.$$
3. Assemble the vectors r_k as the columns of a matrix R in the order that you found them.
4. Define $\Lambda = \operatorname{diag}(J_1, \dots, J_m)$.

q is the smallest number such that $\big(A - \lambda_k I\big)^{q+1} = 0$.

2.3.2 The solution of the linear system

Let $A = R\Lambda R^{-1}$ be the Jordan normal form of A. Just as for diagonalizable matrices, we can easily show that the matrix exponential of A is

$$e^{tA} = Re^{t\Lambda}R^{-1}, \qquad \text{where } e^{t\Lambda} = \begin{pmatrix} e^{tJ_1} & & & \\ & e^{J_2} & & \\ & & \ddots & \\ & & & e^{tJ_m} \end{pmatrix}$$

for any $t \in \mathbb{R}$. The question is therefore how to compute the matrix exponential of a Jordan block. Dropping the k subscript for the moment, let $J = \lambda I_q + N_q \in \mathbb{C}^{q\times q}$ be such a block. By computing powers of J and noting that $N_q^2 = \big(\delta_{i,j-2}\big)_{i,j}$, $N_q^3 = \big(\delta_{i,j-3}\big)_{i,j}$, and so on up to $N_q^q = 0$, it is straightforward to show that

A matrix whose qth power is zero (for a sufficiently large integer q) is called *nilpotent*.

$$e^{tJ} = \sum_{\ell=0}^{q-1} \frac{e^{t\lambda}}{\ell!} N_q^\ell = e^{t\lambda} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{q-1}}{(q-1)!} \\ & 1 & t & \cdots & \frac{t^{q-2}}{(q-2)!} \\ & & \ddots & & \vdots \\ & & & 1 & t \\ & & & & 1 \end{pmatrix}$$

(where q is the dimension of the Jordan block).

Example 2.10. Let $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. Then A is already in Jordan normal form with $R = I_2$. We get

$$e^{tA} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Therefore, the solution to the problem $\dot{u} = Au$, $u(0) = (x_0, y_0)^\mathsf{T}$ is

$$u(t) = e^{\lambda t} \begin{pmatrix} x_0 + ty_0 \\ y_0 \end{pmatrix}. \qquad \triangle$$

2.4 Two-dimensional systems

(put this as a separate chapter?)

2.5 Stability of linear systems

Definition 2.11. Consider the linear system $\dot{u} = Au$, where $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and eigenvectors $r_1, \dots, r_n \in \mathbb{C}^n$. We define the *stable subspace*, *centre subspace* and *unstable subspace* as

$$\begin{aligned} \mathbb{E}^s &= \operatorname{span}\big\{\mathrm{Re}(r_i), \mathrm{Im}(r_i) : \mathrm{Re}(\lambda_i) < 0\} \\ \mathbb{E}^c &= \operatorname{span}\big\{\mathrm{Re}(r_i), \mathrm{Im}(r_i) : \mathrm{Re}(\lambda_i) = 0\} \\ \mathbb{E}^u &= \operatorname{span}\big\{\mathrm{Re}(r_i), \mathrm{Im}(r_i) : \mathrm{Re}(\lambda_i) > 0\}. \end{aligned}$$

TODO: Decomposition of u into $\mathbb{E}^s, \mathbb{E}^c, \mathbb{E}^u$.

Proposition 2.12. *Consider the linear system $\dot{u} = Au$, $u(0) = u_0 \neq 0$. Then*

(i) $u(t) \to 0$ as $t \to \infty$ if and only if $u_0 \in \mathbb{E}^s$

(ii) $u(t) \to 0$ as $t \to -\infty$ if and only if $u_0 \in \mathbb{E}^u$

(iii) $u(t)$ is bounded for all $t \in \mathbb{R}$ if and only if $u_0 \in \mathbb{E}^c$.

2.6 Inhomogeneous linear systems

Consider now the inhomogeneous linear system with constant coefficients,

$$\begin{cases} \dot{u} = Au + h(t) \\ u(0) = u_0 \end{cases} \tag{2.9}$$

for some $u_0 \in \mathbb{R}^n$ and $h\colon \mathbb{R} \to \mathbb{R}^n$. Recalling that the solution to the homogeneous problem (2.2) is $u(t) = M(t)u_0$, where $M(t) := e^{tA}$, we make the educated guess that the solution of (2.9) is $u(t) = M(t)g(t)$, for some function $g\colon \mathbb{R} \to \mathbb{R}^n$. Then $u(0) = M(0)g(0) = g(0)$, so the initial data forces $g(0) = u_0$. Next,

$$\dot{u} = \dot{M}g + M\dot{g} = AMg + M\dot{g} = Au + M\dot{g}.$$

Thus, in order to satisfy (2.9), we need $M\dot{g} = h$, or $\dot{g}(t) = M(t)^{-1}h(t)$. Simply integrating with respect to t yields

$$g(t) = g(0) + \int_0^t M(s)^{-1}h(s)\,ds = u_0 + \int_0^t e^{-sA}h(s)\,ds.$$

Inserting into our definition of u yields

$$u(t) = M(t)g(t) = e^{tA}u_0 + e^{tA}\int_0^t e^{-sA}h(s)\,ds,$$

which, upon bringing e^{tA} inside the integral, yields the solution formula

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}h(s)\,ds. \tag{2.10}$$

The above procedure for solving inhomogeneous differential equations is an example of *Duhamel's principle*.

Chapter 3

Well-posedness

We now move on to general, possibly nonlinear, differential equations

$$\begin{cases} \dot{u} = F(u, t) \\ u(0) = u_0. \end{cases} \tag{3.1}$$

Before we start to analyse solutions to this problem, we need to establish *well-posedness*. We say that a differential equation is well-posed if

(i) a solution exists

(ii) the solution is unique

(iii) the solution depends continuously on the data.

Existence and uniqueness should be obvious requirements — how can we treat (3.1) as a model of a real-world phenomenon if a solution doesn't exist, or if it predicts several different, conflicting outcomes?

Requirement (iii) deserves a closer look. If we denote the solution of (3.1) by $\varphi(t; u_0) = u(t)$, then *continuous dependence on the data* means that $\varphi(t; u_0)$ should be a continuous function of u_0, for every $t \in \mathbb{R}$. In other words, a small change in the data u_0 should only lead to a (relatively) small change in $\varphi(t; u_0)$. This requirement will have important consequences later on, but for now we note what might happen if condition (iii) is not satisfied. In real life, measurement errors are inevitable. Thus, if $\varphi(t; u_0)$ depends *discontinuously* on the data, then these measurement errors would lead to completely erroneous predictions at time t. In this chapter we show how a reasonable condition on F will guarantee that (3.1) is well-posed.

We end this introduction with several counterexamples to well-posedness.

Example 3.1. The equation

$$\begin{cases} \dot{u} = F(u) \\ u(0) = 0, \end{cases} \qquad \text{where } F(u) = \begin{cases} 1 & \text{if } u < 0 \\ -1 & \text{if } u \geqslant 0 \end{cases}$$

does not have a solution. △

Example 3.2. The equation

$$\dot{u} = u^2, \qquad u(0) = 1$$

does not have a solution for all $t \in \mathbb{R}$. △

Example 3.3. The equation

$$\dot{u} = \sqrt{|u|}, \qquad u(0) = 0$$

has infinitely many solutions. △

3.1 Lipschitz continuity

Definition 3.4. A function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is *(globally) Lipschitz continuous in the first variable* if there is some $L \geqslant 0$ such that

$$\|F(u,t) - F(v,t)\| \leqslant L\|u - v\| \qquad \forall\, u, v \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{3.2}$$

The smallest constant L for which the above is true is the *Lipschitz constant of F*.

If F is autonomous we say that it is *Lipschitz continuous* if the above holds.

Lemma 3.5. *Assume that $F(\cdot, t) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ for every $t \in \mathbb{R}$. Let $L \in (0, \infty)$. Then the following are equivalent:*

(i) F is Lipschitz continuous in the first variable, with Lipschitz constant L

(ii) $\|D_u F(u,t)\|_{\mathcal{L}} \leqslant L$ for all $u \in \mathbb{R}^n$, $t \in \mathbb{R}$.

3.2 Global well-posedness

Theorem 3.6. *Assume that $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous, and is Lipschitz continuous in the first variable. Then for every $u_0 \in \mathbb{R}^n$ there is a unique solution u of* (3.1) *defined for all $t \in \mathbb{R}$. If u, v are two solutions with initial data $u(0) = u_0$, $v(0) = v_0$, then*

$$\|u(t) - v(t)\| \leqslant \|u_0 - v_0\| e^{L|t|} \tag{3.3}$$

for all t, where L is the Lipschitz constant of F.

3.3 Local well-posedness

Definition 3.7. A function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is *locally Lipschitz continuous in the first variable* if for every $M > 0$, there is some $L \geqslant 0$ such that

$$\|F(u,t) - F(v,t)\| \leqslant L\|u - v\| \qquad \forall\, u, v \in B_M(0),\ t \in \mathbb{R}. \tag{3.4}$$

It is straightforward to prove a “local” variant of Lemma 3.5: A C^1 function is locally Lipschitz continuous if and only if its Jacobian is locally bounded. If F is autonomous then we can prove the following stronger statement.

Lemma 3.8. *Every $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz continuous.*

Theorem 3.9 (Local existence)**.** *Assume that $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous, and is locally Lipschitz continuous in the first variable. Then for every $u_0 \in \mathbb{R}^n$ there are numbers $a < 0 < b$ and a unique function $u : (a, b) \to \mathbb{R}^n$ solving* (3.1).

Definition 3.10. The largest interval (a, b) on which a solution of (3.1) exists is called the *maximal interval of existence.*

Theorem 3.11. *Assume that $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous, and is locally Lipschitz continuous in the first variable. Let $u_0 \in \mathbb{R}^n$, and let $u: (a, b) \to \mathbb{R}^n$ solve* (3.1)*. Then the following are equivalent:*

(i) for any $\varepsilon > 0$, there exists no solution of (3.1) *defined on $(a, b + \varepsilon)$*

(ii) $\lim_{t \uparrow b} \|u(t)\| = \infty$.

The result says that the only thing that can prevent us from extending the interval of existence beyond $t = b$, is that the solution "blows up" at $t = b$. A similar result holds at the lower limit $t = a$.

Chapter 4

The flow

Henceforth we consider an autonomous dynamical system

$$\begin{cases} \dot{u} = F(u) \\ u(0) = u_0 \end{cases} \tag{4.1}$$

and we assume that F is such that (4.1) has a unique solution, for any choice of $u_0 \in \mathbb{R}^n$. We will often (implicitly) assume that the maximal interval of existence is $\mathbb{R}$.

4.1 The flow

Definition 4.1. The *flow* of (4.1) is the function $\varphi = \varphi(t; u_0)$ which solves (4.1), that is,

$$\begin{cases} \frac{\partial \varphi}{\partial t}(t; u_0) = F(\varphi(t; u_0)) \qquad \forall\, t \\ \varphi(0; u_0) = u_0. \end{cases}$$

Definition 4.2. A point $u_0 \in \mathbb{R}^n$ is a *fixed point* if $\varphi(t; u_0) = u_0$ for all $t \in \mathbb{R}$, or equivalently, if $F(u_0) = 0$.

Definition 4.3. Let $u_0 \in \mathbb{R}^n$. We define

- the *forward orbit at* u_0 as $\Gamma^+(u_0) := \{\varphi(t; u_0) : t \geqslant 0\}$
- the *backward orbit at* u_0 as $\Gamma^-(u_0) := \{\varphi(t; u_0) : t \leqslant 0\}$
- the *orbit through* u_0 as $\Gamma(u_0) := \Gamma^+(u_0) \cup \Gamma^-(u_0) = \{\varphi(t; u_0) : t \in \mathbb{R}\}$.

The function $t \mapsto \varphi(t; u_0)$ parametrizes each of these sets, and is sometimes called the *trajectory through* u_0.

We now derive an important consequence of the uniqueness of solutions of the ODE. Let $u_0 \in \mathbb{R}^n$, let $T \in \mathbb{R}$ and denote $u(t) = \varphi(t+T; u_0)$ and $v(t) = \varphi(t; \varphi(T; u_0))$. Then $\dot{u} = F(u)$, $\dot{v} = F(v)$, and $u(0) = \varphi(T; u_0)$, $v(0) = \varphi(0; \varphi(T; u_0)) = \varphi(T; u_0)$. Thus, u and v are two solutions of the same ODE with the same initial data. By uniqueness of solutions, we must have $u = v$, that is:

$$\varphi(t; \varphi(T; u_0)) = \varphi(t + T; u_0) \qquad \forall\, t, T \in \mathbb{R}. \tag{4.2}$$

This is the *group property of the flow*. Here are some consequences of the group property:

Proposition 4.4. *Let $u_0 \in \mathbb{R}^n$. Then:*

(i) *For every $t \in \mathbb{R}$, the function $u_0 \mapsto \varphi(t; u_0)$ is invertible, with inverse $v \mapsto \varphi(-t; v)$.*

(ii) *$\Gamma(\varphi(t; u_0)) = \Gamma(u_0)$ for any $t \in \mathbb{R}$.*

(iii) *If $v_0 \in \Gamma(u_0)$ then $\Gamma(u_0) = \Gamma(v_0)$.*

(iv) *Distinct orbits cannot cross. That is, if $u_0, v_0 \in \mathbb{R}^n$ are such that $\Gamma(u_0) \neq \Gamma(v_0)$, then $\Gamma(u_0) \cap \Gamma(v_0) = \emptyset$.*

We leave the proof as an exercise.

4.2 Periodic orbits

Another consequence of the group property relates to so-called periodic solutions. Let $u_0 \in \mathbb{R}^n$, and assume that there is some nonzero $T \in \mathbb{R}$ such that $\varphi(T; u_0) = u_0$, i.e., the trajectory returns to u_0 after T seconds. If $v_0 \in \Gamma(u_0)$ is an arbitrary point on the orbit through u_0, let $t \in \mathbb{R}$ be such that $\varphi(t; u_0) = v_0$ (why is there such a number t?). Then

$$\varphi(T; v_0) = \varphi(T; \varphi(t; u_0)) = \varphi(T + t; u_0) = \varphi(t; \varphi(T; u_0)) = \varphi(t; u_0) = v_0.$$

Hence, if there is a point u_0 on an orbit such that φ returns to u_0 after T seconds, then it will also return to *any other point* on the orbit after T seconds. We may also assume that $T > 0$, since if $T < 0$ then $\varphi(-T; u_0) = \varphi(T; \varphi(-T; u_0)) = u_0$, showing that we may replace T by $-T$. We also see that for any $k \in \mathbb{Z}$,

$$\varphi(kT; u_0) = \varphi(T + \cdots + T; u_0) = \varphi(T; \varphi(T; \ldots \varphi(T; u_0) \ldots)) = u_0.$$

Definition 4.5. An orbit $\Gamma(u_0)$ for which there is some nonzero $T \in \mathbb{R}$ for which $\varphi(T; u_0) = u_0$ is a *periodic orbit.* The smallest positive number $T > 0$ for which this holds is called the *period* of the orbit.

4.3 ω-limits

We can roughly divide the long-term behaviour of the trajectory through u_0 into three types:

1. u_0 is a fixed point, or $\varphi(t; u_0)$ approaches a fixed point as $t \to \infty$,

2. u_0 lies on a periodic orbit, or $\varphi(t; u_0)$ approaches a periodic orbit as $t \to \infty$,

3. $\varphi(t; u_0)$ does not settle into a clear pattern as $t \to \infty$.

Although these three types of behaviour are qualitatively very different, we can study all of them by finding the set of points that the trajectory $\varphi(t; u_0)$ approaches as $t \to \infty$.

Definition 4.6. Let $u_0 \in \mathbb{R}^n$. A point $v \in \mathbb{R}^n$ is an *ω-limit point for u_0* if there is an increasing sequence of times $t_k \to \infty$ such that $\varphi(t_k; u_0) \to v$ as $k \to \infty$. We define

$$\omega(u_0) := \{\text{all } \omega\text{-limit points of } u_0\}.$$

Likewise, $v \in \mathbb{R}^n$ is an *α-limit point for u_0* if it is an ω-limit point for u_0 for the time-reversed problem $\dot{u} = -F(u)$. We let $\alpha(u_0)$ denote the set of all such points.

Example 4.7. If u^* is a fixed point then $\omega(u^*) = \{u^*\}$. If $\Gamma(u_0)$ is a periodic orbit then $\omega(u_0) = \Gamma(u_0)$. We leave the proofs as exercises for the reader. △

The following result provides a nice characterization of $\omega(u_0)$.

Proposition 4.8. *Let $u_0 \in \mathbb{R}^n$ and assume $\Gamma_+(u_0)$ is bounded. Then*

See Definition A.1 in the appendix for the definition of dist.

$$\operatorname{dist}\big(\varphi(t; u_0), \omega(u_0)\big) \to 0 \qquad \text{as } t \to \infty.$$

Moreover, $\omega(u_0)$ is the smallest closed set with this property, in the sense that if E is any closed set such that $\operatorname{dist}\big(\varphi(t; u_0), E\big) \to 0$, then $\omega(u_0) \subset E$.

Proof. Assume conversely that $\operatorname{dist}\big(\varphi(t; u_0), \omega(u_0)\big) \not\to 0$. Then there is some $\varepsilon > 0$ and a sequence $\{t_k\}_k$ converging to ∞ such that $\operatorname{dist}\big(\varphi(t_k; u_0), \omega(u_0)\big) \geqslant \varepsilon$ for all $k \in \mathbb{N}$. Since $\{\varphi(t_k; u_0)\}_k$ is a bounded sequence, it has a convergent subsequence, $\varphi(t_{k_l}; u_0) \to v$ as $l \to \infty$ for some $v \in \mathbb{R}^n$. But then both $\operatorname{dist}\big(v, \omega(u_0)\big) \geqslant \varepsilon$ and $v \in \omega(u_0)$, a contradiction.

To see that $\omega(u_0)$ is the smallest closed set with this property, let E be any closed set in $\mathbb{R}^n$ such that $\operatorname{dist}\big(\varphi(t; u_0), E\big) \to 0$ as $t \to \infty$. Let $v \in \omega(u_0)$ and let $\{t_k\}_k$ be such that $t_k \to \infty$ and $\varphi(t_k; u_0) \to v$ as $k \to \infty$. Then

$$0 = \lim_{t\to\infty} \operatorname{dist}\big(\varphi(t; u_0), E\big) = \lim_{k\to\infty} \operatorname{dist}\big(\varphi(t_k; u_0), E\big) = \operatorname{dist}\big(v, E\big),$$

so v lies in the closure of E, which is E itself. This proves that $\omega(u_0) \subset E$.

□

Remark 4.9. The above result can be generalized to arbitrary, unbounded orbits, by restricting to bounded sets: Assume that $\omega(u_0) \neq \emptyset$ (see Theorem 4.10(v)). Then

$$\lim_{\substack{t\to\infty \\ \varphi(t;u_0)\in B_M(u_0)}} \operatorname{dist}\big(\varphi(t; u_0), \omega(u_0)\big) = 0$$

for every $M > 0$. We leave the proof to the interested reader. △

We conclude the section by collecting several useful properties of $\omega(u_0)$.

Theorem 4.10. *Let $u_0 \in \mathbb{R}^n$. Then*

(i) $\omega(u_0) = \omega(v_0)$ *for every* $v_0 \in \Gamma(u_0)$.

(ii) $\omega(u_0)$ *is invariant.*

(iii) $\omega(u_0)$ *is closed.*

(iv) If $v \in \omega(u_0)$ *then* $\omega(v) \subset \omega(u_0)$.

(v) $\omega(u_0) = \emptyset$ *if and only if* $\varphi(t; u_0)$ *diverges as* $t \to \infty$ *— that is, if for every* $M > 0$*, there is some* $t_0 > 0$ *such that* $\|\varphi(t; u_0)\| \geqslant M$ *for all* $t \geqslant t_0$.

(vi) *$\omega(u_0)$ contains a single point u^* if and only if $\varphi(t;u_0) \to u^*$ as $t \to \infty$. In either case, u^* is a fixed point.*

Moreover, if $\Gamma_+(u_0)$ is bounded then also:

(vii) *$\omega(u_0)$ is nonempty.*

(viii) *$\omega(u_0)$ is connected.*

Remark 4.11. The same results — with obvious modifications — hold for the α-limits of u_0. △

Remark 4.12. (vii) can be sharpened as follows: $\omega(u_0)$ is empty if and only if $\|\varphi(t;u_0)\| \to \infty$ as $t \to \infty$. △

Proof of Theorem 4.10.

(i) Exercise for the reader.

(ii) If $v_0 \in \omega(u_0)$ then there is an increasing sequence of times $t_k \to \infty$ such that $\varphi(t_k;u_0) \to v_0$ as $k \to \infty$. Let $t \in \mathbb{R}$; we claim that $w := \varphi(t;v_0) \in \omega(u_0)$. Indeed, if $s_k = t_k - t$ then $s_k \to \infty$ and $\varphi(s_k;w) = \varphi(t_k;v_0) \to u_0$ as $k \to \infty$, so $w \in \omega(u_0)$.

(iii) Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in $\omega(u_0)$ converging to $v \in \mathbb{R}^n$. We claim that $v \in \omega(u_0)$. Let $t_{k,l} \in \mathbb{R}$ be such that $t_{k,l} \to \infty$ and $\varphi(t_{k,l};u_0) \to y_k$ as $l \to \infty$, for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, let $l(k) \in \mathbb{N}$ be such that $t_{k,l(k)} \geqslant k$, and $\|\varphi(t_{k,l(k)};u_0) - v_k\| \leqslant \frac{1}{k}$. Let $s_k = t_{k,l(k)}$. Then $s_k \to \infty$ and $\varphi(s_k;u_0) \to v$ as $k \to \infty$, so $v \in \omega(u_0)$.

(iv) If $v \in \omega(u_0)$ then, by (ii) and (iii), the trajectory $\varphi(t;v)$ lies in the closed set $\omega(u_0)$ for every $t \in \mathbb{R}$, so any ω-limit point of v must necessarily also lie in $\omega(u_0)$.

(v) If $\varphi(t;u_0)$ diverges as $t \to \infty$ then all subsequences $\{\varphi(t_k;u_0)\}_{k\in\mathbb{N}}$ also diverge, and so cannot converge. Hence, $\omega(u_0) = \emptyset$.

If $\varphi(t;u_0)$ does *not* diverge as $t \to \infty$ then there is some $M > 0$ and an increasing sequence of times $s_k \to \infty$ such that $\|\varphi(s_k;u_0)\| \leqslant M$ for all $k \in \mathbb{N}$. By the Bolzano–Weierstrass theorem, there is a subsequence $\{t_k\}_k$ of $\{s_k\}_k$ such that $\{\varphi(t_k;u_0)\}_k$ converges some $v \in \mathbb{R}$. In particular, $v \in \omega(u_0)$, so $\omega(u_0)$ is nonempty.

(vi) If $\varphi(t;u_0) \to u^*$ as $t \to \infty$ then the same is clearly true along any subsequence $t_k \to \infty$, so $\omega(u_0) = \{u^*\}$.

Conversely, assume that $\omega(u_0) = \{u^*\}$, and let $t_k \to \infty$ be such that $\varphi(t_k;u_0) \to u^*$ as $k \to \infty$. If $\varphi(t;u_0) \not\to u^*$ as $t \to \infty$ then there is some $\varepsilon > 0$ and a sequence $s_k \to \infty$ such that $\|\varphi(s_k;u_0) - u^*\| \geqslant \varepsilon$ for every s_k. By removing some elements of the sequences $\{t_k\}_k$ and $\{s_k\}_k$, we may assume that $t_k < s_k < t_{k+1}$ for every $k \in \mathbb{N}$. Let $N \in \mathbb{N}$ be large enough that $\|\varphi(t_k;u_0) - u^*\| < \varepsilon$ for every $k \geqslant N$. By continuity of φ, there are $\tau_k \in (t_k, s_k]$ for every $k \geqslant N$ such that $\varepsilon \leqslant \|\varphi(\tau_k;u_0) - u^*\| \leqslant 2\varepsilon$. Thus, the sequence $\{\varphi(\tau_k;u_0)\}_k$ lies in the closed, bounded set $K := \{v : \|v - u^*\| \in [\varepsilon, 2\varepsilon]\}$, and therefore has a subsequence $\varphi(\tau_{k(l)};u_0)$ converging to some $v \in K$ as $l \to \infty$; in particular, v is an ω-limit of u_0. Since $u^* \notin K$, the point v is distinct from u^*, but at the same time, $v \in \omega(u_0) = \{u^*\}$, a contradiction.

We conclude by showing that u^* is a fixed point. Indeed, if $\varphi(t; u_0) \to u^*$ as $t \to \infty$ then $\frac{d}{dt}\varphi(t; u_0) = F(\varphi(t; u_0)) \to F(u^*)$. If $n := F(u^*) \neq 0$, let $t_0 > 0$ be large enough that $\|F(\varphi(t; u_0)) - F(u^*)\| < \frac{1}{2}\|n\|$ for all $t \geqslant t_0$. Then also $n \cdot F(\varphi(t; u_0)) \geqslant \frac{1}{2}\|n\|^2$ for all $t \geqslant t_0$. If $t_1 > t_0$ and $\tau > 0$ then

$$\begin{aligned} n \cdot \big(\varphi(t_1 + \tau; u_0) - \varphi(t_1; u_0)\big) &= n \cdot \int_{t_1}^{t_1+\tau} \frac{d}{dt}\varphi(t; u_0)\, dt \\ &= \int_{t_1}^{t_1+\tau} n \cdot F(\varphi(t; u_0))\, dt \geqslant \int_{t_1}^{t_1+\tau} \frac{1}{2}\|n\|^2\, dt \\ &= \frac{\tau}{2}\|n\|^2. \end{aligned}$$

The left-hand side converges to $n \cdot \big(u^* - \varphi(t_1; u_0)\big)$ as $\tau \to \infty$, but the right-hand side diverges as $\tau \to \infty$, a contradiction.

(vii) This follows from (v).

(viii) Assume conversely that there are disjoint, nonempty open sets $U_1, U_2 \subset \mathbb{R}^n$ such that $\omega(u_0) \subset U_1 \cup U_2$ and $\omega(u_0) \cap U_j \neq 0$ for $j = 1, 2$. Select any $v \in \omega(u_0) \cap U_1$ and $w \in \omega(u_0) \cap U_2$, and let $\{s_k\}_k, \{t_k\}_k$ be sequences of times going to ∞ as $k \to \infty$ such that $\varphi(s_k; u_0) \to v$ and $\varphi(t_k; u_0) \to w$ as $k \to \infty$. By removing some of the elements of $\{s_k\}_k, \{t_k\}_k$, we may assume that $s_1 < t_1 < s_2 < t_2 < \ldots$, and therefore the corresponding values $\varphi(s_1; u_0), \varphi(t_1; u_0), \varphi(s_2; u_0), \ldots$ alternate between lying in U_1 and U_2. Since $t \mapsto \varphi(t; u_0)$ is continuous, there must be values $\tau_k \in (s_k, t_k)$ such that $\varphi(\tau_k; u_0) \notin U_1 \cup U_2$. Then $\{\varphi(\tau_k; u_0)\}_k$ is a bounded sequence lying in the closed set $\big(U_1 \cup U_2\big)^c$, and therefore has a convergent subsequence $\{\varphi(\tau_{k(l)}; u_0)\}_l$ converging to some $z \in \big(U_1 \cup U_2\big)^c$. This means that z both lies in $\omega(u_0)$ and $\big(U_1 \cup U_2\big)^c$, but these two sets are disjoint — a contradiction. □

4.4 *Liouville's formula

We mention here a few facts about the Jacobian $\nabla_u \varphi$ which we will use later. We start by summarizing some facts, which we will not prove. Assume that F is C^1. Then:

(i) The flow φ is C^1 in u and C^2 in t. This follows from a variant of the well-posedness result in Chapter 3.

(ii) The Jacobian $\nabla_u \varphi$ satisfies $\nabla_u \varphi(0; u_0) = I_n$, and $\nabla_u \varphi(t; u_0)$ is invertible for all $t \in \mathbb{R}$. Indeed, if $\nabla_u \varphi(t; u_0)$ were not invertible then, by the implicit function theorem, $u_0 \mapsto \varphi(t; u_0)$ could not be invertible, which contradicts Proposition 4.4(i).

(iii) The Jacobian determinant $J(t; u_0) := \det\big(\nabla_u \varphi(t; u_0)\big)$ is always positive. Indeed, J is continuous in t, satisfies $J(0; u_0) = \det I_n = 1$, and the Jacobian is invertible, so J cannot cross from positive into non-positive numbers.

Proposition 4.13. *Define* $J(t;u) := \det(\nabla_u \varphi(t;u))$. *Then*

$$\frac{\partial}{\partial t} J(t;u) = J(t;u)\,\mathrm{div} F(\varphi(t;u)).$$

As a consequence,

$$J(t) = \exp\biggl(\int_0^t \mathrm{div} F(\varphi(s;u))\,ds\biggr).$$

Proof. We will need *Jacobi's formula*, which says that for any differentiable function $A\colon \mathbb{R} \to \mathbb{R}^{n\times n}$,

$$\frac{d}{dt}\det A(t) = (\det A(t))\,\mathrm{tr}\biggl(\frac{dA}{dt}(t)A(t)^{-1}\biggr).$$

If $A(t) := \nabla_u \varphi(t;u)$ we have $\det A(t) = J(t)$ and

$$\begin{aligned}\frac{dA}{dt}(t) &= \nabla_u \dot{\varphi}(t;u) = \nabla_u F(\varphi(t;u))\\ &= \nabla F(\varphi(t;u))\nabla_u \varphi(t;u) = \nabla F(\varphi(t;u))A(t).\end{aligned}$$

Thus,

$$\frac{d}{dt}J(t) = J(t)\,\mathrm{tr}\Bigl(\nabla F(\varphi(t;u)\Bigr) = J(t)\,\mathrm{div} F(\varphi(t;u)).$$

□

Corollary 4.14. *If* $\Omega \subset \mathbb{R}^n$, $\Omega(t) = \varphi(t;\Omega)$ *and* $V(t) = \mathrm{vol}(\Omega(t))$ *then*

$$\dot{V}(t) = \int_{\Omega(t)} \mathrm{div} F(v)\,dv = \int_{\partial\Omega(t)} F(v)\cdot n\,dS(v)$$

(where $\partial\Omega(t)$ *is the boundary of* $\Omega(t)$ *and* n *is the outward pointing normal).*

Proof. Let $J(t;u) := \det \nabla_u(\varphi(t;u))$. Then

$$\begin{aligned}V(t) &= \int_{\mathbb{R}^n} \mathbb{1}_{\Omega(t)}(v)\,dv && \text{(where } \mathbb{1}_A(v) = 1 \text{ if } v \in A \text{ and } 0 \text{ otherwise)}\\ &= \int_{\mathbb{R}^n} \mathbb{1}_{\Omega}\bigl(\varphi(-t;v)\bigr)\,dv && \text{(since } v \in \Omega(t) \text{ iff } \varphi(-t;v) \in \Omega)\\ &= \int_{\mathbb{R}^n} \mathbb{1}_{\Omega}(w)\bigl|\det\bigl(\nabla_w \varphi(t;w)\bigr)\bigr|\,dw && \text{(change of variables } w = \varphi(-t;v))\\ &= \int_{\Omega} J(t;w)\,dw. && \text{(by definition of } J\text{, and since } J > 0)\end{aligned}$$

Thus, by Liouville's formula,

$$\begin{aligned}\dot{V}(t) &= \int_{\Omega} \frac{\partial}{\partial t} J(t;w)\,dw = \int_{\Omega} J(t;w)\,\mathrm{div} F(\varphi(t;w))\,dw\\ &= \int_{\Omega(t)} \mathrm{div} F(w)\,dw. && \text{(change of variable } v = \varphi(t;w))\end{aligned}$$

This proves the first equality. The second equality is an application of the divergence theorem. □

4.5 Problems

Problem 4.1. Find the flow of the scalar ODE $\dot{u} = Au$, where $A \in (0, \infty)$.
Find the orbit $\Gamma(u_0)$ for every $u_0 \in \mathbb{R}$.

Problem 4.2. Prove Proposition 4.4.

Problem 4.3. Prove that there are no periodic orbits for scalar equations ($n = 1$).

Chapter 5

Phase portraits

5.1 Phase portraits in one dimension

5.2 Phase portraits in two dimensions

Nullclines. Informally about stability.

5.3 Problems

Problem 5.1. Consider the problem $\dot{u} = Au$, where $A = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$ (TODO!). Plot a selection of orbits $\Gamma(u_0)$ for u_0 lying in the first quadrant, i.e. $u_0 = (x, y)^\top$ with $x, y \geqslant 0$.

Problem 5.2. Let $\Gamma(u_0)$ be a periodic orbit with period $T > 0$. Show that

$$\{t : \varphi(t; u_0) = u_0\} = \{kT : k \in \mathbb{Z}\}.$$

Problem 5.3. Let $n = 2$, and let $\Gamma(u_0)$ be a periodic orbit. Since $\Gamma(u_0)$ is a Jordan curve (a closed, non-self-intersecting curve), it splits $\mathbb{R}^2$ in two parts, an "inside" and an "outside". Let $v_0 \in \mathbb{R}^2$ lie "inside" the periodic orbit. Show that the orbit $\Gamma(v_0)$ is a bounded set, and use Theorem 3.11 to conclude that the solution of the ODE starting at v_0 exists for all times $t \in \mathbb{R}$.

Chapter 6

Stability of fixed points

6.1 Types of stability

Definition 6.1. Let $u^* \in \mathbb{R}^n$ be a fixed point. The *stable manifold at* u^* is the set

$$W^s(u^*) := \big\{u_0 \in \mathbb{R}^n : \lim_{t\to\infty} \varphi(t;u_0) = u^*\big\}. \tag{6.1}$$

The *unstable manifold at* u^* is the set

$$W^u(u^*) := \big\{u_0 \in \mathbb{R}^n : \lim_{t\to-\infty} \varphi(t;u_0) = u^*\big\}. \tag{6.2}$$

A *manifold* is a subset of $\mathbb{R}^n$, such as a curve or a surface, which is "smooth" enough — it can be (locally) parametrized by a smooth function. Manifolds are labelled by their dimension, which is the number of parameters needed to parametrize the set. Thus, a curve is one-dimensional, a surface two-dimensional, and so on. A trivial example of a manifold in $\mathbb{R}^n$ is an open set $U \subset \mathbb{R}^n$ — this set can be parametrized by the identity map $u \mapsto u$ in $\mathbb{R}^n$, and therefore has dimension n.

The stable manifold collects all those initial data that will tend to u^* as time evolves. Thus, we can informally say that the larger this set is, the more stable the fixed point is. Although the stable (and unstable) manifold can be difficult to compute exactly, one of the goals of this section will be to prove that these sets are indeed manifolds, and to see that we can easily find their dimension and find out approximately how they look close to u^*.

The term *basin of attraction* is sometimes used synonymously with stable manifold. Some authors choose to use this term in a more narrow sense, namely in the case where the stable manifold is a neighbourhood of u^*. (By our discussion above, the stable manifold must then be, or contain, a manifold of dimension n.) Others use it more broadly as any forward invariant set $S \subset \mathbb{R}^n$ such that $\varphi(t;u_0) \to u^*$ as $t \to \infty$ for all $u_0 \in S$. (As seen in the next proposition, the stable/unstable manifolds are invariant.)

We summarize a few basic facts about the stable manifold, and leave the proof as an exercise (Problem 6.2). Analogous statements about the unstable manifold follow similarly.

Proposition 6.2. *Let u^* be a fixed point. Then:*

(i) The stable manifold contains u^*, and hence is never empty.

(ii) The stable manifold is invariant.

(iii) $u_0 \in W^s(u^*)$ *if and only if* $\omega(u_0) = \{u^*\}$.

Definition 6.3. Let $u^* \in \mathbb{R}^n$ be a fixed point.

(i) u^* is *Lyapunov stable* (or *L-stable*) if for all $\varepsilon > 0$, there is some $\delta > 0$ such that $\|\varphi(t; u_0) - u^*\| < \varepsilon$ for all $t \geqslant 0$ whenever $\|u_0 - u^*\| < \delta$. Put in different terms, for every $\varepsilon > 0$ there is some $\delta > 0$ such that $\Gamma^+(u_0) \subset B_\varepsilon(u^*)$ for any $u_0 \in B_\delta(u_0)$.

Informally speaking, L-stability means that by starting sufficiently close to u^*, you will stay arbitrarily close to u^*.

(ii) u^* is *unstable* if it is not Lyapunov stable — that is, there is some $\varepsilon > 0$ such that for any $\delta > 0$, there is some $u_0 \in B_\delta(u^*)$ and $t > 0$ such that $\|\varphi(t; u_0) - u^*\| \geqslant \varepsilon$.

(iii) u^* is *ω-attracting* if there is some $\delta > 0$ such that $\lim_{t\to\infty} \varphi(t; u_0) = u^*$ for every $u_0 \in B_\delta(u^*)$. Put in different terms, $B_\delta(u^*) \subset W^s(u^*)$.

(iv) u^* is *asymptotically stable* (sometimes called a *sink*) if it is both Lyapunov stable and ω-attracting.

(v) u^* is *globally asymptotically stable* if it is asymptotically stable and $\lim_{t\to\infty} \varphi(t; u_0) = u^*$ for *every* $u_0 \in \mathbb{R}^n$.

The latter statement is the same as saying that $W^s(u^*) = \mathbb{R}^n$.

(vi) u^* is *repelling* if it is asymptotically stable backwards in time.

An ω-attracting fixed point u^* is a point which “attracts” all points sufficiently close to it — in other words, its stable manifold $W^s(u^*)$ is a neighbourhood of u^* (that is, it contains an open set containing u^*).

6.2 Stability for scalar equations

6.3 Linearized stability

In this and the next section we study the stability of fixed points in terms of the linearization around the fixed point. If u^* is a fixed point, then, according to Taylor's theorem,

$$\dot{u} = F(u) = F(u^*) + DF(u^*)(u - u^*) + g(u - u^*)$$

for some function g such that $g(v) = O(\|v\|^2)$ for v close to 0. Letting $v = u - u^*$ and $A := DF(u^*)$ and noting that $F(u^*) = 0$, we get

$$\dot{v} = Av + g(v). \tag{6.3}$$

The key observation for proving the results in this and the next section is that when v is close to 0, $g(v)$ is very small (of the order of $\|v\|^2$), and therefore most of the behaviour of v (and therefore u) is therefore determined by the linear part Av.

Definition 6.4. Let u^* be a fixed point and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be the eigenvalues of $DF(u^*)$. We refer to $\lambda_1, \dots, \lambda_n$ as *the eigenvalues of* u^*. We say that u^* is *hyperbolic* if none of the eigenvalues of u^* have zero real part. If u^* is hyperbolic then we call it a *saddle* if at least one eigenvalue has negative real part, and at least one has positive real part.

Theorem 6.5. *Let u^* be a hyperbolic fixed point.*

(i) If all of the eigenvalues of u^* have negative real part, then u^* is asymptotically stable.

(ii) If at least one eigenvalue of u^* has positive real part, then u^* is unstable.

(iii) If all of the eigenvalues of u^* have positive real part, then u^* is repulsive.

Proof of Theorem 6.5(i). Letting $v(t) = u(t) - u^*$, we have seen that v satisfies (6.3). Viewing this as a linear, inhomogeneous system with constant coefficients and recalling Duhamel's formula (2.10), we can write

$$v(t) = e^{tA}v_0 + \int_0^t e^{(t-s)A} g(v(s))\, ds$$

where $v_0 = v(0) = u_0 - u^*$. By assumption, $\mathrm{Re}(\lambda_k) < 0$ for all k, so from Theorem 2.4(vii) we deduce that there are constants $\alpha, K > 0$ such that

$$\|e^{tA}\|_{\mathcal{L}} \leqslant Ke^{-\alpha t} \qquad \forall t \geqslant 0,$$

and therefore,

$$\|v(t)\| \leqslant Ke^{-\alpha t}\|v_0\| + K\int_0^t e^{-\alpha(t-s)}\|g(v(s))\|\, ds.$$

Next, by the properties of g, there is some $\delta_0 > 0$ and $C > 0$ such that $\|g(v)\| \leqslant C\|v\|^2$ whenever $\|v\| \leqslant \delta_0$. Let now $\delta_1 \leqslant \delta_0$. If we can show that

$$\|v(s)\| \leqslant \delta_1 \qquad \text{for every } s \geqslant 0, \tag{6.4}$$

we can deduce that

$$\begin{aligned} \|v(t)\| &\leqslant Ke^{-\alpha t}\|v_0\| + KC\int_0^t e^{-\alpha(t-s)}\|v(s)\|^2\, ds \\ &\leqslant Ke^{-\alpha t}\|v_0\| + KC\delta_1\int_0^t e^{-\alpha(t-s)}\|v(s)\|\, ds. \end{aligned}$$

Multiplying the above by $e^{\alpha t}$ and letting $\psi(t) = e^{\alpha t}\|v(t)\|$ yields

$$\psi(t) \leqslant K\|v_0\| + KC\delta_1\int_0^t \psi(s)\, ds.$$

By the integral form of Gronwall's inequality we deduce that $\psi(t) \leqslant K\|v_0\|e^{KC\delta_1 t}$, or equivalently,

$$\|v(t)\| \leqslant K\|v_0\|e^{(KC\delta_1-\alpha)t}$$

Hence, if δ_1 is chosen small enough that $KC\delta_1 - \alpha < 0$, and $\|v_0\|$ is small enough that $K\|v_0\| \leqslant \delta_1$, then $\|v(t)\| \leqslant \delta_1$ for all t, so (6.4) is true.

This now proves that u^* is ω-attracting, since if $\|u_0 - u^*\| \leqslant \delta_1/K$ then $\|u(t) - u^*\| \leqslant K\|v_0\|e^{(KC\delta_1-\alpha)t} \to 0$ as $t \to \infty$. It also shows that u^* is Lyapunov stable, since if $\varepsilon > 0$, then choosing $\delta = \min(\delta_1/K, \varepsilon/K)$ and assuming that $\|u_0 - u^*\| < \delta$ yields $\|u(t) - u^*\| \leqslant K\|u_0 - u^*\| \leqslant \varepsilon$ for all $t \geqslant 0$. □

Proof of Theorem 6.5(ii). We will assume, for the sake of simplicity, that $A = DF(u^*)$ is diagonalizable (although the result is still true otherwise). Write $A = R\Lambda R^{-1}$ for $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. As in the previous proof we let $v = u - u^*$, which solves (6.3). If $w = R^{-1}v$ then

$$\dot{w} = \Lambda w + h(w)$$

where $h(w) = R^{-1}g(Rw)$. Just as for g, there is some $\delta_0 > 0$ and $C > 0$ such that $\|h(w)\| \leqslant C\|w\|^2$ whenever $\|w\| \leqslant \delta_0$.

Assume that $\lambda_1, \dots, \lambda_n$ have been ordered with decreasing real part, with $m \geqslant 1$ positive real parts and $n - m \geqslant 0$ negative real parts. Thus, $\mathrm{Re}\lambda_1 \geqslant \mathrm{Re}\lambda_2 \geqslant \dots \geqslant \mathrm{Re}\lambda_m > 0 > \mathrm{Re}\lambda_{m+1} \geqslant \dots \geqslant \mathrm{Re}\lambda_n$. Define $\alpha := \min_k |\mathrm{Re}(\lambda_k)| > 0$.

Define

$$\begin{aligned} L(w) &= \frac{1}{2}\Big(\big|w^{(1)}\big|^2 + \dots + \big|w^{(m)}\big|^2 - \big|w^{(m+1)}\big|^2 - \dots - \big|w^{(n)}\big|^2\Big) \\ &= \frac{1}{2}\sum_{k=1}^{n} \mathrm{sign}(\mathrm{Re}(\lambda_k))\big|w^{(k)}\big|^2 \end{aligned}$$

The sign function is defined as

$$\mathrm{sign}(r) = \begin{cases} 1 & \text{if } r \geqslant 0 \\ 0 & \text{if } r < 0 \end{cases}$$

Note in particular that

$$L(w) \leqslant \|w\|^2 \qquad \forall\ w \in \mathbb{R}^n. \tag{6.5}$$

The k-th component $w^{(k)}$ of w satisfies $\dot{w}^{(k)} = \lambda_k \dot{w}^{(k)} + h^{(k)}(w)$, so a straightforward computation yields

$$\frac{d}{dt}\big|w^{(k)}\big|^2 = \frac{d}{dt}\big(w^{(k)}\overline{w^{(k)}}\big) = 2\mathrm{Re}(\lambda_k)\big|w^{(k)}\big|^2 + 2\mathrm{Re}\big(w^{(k)}h^{(k)}(w)\big).$$

Hence,

$$\begin{aligned} \frac{d}{dt}L(w(t)) &= \sum_{k=1}^{n} \mathrm{sign}(\mathrm{Re}(\lambda_k))\Big(\mathrm{Re}(\lambda_k)\big|w^{(k)}\big|^2 + \mathrm{Re}\big(w^{(k)}h^{(k)}(w)\big)\Big) \\ &\geqslant \sum_{k=1}^{n}\big|\mathrm{Re}(\lambda_k)\big|\big|w^{(k)}\big|^2 - \sum_{k=1}^{n}\big|w^{(k)}\big|\big|h^{(k)}(w)\big| \\ &\geqslant \sum_{k=1}^{n}\alpha\big|w^{(k)}\big|^2 - \|w\|\|h(w)\| \\ &= \alpha\|w\|^2 - \|w\|\|h(w)\|. \end{aligned}$$

Let now $\delta > 0$ and set $w_0 = \begin{pmatrix}\delta & 0 & \dots & 0\end{pmatrix}^\mathsf{T}$, so that $L(w_0) = \delta^2/2 > 0$. If $\delta < \delta_0$ then $\|w_0\| = \delta < \delta_0$, so by our upper bound on $h(w)$ we get

$$\begin{aligned} \frac{d}{dt}L(w(t)) &\geqslant \alpha\|w\|^2 - \|w\|\|h(w)\| \geqslant \alpha\|w\|^2 - C\|w\|^3 = \|w\|^2\big(\alpha - C\|w\|\big) \\ &> \frac{\alpha}{2}\|w\|^2 \end{aligned}$$

provided $\|w\| < \alpha/(2C)$. Thus, no matter how small δ is, $L(w(t))$ (and hence also $\|w(t)\|$, because of (6.5)) will continue to increase until either $\|w(t)\| = \delta_0$ or $\|w(t)\| = \alpha/(2C)$. In particular, we cannot force $\|w(t)\|$ to be arbitrarily small for all $t \geqslant 0$. $\square$

Proof of 6.5(iii). In the time-reversed problem, u^* has eigenvalues with strictly negative real part, so by part (i) it is asymptotically stable. Hence, u^* in the original problem is repulsive. $\square$

Example 6.6. If the fixed point is non-hyperbolic (i.e., one eigenvalue has zero real part) then we cannot deduce anything about stability from the eigenvalues alone. Indeed, let $n = 2$ and consider the two problems

$$\dot{u} = F(u), \qquad \dot{v} = G(v),$$

where

$$F(u) = Au - \|u\|^2, \qquad G(v) = Av + \|v\|^2, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The origin $u^* = v^* = 0$ is a fixed point for both problems, and the linearization of both equations around 0 is $\dot{w} = Aw$. The eigenvalues of this system are $\pm i$. We claim that u^* is asymptotically stable, while v^* is repulsive. Indeed,

$$\frac{d}{dt}\|u\|^2 = 2u^\mathsf{T} Au - 2\|u\|^2 = -2\|u\|^2$$

(since $u^\mathsf{T} Au = 0$ for any vector u), so $\|u(t)\|^2 = \|u_0\|^2 e^{-2t} \to 0$ as $t \to \infty$. On the other hand,

$$\frac{d}{dt}\|v\|^2 = 2\|v\|^2$$

so $\|v(t)\|^2 = \|v_0\|^2 e^{2t} \to \infty$ as $t \to \infty$ for any $v_0 \neq 0$. $\triangle$

6.4 The stable manifold theorem

Definition 6.7. Let u^* be a fixed point. The *local stable manifold for u^* of radius $\delta > 0$* is the set

$$W_\delta^s(u^*) := \big\{u_0 \in W^s(u^*) : \|\varphi(t; u_0) - u^*\| < \delta \ \forall\ t \geqslant 0\big\}.$$

Similarly, The *local unstable manifold for u^* of radius $\delta > 0$* is the set

$$W_\delta^u(u^*) := \big\{u_0 \in W^u(u^*) : \|\varphi(t; u_0) - u^*\| < \delta \ \forall\ t \leqslant 0\big\}.$$

Theorem 6.8 (The stable manifold theorem)**.** *Let u^* be a hyperbolic fixed point. Then there is some $\delta > 0$ such that $W_\delta^s(u^*)$ and $W_\delta^u(u^*)$ are manifolds that are tangent to $\mathbb{E}^s$ and $\mathbb{E}^u$ at u^*, respectively.*

Sketch of proof. We will sketch a proof for $W^s(u^*)$ only, and we will assume that $n = 2$ and u^* is a saddle. Then the eigenvalues of u^* are real and satisfy $\lambda_1 < 0 < \lambda_2$. Let $A = R\Lambda R^{-1}$ be the diagonalization of A. Recalling that $v := u - u^*$ satisfies (6.3), we have

$$\dot{w} = \Lambda w + h(w) \tag{6.6}$$

where $w = R^{-1}v$ and $h(w) = R^{-1}g(Rw)$. We aim to show that $W_\delta^s(u^*)$ can be parametrized as a graph over the line $\mathbb{E}^s$, and is tangent to $\mathbb{E}^s$ at $u = u^*$. Our change of variables $u \mapsto w$ maps $u = u^*$ to $w^* = 0$ and $\mathbb{E}^s$ to the $w^{(1)}$-axis. For the sake of notational simplicity we label the $w^{(1)}$ component of w as $a \in \mathbb{R}$. Thus, we wish to show that there is a curve $(a, \psi(a))$ which (for small $|a|$) parametrizes the stable manifold of the fixed point $w^* = 0$ of (6.6). The statement that $W^s(u^*)$ is tangent to $\mathbb{E}^s$ at u^* is then equivalent to stating that $\psi(0) = 0$ and that the curve $(a, \psi(a))$ is tangent to the a-axis at $a = 0$, which again is equivalent to $\psi'(0) = 0$.

Write w as a function of the first component a of w_0, $w = w(t; a)$, and use a variant of Duhamel's formula:

$$\begin{aligned} w(t;a) &:= \begin{pmatrix} e^{\lambda_1 t} a \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{\lambda_1 (t-s)} & 0 \\ 0 & 0 \end{pmatrix} h(w(s;a))\, ds \\ &\quad - \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{\lambda_2 (t-s)} \end{pmatrix} h(w(s;a))\, ds \\ &= \begin{pmatrix} e^{\lambda_1 t} a \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{\lambda_1 (t-s)} h^{(1)}(w(s;a)) \\ 0 \end{pmatrix} ds \\ &\quad - \int_t^\infty \begin{pmatrix} 0 \\ e^{\lambda_2 (t-s)} h^{(2)}(w(s;a)) \end{pmatrix} ds. \end{aligned} \tag{6.7}$$

(The existence of a function satisfying (6.7) for a in an interval $(-\delta, \delta)$ follows the same argument as the existence theorem in Chapter 3.) The proof that the above function satisfies (6.6) is similar to that of Duhamel's formula. We also see that

$$w(0;a) = \begin{pmatrix} a \\ \psi(a) \end{pmatrix}, \qquad \text{where } \psi(a) := -\int_0^\infty e^{-\lambda_2 s} h^{(2)}(w(s;a))\, ds.$$

We first claim that there is some $C > 0$ such that $\|w(t;a)\| \leqslant C|a|e^{\lambda_1 t}$ for $t \geqslant 0$. The proof is similar to that of Theorem 6.5(i). From this is follows that $w(t;a) \to w^* = 0$ as $t \to \infty$, so that $w(0;a) \in W^s(w^*)$ for every a. It also follows that $w(0;0) = 0$, so that $\psi(0) = 0$. Next, we have

$$\psi'(a) = -\int_0^\infty e^{-\lambda_2 s} Dh^{(2)}(w(s;a)) \cdot \frac{\partial w}{\partial a}(s;a)\, ds.$$

But $w(s;0) = 0$, so $Dh^{(2)}(w(s;0)) = Dh^{(2)}(0) = 0$ (the last equality following from the fact that h is the Taylor remainder term, which is quadratic near 0). Hence, $\psi'(0) = 0$. □

6.5 The Hartman–Grobman theorem

Definition 6.9. A function $H : U \to V$ (for open sets $U, V \subset \mathbb{R}^n$) is a *homeomorphism* if H is bijective and both H and its inverse H^{-1} are continuous.

Theorem 6.10 (The Hartman–Grobman theorem)**.** *Let u^* be a hyperbolic fixed point of* (1)*, let φ be the flow of* (1)*, and let $\psi(t; v_0) := e^{tA} v_0$ (where $A := \nabla F(u^*)$) is the flow of the linearized equation. Then there is a homeomorphism $H : U \to V$ such that*

$$H(\varphi(t;u_0)) = \psi(t; H(u_0)) \qquad \forall\, u_0 \in U,\ t \in I_0 \tag{6.8}$$

where $I_0 \subset \mathbb{R}$ is an interval containing $t = 0$, and $U, V \subset \mathbb{R}^n$ are open sets containing u^* and 0*, respectively.*

Sketch of proof. First, by making the change of variables $u \mapsto u - u^*$, we may assume that $u^* = 0$.

Let us denote $\varphi_t(u_0) = \varphi(t; u_0)$ and $\psi_t(u_0) = \psi(t; u_0)$, and note that (6.8) can be equivalently written

$$H \circ \varphi_t = \psi_t \circ H \tag{6.8'}$$

(where "$\circ$" means composition, e.g. $\psi_t \circ H(u_0) = \psi_t(H(u_0))$). Let us assume for the moment that we have found a homeomorphism $\bar{H}: U \to V$, where $U, V \subset \mathbb{R}^n$ are neighbourhoods of 0, satisfying

The time $t = 1$ has been chosen merely for convenience.

$$\bar{H} \circ \varphi_1 = \psi_1 \circ \bar{H} \qquad \text{in } U \tag{6.9}$$

(compare with (6.8')). Since φ_1 is invertible with inverse φ_{-1}, we can rewrite (6.9) as

$$\psi_1 \circ \bar{H} \circ \varphi_{-1} = \bar{H}. \tag{6.9'}$$

Next, define

$$H := \int_0^1 \psi_{-s} \circ \bar{H} \circ \varphi_s \, ds \tag{6.10}$$

Then

$$\begin{aligned}
\psi_t \circ H &= e^{tA} \int_0^1 e^{-sA} \bar{H} \circ \varphi_s \, ds \\
&= \int_0^1 e^{(t-s)A} \bar{H} \circ \varphi_s \, ds \\
&= \int_0^1 \psi_{t-s} \circ \bar{H} \circ \varphi_{s-t} \, ds \circ \varphi_t \\
&= \int_{-t}^{1-t} \psi_{-s} \circ \bar{H} \circ \varphi_s \, ds \circ \varphi_t \\
&= \int_{-t}^{0} \psi_{-s} \circ \bar{H} \circ \varphi_s \, ds \circ \varphi_t + \int_0^{1-t} \psi_{-s} \circ \bar{H} \circ \varphi_s \, ds \circ \varphi_t \\
&= \int_{1-t}^{1} \psi_{1-s} \circ \bar{H} \circ \varphi_{s-1} \, ds \circ \varphi_t + \int_0^{1-t} \psi_{-s} \circ \bar{H} \circ \varphi_s \, ds \circ \varphi_t .
\end{aligned}$$

Since $\varphi_{s-t} \circ \varphi_t = \varphi_s$.

Change of variables $s \mapsto s + t$.

Split the integral in two.

Change of variables $s \mapsto s - 1$ in the first integral.

We can write the first integrand as

$$\psi_{1-s} \circ \bar{H} \circ \varphi_{s-1} = \psi_{-s} \circ \psi_1 \circ \bar{H} \circ \varphi_{-1} \circ \varphi_s = \psi_{-s} \circ \bar{H} \circ \varphi_s$$

because of (6.9'). Thus,

$$\begin{aligned}
\psi_t \circ H = &= \int_{1-t}^{1} \psi_{-s} \circ \bar{H} \circ \varphi_s \, ds \circ \varphi_t + \int_0^{1-t} \psi_{-s} \circ \bar{H} \circ \varphi_s \, ds \circ \varphi_t \\
&= \int_0^1 \psi_{-s} \circ \bar{H} \circ \varphi_s \, ds \circ \varphi_t = H \circ \varphi_t .
\end{aligned}$$

It remains to prove the existence of $\bar{H}$. The matrix A is similar to a matrix of the form

$$\begin{pmatrix} N & 0 \\ 0 & P \end{pmatrix}$$

where $N \in \mathbb{R}^{k \times k}$ and $P \in \mathbb{R}^{(n-k)\times(n-k)}$ have only eigenvalues with negative and positive real part, respectively. Indeed, let $A = R \Lambda R^{-1}$ be the Jordan normal form of A. Ordering the eigenvalues $\lambda_1, \dots, \lambda_m$ so that $\text{Re}(\lambda_1) \leqslant \dots \leqslant \text{Re}(\lambda_{m'}) <$

$0 < \mathrm{Re}(\lambda_{m'+1}) \leqslant \ldots \leqslant \mathrm{Re}(\lambda_m)$, we can set $N = \mathrm{diag}(J_1, \ldots, J_{m'})$ and $P = \mathrm{diag}(J_{m'+1}, \ldots, J_m)$. See Section 2.3. Thus, we may assume that A is a matrix of this form. Write $u = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$. Denoting the corresponding components of $\bar{H}$ by $\bar{H}_x, \bar{H}_y$, condition (6.9) now reads

$$\begin{cases} \bar{H}_x\big(\varphi_1(u)\big) = e^N \bar{H}_x(u) \\ \bar{H}_y\big(\varphi_1(u)\big) = e^P \bar{H}_y(u) \end{cases} \tag{6.9''}$$

We can Taylor expand φ_1 to get

$$\varphi_1(u) = \underbrace{\varphi_1(0)}_{=0} + \underbrace{D\varphi_1(0)}_{=e^A} u + \xi(u) = e^A u + \xi(u)$$

where $\|\xi(u)\| \leqslant C\|u\|^2$ for some $C > 0$ for small u. Thus, the second equation of (6.9”), after multiplying by e^{-P}, reads

$$\bar{H}_y(u) = e^{-P}\bar{H}_y\big(e^A u + \xi(u)\big)$$

The proof of existence of such a function $\bar{H}_y$ — which we will not give in full here — goes via the fixed point iteration

$$\bar{H}_{y,1}(u) = u, \qquad \bar{H}_{y,j+1}(u) = e^{-P}\bar{H}_{y,j}\big(e^A u + \xi(u)\big), \qquad j = 1, 2, \ldots.$$

To prove that this iteration is a contraction, one uses the fact that $\xi(u)$ is small for u close to 0, and that $\|e^{-P}\|_{\mathcal{L}} < 1$, because the eigenvalues of $-P$ are negative. A similar argument works for the first component of (6.9”) when written in the form $\bar{H}_x(u) = e^N \bar{H}_x(\varphi_{-1}(u))$. Further details are provided in e.g. [Per01]. □

6.6 Phase portraits, revisited

TODO: Picture-by-picture example on drawing 2d phase portraits, with knowledge of stability of fixed points, stable/unstable manifolds. Lotka–Volterra example?

6.7 Problems

Problem 6.1. Consider the linear system $\dot{u} = Au$. For each of the following choices of A, determine

(i) whether the system is a stable node, unstable node, stable focus, unstable focus, centre, saddle, or neither of the above, and

(ii) whether the fixed point $u^* = 0$ is Lyapunov stable, ω-attracting, asymptotically stable, globally attracting, or repelling.

(a) $\begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 & 5 \\ -2 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} -3 & -2 \\ 5 & 2 \end{pmatrix}$ **(f)** $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

Problem 6.2. Prove Proposition 6.2.

Chapter 7

Analysis via scalar functions

7.1 Conserved quantities

Conserved quantities are sometimes referred to as *energies* or *Hamiltonians*.

Definition 7.1. A function $E\colon \mathbb{R}^n \to \mathbb{R}$ is a *conserved quantity* for (1) if E is constant along orbits of (1), or in other words,

$$\frac{d}{dt} E\big(\varphi(t;u_0)\big) = 0 \qquad \forall\, t \in \mathbb{R}, \forall\, u_0 \in \mathbb{R}^n. \tag{7.1}$$

Recalling the definition of *level sets*,

$$E^{-1}(E_0) = \{v \in \mathbb{R}^n : E(v) = E_0\} \qquad \text{for some } E_0 \in \mathbb{R},$$

we see that the trajectory $\varphi(t;u_0)$ must move along the level set $E^{-1}(E(u_0))$. Using the chain rule, we see that

$$\frac{d}{dt} E\big(\varphi(t;u_0)\big) = \nabla E\big(\varphi(t;u_0)\big) \cdot \dot{\varphi}(t;u_0) = \nabla E\big(\varphi(t;u_0)\big) \cdot F\big(\varphi(t;u_0)\big).$$

Thus, E is a conserved quantity *if and only if*

$$\nabla E(v) \cdot F(v) = 0 \qquad \forall\, v \in \mathbb{R}^n, \tag{7.2}$$

so the trajectory moves in a direction orthogonal to the vector field ∇E. From Calculus we recall that $\nabla E(v)$ is again orthogonal to the level set of E at v, so the trajectory indeed moves along level sets of E.

It can be very difficult, or impossible, to find conserved quantities for a given dynamical system, but sometimes it is possible with a bit of algebra.

Example 7.2. Consider the predator–prey system

$$\begin{cases} \dot{x} = x(a - by) \\ \dot{y} = y(-c + dx) \end{cases}$$

for $a, b, c, d > 0$. We note that this system has two fixed points: $(0,0)$ and $\big(a/b, c/d\big)$. (Derive conserved quantity, plot orbits, make analysis) △

Example 7.3. Particle moving through a conservative force field △

Example 7.4. Pendulum

$$\begin{cases} \dot{\theta} = v \\ \dot{v} = -\frac{g}{\ell} \sin\theta \end{cases} \tag{7.3}$$

△

7.2 Stability analysis by Lyapunov functions

Definition 7.5. Let u^* be a fixed point for (1). A function $L\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is an open set containing u^*, is a *weak Lyapunov function for* u^* if

(i) u^* is a strict minimum:

$$L(u) > L(u^*) \qquad \forall\, u \in U \setminus \{u^*\}. \tag{7.4}$$

(ii) L decreases along the trajectory:

$$\frac{d}{dt} L(\varphi(t; u_0)) \leqslant 0 \qquad \forall\, u_0 \in U, \forall\, t > 0 \text{ such that } \varphi(t; u_0) \in U. \tag{7.5}$$

We say that L is a *(strict) Lyapunov function for* u^* if property (ii) is replaced by $\frac{d}{dt} L(\varphi(t; u_0)) < 0$ for all $u_0 \neq u^*$.

As for conserved quantities, we can apply the chain rule and see that property (ii) is equivalent to

$$\nabla L(v) \cdot F(v) \leqslant 0 \qquad \forall\, v \in U. \tag{7.5'}$$

We note also that if L is a strict Lyapunov function, then

$$\nabla L(v) \cdot F(v) < 0 \qquad \forall\, v \in U \setminus \{u^*\}. \tag{7.6}$$

Remark 7.6. Condition (7.5) is somewhat cumbersome since we have to make sure that we do not evaluate L at points $\varphi(t; u_0)$ outside of U, where L is not defined. We can simplify things by replacing U by a smaller, forward invariant set $V \subset U$, which can be constructed as follows. We note that the constructed set is open, bounded, forward invariant and contains u^*.

Since U is open, there is some $r > 0$ such that $\overline{B}_r(u^*) \subset U$. Let $m := \min_{v \in \partial B_r(u^*)} L(v)$ and let L_0 be any number satisfying $L(u^*) < L_0 < m$. Define the set

A set of this form is a *strict sublevel set* of L.

$$V := \{u \in B_r(u^*) : L(u) < L_0\},$$

which clearly contains u^*, and which is open, since it is the intersection of the open sets $B_r(u^*)$ and $L^{-1}\big((-\infty, L_0)\big)$. We claim that V is forward invariant. Let $u_0 \in V$, and assume for contradiction that there is some $t > 0$ such that $\varphi(t; u_0) \notin V$; we may assume that t is the smallest such value. Then either $\varphi(t; u_0) \notin B_r(u^*)$ or $\varphi(t; u_0) \notin L^{-1}\big((-\infty, L_0)\big)$. In the first case, by continuity of φ, there would have to be some $t_0 \in (0, t)$ such that $\varphi(t; u_0) \in \partial B_r(u^*)$, but using the fact that L decreases along trajectories, we get $m > L(u_0) = L(\varphi(0; u_0)) \geqslant L(\varphi(t_0; u_0)) \geqslant m$, a contradiction. On the other hand, if $\varphi(t; u_0)$ lies in $B_r(u^*)$ but not $L^{-1}\big((-\infty, L_0)\big)$ then $L_0 > L(u_0) \geqslant L(\varphi(t; u_0)) \geqslant L_0$, another contradiction. This proves our claim. △

Theorem 7.7. *Let u^* be a fixed point for* (1).

(i) If there exists a weak Lyapunov function for u^*, then u^* is Lyapunov stable.

(ii) If there exists a Lyapunov function for u^*, then u^* is asymptotically stable.

Proof of (i). Let $L\colon U \to \mathbb{R}$ be a weak Lyapunov function for u^*, and let $\varepsilon > 0$. By letting ε be even smaller, we may assume that $B_\varepsilon(u^*) \subset U$. By the same procedure as in Remark 7.6, we know that there exists some open, forward invariant set $V \subset B_\varepsilon(u^*)$ containing u^*. Since V is open, there is some $\delta > 0$ such that $B_\delta(u^*) \subset V$. For any $u_0 \in B_\delta(u^*)$ we then get $\varphi(t; u_0) \in V \subset B_\varepsilon(u^*)$, which proves the claim. □

Proof of (ii). Let $L\colon U \to \mathbb{R}$ be a Lyapunov function for u^*. As described in Remark 7.6, there exists some open, bounded, forward invariant set $V \subset U$ containing u^*. If $u_0 \in V$ then either $u_0 = u^*$, or $\{L(\varphi(t;u_0))\}_{t \geqslant 0}$ is a decreasing sequence of numbers bounded from below by $L(u^*)$, so there is some number $\overline{L}(u_0) \geqslant L(u^*)$ such that $L(\varphi(t;u_0)) \to \overline{L}(u_0)$ as $t \to \infty$. Then also

$$\nabla L(\varphi(t;u_0)) \cdot F(\varphi(t;u_0)) = \frac{d}{dt} L(\varphi(t;u_0)) \to 0 \qquad \text{as } t \to \infty.$$

The forward orbit at u_0 is bounded, since it lies in the bounded set V, so by Theorem 4.10, $\omega(u_0)$ is nonempty. If $\bar{u} \in \omega(u_0)$ then, by the above computation, $\nabla L(\bar{u}) \cdot F(\bar{u}) = 0$, so by (7.6), we must have $\bar{u} = u^*$. This proves that $\omega(u_0) = \{u^*\}$, which by Theorem 4.10 is equivalent to stating that $\varphi(t;u_0) \to u^*$ as $t \to \infty$. Letting now $\delta > 0$ be such that $B_\delta(u^*) \subset V$ concludes the proof of ω-stability, and hence asymptotic stability. □

Corollary 7.8. *Let u^* be a fixed point for* (1)*, and assume that there exists a Lyapunov function for u^* for the time-reversed problem $\dot{v} = -F(v)$. Then u^* is repelling for* (1).

Example 7.9. $\dot{u} = Au - u\|u\|^2$, $\dot{v} = Av + v\|v\|^2$. Then $L(u) = \|u\|^2$ is a Lyapunov function for $u^* = 0$, and time-reversed for v^*. Stability analysis by linearization doesn't work. △

Theorem 7.10. *Let u^* be a fixed point for* (1) *with a weak Lyapunov function $L\colon U \to \mathbb{R}$. Define the set*

$$Z := \big\{u \in U : \nabla L(u) \cdot F(u) = 0\big\},$$

and assume that for every $u_0 \in Z \setminus \{u^*\}$, there is some $t > 0$ such that $\varphi(t;u_0) \notin Z$. Then u^* is asymptotically stable.

Example 7.11. Apply the above to the pendulum with friction. △

Surprisingly, there is a converse to Theorem 7.7:

Theorem 7.12. *Let u^* be a fixed point for* (1).

(i) If u^* is Lyapunov stable then there exists a weak Lyapunov function for u^*.

(ii) If u^* is asymptotically stable then there exists a Lyapunov function for u^*.

Proof. In both cases u^* is Lyapunov function, so there is some $\delta > 0$ such that $\|\varphi(t;u_0) - u^*\| < 1$ for all $u_0 \in U := B_\delta(u^*)$. Define the function $\lambda\colon U \to \mathbb{R}$ by

$$\lambda(u) := \sup_{s \geqslant 0} \|\varphi(t;u) - u^*\|.$$

Then $\lambda(u) \in [0,1]$ for all $u \in U$, and it can be shown that λ is continuous (we skip the proof here). Moreover,

$$\lambda(u^*) = 0, \qquad \text{and if } u \neq u^* \text{ then } \lambda(u) > 0.$$

If $t > 0$ and $u \in U$ then

$$\begin{aligned}\lambda(\varphi(t;u)) &= \sup_{s \geqslant 0} \|\varphi(s;\varphi(t;u)) - u^*\| = \sup_{\tau \geqslant s} \|\varphi(\tau;u) - u^*\| \\ &\leqslant \sup_{\tau \geqslant 0} \|\varphi(\tau;u) - u^*\| = \lambda(u).\end{aligned}$$

It follows that λ is a weak Lyapunov function for u^*.

Assume now that u^* is asymptotically stable. Then there is some $\varepsilon > 0$ such that $\varphi(t; u_0) \to u^*$ as $t \to \infty$ for all $u_0 \in B_\varepsilon(u^*)$. Let $V := U \cap B_\varepsilon(u^*)$, and define

$$L(u) := \int_0^\infty e^{-s} \lambda(\varphi(s; u))\, ds \qquad \text{for } u \in V.$$

We claim that L is a strict Lyapunov function. Indeed, if $t \geqslant 0$ and $u \in V$ then

$$L(\varphi(t; u)) = \int_0^\infty e^{-s} \lambda(\varphi(t + s; u))\, ds \leqslant \int_0^\infty e^{-s} \lambda(\varphi(s; u))\, ds = L(u)$$

(since λ is a weak Lyapunov function), and hence L is a weak Lyapunov function. We claim that there is strict inequality in the above computation whenever $u \neq u^*$. Indeed, if there is equality in the above computation for some choice of t and u then necessarily $\lambda(\varphi(t + s; u)) = \lambda(\varphi(s; u))$ for all $s \geqslant 0$. Thus,

$$\lambda(u) = \lambda(\varphi(0; u)) = \lambda(\varphi(t; u)) = \cdots = \lambda(\varphi(kt; u))$$

for any $k \in \mathbb{N}$. But $\varphi(kt; u) \to u^*$ as $k \to \infty$, so

$$\lambda(u) = \lim_{k\to\infty} \lambda(\varphi(kt; u)) = \lambda(u^*) = 0,$$

so $u = u^*$. □

7.3 Gradient flows

Definition 7.13. We say that a dynamical system is a *gradient flow* if it can be written as

$$\dot{u} = -\nabla G(u) \tag{7.7}$$

for some C^2 function $G\colon \mathbb{R}^n \to \mathbb{R}$.

Gradient flows are also called *gradient systems*.

We make several observations:

1. The minus sign in (7.7) is purely for convenience. If your system is of the form $\dot{u} = \nabla G(u)$, then replacing G by $-G$ will put it in the form (7.7).
2. The fixed points for a gradient flow are precisely the *critical points* of G, i.e. the points u^* where $\nabla G(u^*) = 0$.
3. Recalling that the level sets of a function are orthogonal to its gradient, we see that the trajectories of a gradient flow are orthogonal to its level curves.

The proof of the following result is left as an exercise:

Proposition 7.14. *The solution of the gradient flow* (7.7) *satisfies*

$$\frac{d}{dt} G(\varphi(t; u_0)) < 0$$

for all u_0 that are not fixed points.

Example 7.15. $G(x, y) = x^2(x - 1)^2 + u^2$. TODO: Find fixed points, plot level curves, recollecting the above observation about level curves. △

Theorem 7.16. *Let u^* be an* isolated *fixed point for the gradient flow* (7.7)*, that is, a point where $\nabla G(u^*) = 0$, and such that there are no other such points in some neighbourhood of u^*. Then:*

(i) If u^* is a local minimum for G, then u^* is asymptotically stable.

(ii) If u^* is a local maximum for G, then u^* is repelling.

(iii) If u^* is a saddle for G then u^* is a saddle fixed point; in particular, it is unstable.

Proof.

(i) Let $\varepsilon > 0$ be such that G is the only fixed point in $B_\varepsilon(u^*)$. Since we have assumed that u^* is both an isolated fixed point and a local minimum, it is also a strict local minimum, $G(u^*) < G(u)$ for all $u \in B_\varepsilon(u^*) \setminus \{u^*\}$. Then G is a Lyapunov function for u^* in $U := B_\varepsilon(u^*)$, so the result follows from Theorem 7.7(ii).

(ii) The argument is similar to (i).

(iii) Let $\varepsilon > 0$. Since u^* is a saddle, there is for every $\delta > 0$ some $u_0 \in B_\delta(u^*)$ such that $G(u_0) < G(u^*)$. In particular, the closed set $E := \overline{B}_\varepsilon(u^*) \setminus G^{-1}\big((G(u_0), \infty)\big)$ is nonempty and contains u_0. Let $K := \min_{u \in E} \|\nabla G(u)\|$, which (for sufficiently small ε) is strictly positive since u^* is an isolated critical point. We have

$$\frac{d}{dt} G(\varphi(t; u_0)) = -\|\nabla G(\varphi(t; u_0))\|^2 \leqslant -K^2$$

for every $t > 0$ such that $\varphi(t; u_0) \in E$, so $G(\varphi(t; u_0)) \leqslant G(u_0) - K^2 t$ for all such $t > 0$. But G is lower-bounded in E, so φ must exit E sooner or later. The only way that φ can exit E is by exiting $\overline{B}_\varepsilon(u^*)$. This proves that u^* is unstable.

□

Remark 7.17. We could have simplified the proof of Theorem 7.16 greatly by assuming that u^* is hyperbolic — that is, none of the eigenvalues of the Hessian matrix $\nabla^2 G(u^*)$ have zero real part. (In fact, this matrix is symmetric and hence only has real eigenvalues.) In that case, we could appeal to Theorem 6.5 on stability via linearization. △

Although we cannot in general show that a gradient flow always converges to a fixed point, we can give a rather concrete characterization of its ω-limit points.

Theorem 7.18. *Consider a gradient flow* (7.7) *and let $u_0 \in \mathbb{R}^n$. Then all ω-limits of u_0 are fixed points and have the same G value.*

Proof. If u_0 is itself a fixed point then the conclusion is automatic, so assume that u_0 is not a fixed point. Let $u^* \in \omega(u_0)$, and let $t_1 < t_2 < \cdots \to \infty$ be such that $\varphi(t_k; u_0) \to u^*$ as $k \to \infty$. Then $G(\varphi(t_1; u_0)) < G(\varphi(t_2; u_0)) < \ldots$, and $G(\varphi(t_k; u_0)) \to G(u^*)$ as $k \to \infty$. Then also $G(\varphi(t; u_0)) \to G(u^*)$ as $t \to \infty$, since $t \mapsto G(\varphi(t; u_0))$ is a monotonously decreasing function. In particular,

$$0 = \lim_{t\to\infty} \frac{d}{dt} G(\varphi(t; u_0)) = \lim_{t\to\infty} -\|\nabla G(\varphi(t; u_0))\|^2 = -\|\nabla G(u^*)\|^2,$$

showing that $\nabla G(u^*) = 0$, and therefore u^* is a fixed point. Since $G(\varphi(t; u_0))$ converges to $G(u^*)$ as $t \to \infty$ for any ω-limit u^*, all ω-limits must have the same G value. □

7.4 Hamiltonian systems

Definition 7.19. A system with unknowns $q, p: \mathbb{R} \to \mathbb{R}^m$ (for some $m \geqslant 1$) which is of the form

$$\begin{cases} \dot{q} = \nabla_p H(q, p) \\ \dot{p} = -\nabla_q H(q, p) \end{cases} \tag{7.8}$$

for some C^2 function $H: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, is called a *Hamiltonian system*. The function H is the *Hamiltonian function* or *energy* of (7.8). The variables q and p are called the *position* and *momentum*.

Also called *generalized position* and *generalized momentum*, if the variables do not refer to actual position and momentum of some particle

Example 7.20. Let $H(q, p) = \frac{1}{2}p^2 - \frac{g}{\ell}\cos q$, the energy of the pendulum in Example 7.4. Then $\nabla_p H = p$ and $\nabla_q H = \frac{g}{\ell}\sin q$, so the system

$$\begin{cases} \dot{q} = p \\ \dot{p} = -\frac{g}{\ell}\sin q, \end{cases}$$

which we recognize as the pendulum system (7.3), is a Hamiltonian system. △

In a Hamiltonian system we get at least one conserved quantity "for free":

Proposition 7.21. *The Hamiltonian H is a conserved quantity for* (7.8)*.*

Proof. We use the chain rule to obtain

$$\begin{aligned} \frac{d}{dt}H(q, p) &= \nabla_q H(q, p) \cdot \dot{q} + \nabla_p H(q, p) \cdot \dot{y} \\ &= \nabla_q H(q, p) \cdot \nabla_p H(q, p) - \nabla_p H(q, p) \cdot \nabla_q H(q, p) = 0. \end{aligned}$$

□

Using Liouville's Theorem from Section 4.4, we can show that Hamiltonian systems are *area preserving*.

Proposition 7.22. *Let φ be the flow of the Hamiltonian system* (7.8)*. If $D \subset \mathbb{R}^{2m}$ is any bounded set and $D(t) := \varphi(t; D)$, then the volume of $D(t)$ is constant for all t.*

Proof. As in Corollary 4.14, let $V(t) := \mathrm{vol}(D(t))$. We have

$$\mathrm{div}F(v) = \sum_{i=1}^m \frac{\partial F^{(i)}}{\partial q^{(i)}}(v) + \frac{\partial F^{(i+m)}}{\partial p^{(i)}}(v) = \sum_{i=1}^m \frac{\partial^2 H}{\partial p^{(i)} \partial q^{(i)}} - \frac{\partial^2 H}{\partial q^{(i)} \partial p^{(i)}} = 0.$$

Thus, by Corollary 4.14, $\dot{V}(t) = \int_{D(t)} \mathrm{div}F(v)\,dv = 0$, so $V(t) = V(0)$ for all t. □

The property of preserving area (or volume) in $\mathbb{R}^{2m}$ is somewhat abstract and is not important in itself. However, it hints at a much more powerful structural property of Hamiltonian systems, namely symplecticity, which we explore in the next section. As it turns out, this property is equivalent to being a Hamiltonian system, and by luck, it is a property which is rather simple to check for numerical methods.

7.4.1 Symplectic maps

Assume for the moment that $m = 1$. If $u_1 = (q_1\ p_1)^\mathsf{T}$ and $u_2 = (q_2\ p_2)^\mathsf{T}$ are given vectors then the area of the parallelogram spanned by u_1 and u_2 is given by

$$\omega(u_1, u_2) = q_1 p_2 - q_2 p_1 = u_1^\mathsf{T} J u_2, \qquad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let now $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function $g(u) = Au$ for some $A \in \mathbb{R}^{2\times 2}$. We say that g (or A) is *symplectic* if it preserves the area of all parallelograms — that is, if

$$\omega\big(g(u_1), g(u_2)\big) = \omega(u_1, u_2) \qquad \forall\ u_1, u_2 \in \mathbb{R}.$$

Inserting $g(u) = Au$, we see that g is symplectic *if and only if* $u_1^\mathsf{T} A^\mathsf{T} J A u_2 = u_1^\mathsf{T} J u_2 \mathsf{T}$ for all $u_1, u_2 \in \mathbb{R}^2$, which is possible if and only if

$$A^\mathsf{T} J A = J. \tag{7.9}$$

Problem 7.1. Show that (7.9) holds if and only if $\det(A) = 1$.

Problem 7.2. Let

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

for some $\theta \in [0, 2\pi)$. Show that A and B are symplectic, and that C is not. For each of the matrices $D = A, B, C$, a drawing of some set $\Omega \subset \mathbb{R}^2$ (say, the unit square) and its image $D\Omega = \{Du : u \in \Omega\}$, and interpret your results.

Let now $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an arbitrary, possibly nonlinear function. Noting that the matrix A in (7.9) is the Jacobian of g, we say that g is *symplectic* if the identity (7.9) is true for the matrix $A = \nabla g(u)$, for every $u \in \mathbb{R}^2$. More generally:

Definition 7.23. A C^1 function $g\colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is *symplectic* if (7.9) holds for $A = \nabla g(u)$ for all $u \in \mathbb{R}^{2m}$, where now

$$J := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}. \tag{7.10}$$

We now investigate the flow $\varphi(t; u)$ of the Hamiltonian system (7.8). First, observe that we can write the system (7.8) as

$$\dot{u} = J \nabla H(u) \tag{7.11}$$

where $\nabla H(q, p) = \big(\frac{\partial H}{\partial q}(q, p)\ \ \frac{\partial H}{\partial p}(q, p)\big)^\mathsf{T}$.

Theorem 7.24 (Poincaré, 1899)**.** *Let φ be the flow of the Hamiltonian system* (7.8). *Then for every $t \in \mathbb{R}$, the function $u \mapsto \varphi(t; u)$ is symplectic.*

Proof. We need to check that for every fixed time $t \in \mathbb{R}$ and $u \in \mathbb{R}^{2m}$, the matrix $A(t) := \nabla\varphi(t; u)$ satisfies (7.9). If $t = 0$ then $\varphi(t; u) = u$, so $A(0) = I$, which clearly satisfies (7.9). For a general $t \in \mathbb{R}$ we see from (7.11) that

$$\begin{aligned} \frac{d}{dt} A(t) &= \nabla\big(\tfrac{d}{dt}\varphi(t; u)\big) = \nabla\big(J \nabla H(\varphi(t; u))\big) \\ &= J \nabla^2 H(\varphi(t; u)) \nabla\varphi(t; u) = JMA \end{aligned}$$

where $M(t) := \nabla^2 H(\varphi(t;u))$, the Hessian matrix of H, which consists of all second-order partial derivatives of H. Hence,

$$\begin{aligned}\frac{d}{dt}\big(A^\mathsf{T} JA\big) &= \big(\tfrac{d}{dt}A\big)^\mathsf{T} JA + A^\mathsf{T} J\big(\tfrac{d}{dt}A\big) \\ &= A^\mathsf{T} M^\mathsf{T} J^T JA + A^\mathsf{T} JJMA \\ &= A^\mathsf{T} MA - A^\mathsf{T} MA \\ &= 0\end{aligned}$$

because M is symmetric and $J^{-1} = J^\mathsf{T} = -J$. Since $A^\mathsf{T} JA = J$ when $t = 0$, we conclude that $A^\mathsf{T} JA = J$ for every $t \in \mathbb{R}$. □

Perhaps even more surprising is the fact that Poincaré's theorem holds in the converse direction:

Theorem 7.25. *Let $\varphi\colon \mathbb{R} \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ be the flow of some ODE and assume that for each $t \in \mathbb{R}$, the map $u \mapsto \varphi(t;u)$ is symplectic. Then there is a function $\tilde{H}\colon \mathbb{R}^{2m} \to \mathbb{R}$ such that φ is the flow of the ODE with Hamiltonian $\tilde{H}$.*

Proof. We give a proof for $m = 1$ only. Write the components of F as $F(q,p) = \begin{pmatrix} f(q,p) \\ g(q,p)\end{pmatrix}$. For a fixed $u \in \mathbb{R}^2$ we can let $A(t) := \nabla\varphi(t;u)$ and, as in the proof of Theorem 7.24, find that

$$\frac{d}{dt}A(t) = \nabla F(\varphi(t;u))A(t).$$

Since $A^\mathsf{T} JA = J$ for all t, we can differentiate and get

$$0 = \frac{d}{dt}\big(A^\mathsf{T} JA\big) = A^\mathsf{T}\big(\nabla F^\mathsf{T} J + J\nabla F\big)A.$$

Since $u \mapsto \varphi(t;u)$ is invertible, the matrix $A(t) = \nabla\varphi(t;u)$ is always invertible, so the above is equivalent to $J\nabla F = -\nabla F J$. Writing out the components of this matrix equation, we find that

$$\frac{\partial f}{\partial q}(q,p) = -\frac{\partial g}{\partial p}(q,p).$$

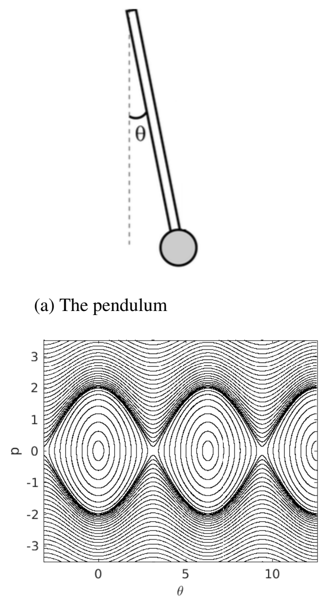
But these are precisely the conditions that ensure that there is a function $\tilde{H}\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(q,p) = \frac{\partial \tilde{H}}{\partial p}(q,p)$ and $g(q,p) = -\frac{\partial \tilde{H}}{\partial q}(q,p)$. □

Thus, in a certain sense, the flow map φ is symplectic *if and only if* the corresponding ODE is Hamiltonian. The proof of the above theorem is outside the scope of these notes.

Example 7.26. Consider a pendulum of length $L > 0$ hanging from a frictionless joint. We can describe its position by the angle θ that the pendulum makes with the downward vertical, see Figure 7.1(a). Using Newton's second law, one can show that the angle behaves according to the ODE

$$L\ddot{\theta} = -g\sin\theta,$$

where $g > 0$ is the gravitational constant. Letting $p = L\dot{\theta}$, we can write the above ODE as a Hamiltonian system with unknown (p,θ) and Hamiltonian function $H(p,\theta) =$



(a) The pendulum

(b) Phase portrait

Figure 7.1: The pendulum in Example 7.26.

$\frac{1}{2L} p^2 - g\cos(\theta)$. A phase portrait of this Hamiltonian system can be seen in Figure 7.1(b). Figure 7.2 shows the evolution of the flow over time, along with a superimposed image where each pixel follows the flow. Although the image is severely distorted over time, its area (and the area of any section of the image) is preserved. △

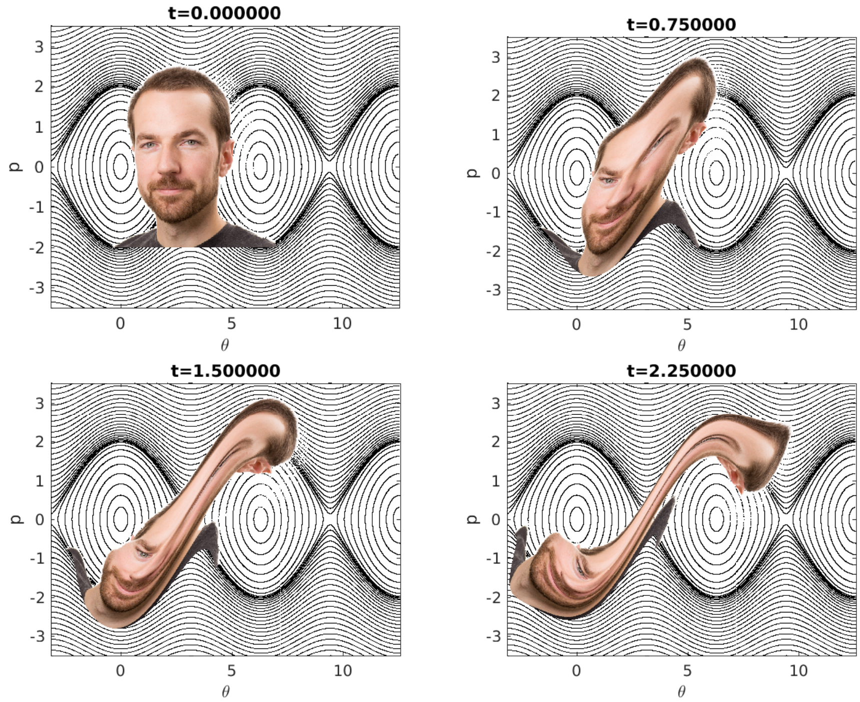


Figure 7.2: The author is transported with the flow φ_t, but his area is preserved over time. The black curves are orbits of the flow.

7.5 Problems

Problem 7.3. Let $G: \mathbb{R}^n \to \mathbb{R}$ be a conserved quantity for (1). Show that G is a Lyapunov function for all of its strict local minima. Be sure to specify the set U in Definition 7.5. (Recall that a point $u^* \in \mathbb{R}^n$ is a *strict local minimum* if there is some $\delta > 0$ such that $G(u^*) < G(u)$ for all $u \in B_\delta(u^*) \setminus \{u^*\}$.)

Problem 7.4. Consider the linear system $\dot{u} = Au$ for $A \in \mathbb{R}^{2\times 2}$. Determine for what A this system is (i) a gradient system and (ii) a Hamiltonian system. Can the system be both?

Problem 7.5. Determine whether each of the following systems are (i) gradient flow, (ii) Hamiltonian systems, or (iii) neither. In case (i) and (ii) use this information to draw a phase portrait. (Use a computer to draw a contour plot, if needed — but try to do it by hand first.)

(**a**) $\dot{x} = -2xy^2$, $\dot{y} = -2x^2y$

(**b**) $\dot{x} = y - 3$, $\dot{y} = 2 - x$

(c) $\dot{x} = 2xy - y^2$, $\dot{y} = x^2 y$

(d) $\dot{x} = x^2 y$, $\dot{y} = -xy^2$

(e) $\dot{x} = x^2 - 1$

(f) $\dot{x} = -\cos(x)\cos(y)$, $\dot{y} = -\sin(x)\sin(y)$

Chapter 8

Periodic orbits

In this chapter we study periodic orbits, that is, orbits $\Gamma(u_0)$ for which there is some $T > 0$ such that $\varphi(T; v_0) = v_0$ for every $v_0 \in \Gamma(u_0)$. Necessarily then, the curve $\Gamma(u_0)$ is a closed, simple curve (a curve looping back on itself and which never crosses itself).

8.1 Stability of periodic orbits

In the same way that we can ask whether a fixed point is stable, we can ask whether a periodic orbit is stable.

Definition 8.1. Let $\gamma = \Gamma(u_0)$ be a periodic orbit.

See Definition A.1 in the appendix for the definition of dist.

- γ is *Lyapunov stable* (or *L-stable*) if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that $\text{dist}(\varphi(t; v_0), \gamma) < \varepsilon$ for all $t > 0$ whenever $\text{dist}(v_0, \gamma) < \delta$.
- γ is *orbitally ω-attracting* if there is some $\delta > 0$ such that if $\text{dist}(v_0, \gamma) < \delta$ then $\varphi(t; v_0) \to \gamma$ as $t \to \infty$.
- γ is *orbitally asymptotically attracting* if it is both Lyapunov stable and orbitally ω-attracting.
- γ is *orbitally repelling* if it is orbitally asymptotically attracting backwards in time.

Example 8.2. $\dot{u} = Au$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ △

Example 8.3. ... △

8.2 The Poincaré–Bendixson theorem

Theorem 8.4 (The Poincaré–Bendixson theorem)**.** *Let $n = 2$ and assume that $\Gamma_+(u_0)$ is bounded. Then either*

A result of this form, where exactly one of two alternatives is true, is a *dichotomy*.

(i) $\omega(u_0)$ contains a fixed point

(ii) $\omega(u_0)$ is a periodic orbit.

Similarly, if $\Gamma_-(u_0)$ is bounded, then the same dichotomy applies to $\alpha(u_0)$. We postpone the proof of the theorem until the next section, and note here some consequences of the theorem. The first corollary is typical for the types of results which follow from the Poincaré–Bendixson theorem.

Corollary 8.5. *Let $C \subset \mathbb{R}^2$ be a closed, bounded and forward invariant set which does not contain any fixed points. Then for every $u_0 \in C$, the trajectory $\varphi(t; u_0)$ converges to a periodic orbit as $t \to \infty$.*

Proof. If $u_0 \in C$ then $\Gamma_+(u_0) \subset C$, since C is forward invariant, and therefore also $\omega(u_0) \subset C$, since C is closed. But C does not contain any fixed points, so by the Poincaré–Bendixson theorem, $\omega(u_0)$ must be a periodic orbit. □

We can also use the idea behind the Poincaré–Bendixson theorem to prove stability of periodic orbits.

Corollary 8.6. *Assume that $u_0 \in \mathbb{R}^2$ does not lie on a periodic orbit, but that $\omega(u_0)$ is periodic. Then the periodic orbit $\omega(u_0)$ is asymptotically stable from the side on which u_0 lies.*

Consult the proof of this result in the next section for a precise formulation.

The next result is somewhat technical, but is needed in the result that follows. We postpone the proof of this result to the next section.

Lemma 8.7. *Let $n = 2$, let $u_0 \in \mathbb{R}^2$, and assume that $\Gamma(u_0)$ is bounded. Then either:*

(i) *u_0 is a fixed point*

(ii) *$\Gamma(u_0)$ is periodic*

(iii) *At least one of $\alpha(u_0)$ and $\omega(u_0)$ is a periodic orbit, and in this case, $\alpha(u_0)$ and $\omega(u_0)$ are disjoint*

(iv) *Both $\alpha(u_0)$ and $\omega(u_0)$ contain a fixed point.*

Corollary 8.8. *Let $n = 2$ and let $\Gamma(u_0)$ be a periodic orbit which encloses the open set $U \subset \mathbb{R}^2$. Then U must contain a fixed point.*

By "enclose U", we mean that U is an open, bounded set with boundary $\Gamma(u_0)$. (The existence and uniqueness of this set is given by Jordan's curve theorem; see the appendix.)

Proof. Assume conversely that U does not contain a fixed point. If U contains a periodic orbit, then it is possible to choose a "smallest" periodic orbit $\Gamma(v_0) \subset U$ so that the set $V \subset U$ which it encloses does not itself contain any periodic orbits. See Lemma 8.15 in the next section for a proof of this fact.

Since V is enclosed by a periodic orbit, it is invariant, so $\omega(w_0) \subset \overline{V}$ for every $w_0 \in V$. We assumed that V does not contain fixed points, so by the Poincaré–Bendixson theorem, $\omega(w_0)$ must be a periodic orbit, and again by assumption, the only periodic orbit in $\bar{V}$ is $\Gamma(u_0)$. Hence, $\omega(w_0) = \Gamma(u_0)$ for all $w_0 \in V$. The exact same argument applies to the α-limits, so $\alpha(w_0) = \Gamma(u_0)$ for all $w_0 \in V$. But this contradicts Lemma 8.7. □

8.3 Proof of the Poincaré–Bendixson theorem

The proof of the theorem proceeds by a sequence of lemmas. As in the previous section, we assume that $n = 2$ in this entire section.

The proof relies heavily on the idea of a *transversal.*

Definition 8.9. Consider $\dot{u} = F(u)$ in $n = 2$ dimensions. A *transversal* ℓ is a closed line segment in $\mathbb{R}^2$ containing no fixed points and no point where F is tangential to ℓ. If $u_0 \in \mathbb{R}^2$ we say that ℓ is a *transversal for* u if ℓ is a transversal and u is an interior point of ℓ.

By *interior point* we mean that $u \in \ell$ and that u is not one of the endpoints of ℓ.

We note that every closed line segment in $\mathbb{R}^2$ can be written as

$$\ell = \{u \in \mathbb{R}^2 : \|u_0 - u\| \leqslant r,\ (u_0 - u) \cdot \nu = 0\}$$

for some $r > 0$, $u_0 \in \mathbb{R}^2$ and a nonzero vector $\nu \in \mathbb{R}^2$. The conditions of being a transversal can then be summarized as (potentially after replacing ν by $-\nu$)

$$F(u) \cdot \nu > 0 \qquad \forall\, u \in \ell.$$

Thus, all orbits that hit ℓ must cross it, and must do so in the direction ν. With this interpretation it would be easy to extend the definition of transversals also to dimensions $n > 2$, but we will only use them when $n = 2$.

Lemma 8.10. *Every $u_0 \in \mathbb{R}^2$ which is not a fixed point has a transversal.*

This result can be generalized to $\mathbb{R}^n$ for $n > 2$.

Proof. Let $\nu := F(u_0)$, which is nonzero. The function F is continuous, so there is some $\delta > 0$ such that $\|F(u_0) - F(u)\| \leqslant \|\nu\|/2$ for all $u \in \overline{B}_\delta(u_0)$. Since $\nu = F(u_0)$ we have

$$\|F(u_0) - F(u)\|^2 = \|\nu\|^2 - 2F(u) \cdot \nu + \|F(u)\|^2 \leqslant \|\nu\|^2/4,$$

while on the other hand, by the inverse triangle inequality, $\|F(u)\| \geqslant \big|\|\nu\| - \|\nu - F(u)\|\big| \geqslant \|\nu\|/2$. Reordering the above estimate yields

$$F(u) \cdot \nu \geqslant \frac{1}{2}\left(\frac{3}{4}\|\nu\|^2 + \|F(u)\|^2\right) \geqslant \frac{1}{2}\left(\frac{3}{4}\|\nu\|^2 + \frac{1}{4}\|\nu\|^2\right) = \frac{1}{2}\|\nu\|^2 > 0.$$

Thus, the set $\ell = \{u \in \mathbb{R}^2 : \|u_0 - u\| \leqslant \delta,\ (u_0 - u) \cdot \nu = 0\}$ is a transversal of u_0. □

The next result says that points sufficiently close *in space* to a transversal, also have trajectories that are close to the transversal *in time.* In particular, we can modify a sequence converging to a point on a transversal such that each element of the sequence lies on the transversal.

Lemma 8.11. *Let u_0 not be a fixed point and let ℓ be a transversal for u_0.*

(i) For every $\varepsilon > 0$, there is some $\delta > 0$ such that for any $v_0 \in \overline{B}_\delta(u_0)$ there is some $t \in [-\varepsilon, \varepsilon]$ such that $\varphi(t; v_0) \in \ell$.

(ii) If $\{v_k\}_k$ is a sequence in $\mathbb{R}^n$ converging to u_0, then there is a sequence $\tau_k \to 0$ such that if $\tilde{v}_k := \varphi(\tau_k; v_k)$, then $\tilde{v}_k \in \ell$ for large k, and still $\tilde{v}_k \to u_0$.

This proof is tedious and can safely be skipped.

Proof of (i). By shrinking ℓ, we may assume that it is of the form

$$\ell = \{u \in \mathbb{R}^2 : \|u_0 - u\| \leqslant r,\ (u_0 - u) \cdot \nu = 0\}$$

for some unit vector $\nu \in \mathbb{R}^2$ and $r > 0$. Define

$$M := \sup_{u \in \overline{B}_r(u_0)} \|F(u)\|, \qquad m := \inf_{u \in \overline{B}_r(u_0)} F(u) \cdot \nu.$$

Since $F(u_0) \cdot \nu > 0$, we may assume that r is small enough that $m > 0$. Let

$$\delta = \min\Big(m\varepsilon, \frac{r}{1 + M/m}\Big)$$

and note that $\delta < r$. Let $v_0 \in \overline{B}_\delta(u_0)$, and assume that v_0 lies in direction $-\nu$ with respect to ℓ (the proof of the opposite direction is analogous). We have

$$\frac{d}{dt}\Big(\big(\varphi(t; v_0) - u_0\big) \cdot \nu\Big) = F(\varphi(t; v_0)) \cdot \nu \geqslant m,$$

for all t such that $\varphi(t; v_0) \in \overline{B}_r(u_0)$. Since $(v_0 - u_0) \cdot \nu \geqslant -\|v_0 - u_0\| \geqslant -\delta$, we get

$$\big(\varphi(t; v_0) - u_0\big) \cdot \nu \geqslant -\delta + tm.$$

Thus, as long as $\varphi(t; v_0) \in \overline{B}_r(u_0)$, there is some $t_0 > 0$ with $t_0 \leqslant \delta/m \leqslant \varepsilon$ such that $\big(\varphi(t_0; v_0) - u_0\big) \cdot \nu = 0$. We claim that indeed $\varphi(t; v_0) \in \overline{B}_r(u_0)$ for $|t| \leqslant \delta/m$. We have

$$\frac{d}{dt}\|\varphi(t; v_0) - u_0\|^2 = \big(\varphi(t; v_0) - u_0\big) \cdot F(\varphi(t; v_0)) \leqslant M\|\varphi(t; v_0) - u_0\|,$$

so $\frac{d}{dt}\|\varphi(t; v_0) - u_0\| \leqslant M$, whence

$$\|\varphi(t; v_0) - u_0\| \leqslant \|v_0 - u_0\| + Mt \leqslant \delta + M\delta/m = \delta(1 + M/m) \leqslant r$$

whenever $t \leqslant \delta/m$, by our choice of δ. This proves our claim. □

The next lemma says that if a trajectory hits a transversal, it must leave it immediately, and in particular, that a periodic orbit can hit the transversal at most once. This result uses the fact that $n = 2$ in a fundamental way.

Lemma 8.12. *Let $n = 2$, let ℓ be a transversal and let $u_0 \in \mathbb{R}^2$. Then:*

(i) If $a < b$ and $u_0 \in \mathbb{R}^2$, then the set $\{t \in [a, b] : \varphi(t; u_0) \in \ell\}$ is finite (or empty).

(ii) If $\Gamma(u_0)$ is periodic then it intersects ℓ in at most one point.

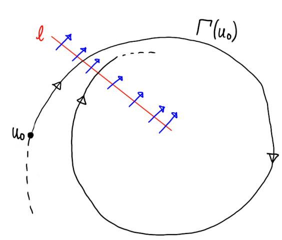
(iii) $\omega(u_0)$ can intersect ℓ in at most one point.

Proof. (i) Assume conversely that there are distinct $t_1, t_2, \dots \in [a, b]$ such that $\varphi(t_k; u_0) \in \ell$ for all $k \in \mathbb{N}$. Since $[a, b]$ is compact, we can take a subsequence $\{t_{k(j)}\}_{j \in \mathbb{N}}$ such that $t_{k(j)} \to t^* \in [a, b]$ as $j \to \infty$. Then also $\varphi(t_{k(j)}; u_0) \to v^* := \varphi(t^*; u_0) \in \ell$ as $j \to \infty$. Let

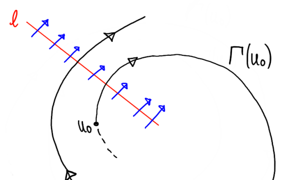
$$v_j := \frac{\varphi(t^*; u_0) - \varphi(t_{k(j)}; u_0)}{t^* - t_{k(j)}}, \qquad j \in \mathbb{N}.$$

On one hand, $v_j \to \frac{\partial \varphi}{\partial t}(t^*; u_0) = F(v^*)$ as $j \to \infty$, which is not parallel to ℓ (since ℓ is a transversal), but on the other hand, v_j is parallel to ℓ for all j — a contradiction.

(ii) Let $\Gamma(u_0)$ be an orbit which intersects ℓ at two distinct points $\varphi(t_1; u_0) \neq \varphi(t_2; u_0)$, at times $t_1 < t_2$. We aim to prove that $\Gamma(u_0)$ cannot be periodic. According to the first part of this lemma, $\Gamma(u_0)$ can intersect ℓ in at most finitely many times $t \in [t_1, t_2]$, so we can let t_2 be the first time of intersection after t_1. Then the flow either looks like case (a) or (b); let us assume that we are in situation (a). Let



(a)

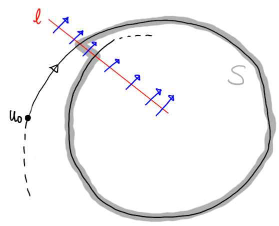


$$S := \{\varphi(t; u_0) : t \in [t_1, t_2]\} \cup S'$$

where S' is the line segment between $\varphi(t_1; u_0)$ and $\varphi(t_2; u_0)$ — note that $S' \subset \ell$. The curve S is a Jordan curve (a closed curve which does not intersect itself), so by the Jordan Curve Theorem (see the appendix), it separates the plane into two open sets U and V, both with S as its boundary, and where U is bounded (the "inside") and V is unbounded (the "outside"). The set U is forward invariant, since no orbit can cross the set $\{\varphi(t; u_0) : t \in [t_1, t_2]\}$, and since every orbit through S' must cross from V into U. Moreover, $F(\varphi(t; u_0))$ points into U, so $\varphi(t; u_0) \in U$ for some $t > t_2$. But then $\varphi(t; u_0) \in U$ for all $t > t_2$, so the orbit can never return to $\varphi(t_1; u_0) \notin U$. Thus, the orbit cannot be periodic.

(iii) is proved similarly to (ii), and is left as an exercise to the interested reader. □

Lemma 8.13. *Let $n = 2$, $u_0 \in \mathbb{R}^2$ and assume that $\Gamma(u_0) \cap \omega(u_0) \neq \emptyset$. Then either u_0 is a fixed point, or $\Gamma(u_0)$ is a periodic orbit.*



The set S

Proof. Assume that u_0 is not a fixed point. By assumption, there is some $s \in \mathbb{R}$ such that $\varphi(s; u_0) \in \omega(u_0)$. Let ℓ be a transversal of $\varphi(s; u_0)$, as prescribed by Lemma 8.10. Let $\{t_k\}_{k \in \mathbb{N}}$ be such that $t_k \to \infty$ and $\varphi(t_k; u_0) \to \varphi(s; u_0)$ as $k \to \infty$. By Lemma 8.11, as long as $k \geqslant N$ for some $N \in \mathbb{N}$, we can modify each t_k slightly so that $\varphi(t_k; u_0) \in \ell$, and still $t_k \to \infty$ and $\varphi(t_k; u_0) \to \varphi(s; u_0)$ as $k \to \infty$. By Lemma 8.12, $\omega(u_0)$ can intersect ℓ in at most one point, and since $\Gamma(\varphi(s; u_0)) \subset \omega(u_0)$, also $\Gamma(\varphi(s; u_0))$ can intersect ℓ in at most one point. Since $\varphi(s; u_0) \in \ell$, this one point must be $\varphi(s; u_0)$, and therefore, $\varphi(t_k; u_0) = \varphi(s; u_0)$ for all $k \geqslant N$. Hence, the orbit is periodic. □

Lemma 8.14. *Let $n = 2$, $u_0 \in \mathbb{R}^2$, and assume that $\Gamma_+(u_0)$ is bounded. If $\omega(u_0)$ contains no critical points, but there is some periodic orbit $\Gamma(v_0) \subset \omega(u_0)$, then $\Gamma(v_0) = \omega(u_0)$.*

Proof. Assume $\omega(u_0)$ does not contain any critical points, and the orbit through $v_0 \in \omega(u_0)$ is periodic, but $\Gamma(v_0) \neq \omega(u_0)$. Then $\Gamma(v_0) \subsetneqq \omega(u_0)$, since $\omega(u_0)$ is invariant (by Theorem 4.10(ii)). Since $\Gamma_+(u_0)$ is bounded, we know that $\omega(u_0)$ is connected (by Theorem 4.10(viii)), so for every $k \in \mathbb{N}$, there is some $v_k \in \omega(u_0) \setminus \Gamma(v_0)$ with $\operatorname{dist}(v_k, \Gamma(v_0)) < \frac{1}{k}$ — otherwise, we could partition $\omega(u_0)$ into two open, disjoint sets. The set $\Gamma(v_0)$ is bounded, so by taking a subsequence $k(j)$, there is some $\bar{v} \in \Gamma(v_0)$ such that $v_{k(j)} \to \bar{v}$ as $j \to \infty$.

Let ℓ be a transversal of $\bar{v}$, as prescribed in Lemma 8.10. By Lemma 8.11, there are times $t_j \to 0$ so that $w_j := \varphi(t_j; v_{k(j)})$ lie on ℓ, and still $w_j \to \bar{v}$. The set $\omega(u_0) \setminus \Gamma(v_0)$ is invariant, so we are guaranteed that $w_j \in \omega(u_0) \setminus \Gamma(v_0)$. But by Lemma 8.12(iii), $\omega(u_0)$ can intersect ℓ in only one point, which must be $\bar{v}$, so $w_j = \bar{v}$ for all j. Thus, w_j both lies in $\Gamma(v_0)$ and in $\omega(u_0) \setminus \Gamma(v_0)$, which is absurd. □

Proof of the Poincaré–Bendixson theorem. Assume that $\omega(u_0)$ does not contain a fixed point. Let $v_0 \in \omega(v_0)$ and let $w_0 \in \omega(v_0)$ (both sets are nonempty, since the orbits are bounded). Note that both $\omega(v_0)$ and $\Gamma(v_0)$ are subsets of $\omega(u_0)$. Let ℓ be a traversal of w_0. By Lemma 8.12(iii), $\omega(u_0)$ intersects ℓ in at most one point, which must be w_0. We have $w_0 \in \omega(v_0)$, so there are $t_1 < t_2 < \cdots \to \infty$ such that $\varphi(t_k; v_0) \to w_0$ as $k \to \infty$. By Lemma 8.11 we can modify each t_k slightly so that $\varphi(t_k; v_0) \in \ell$ for all k, but still $t_k \to \infty$ and $\varphi(t_k; v_0) \to w_0$ as $k \to \infty$. But $\varphi(t_k; v_0) \in \omega(u_0)$ for all k, and $\omega(u_0)$ only intersects ℓ at w_0, so $\varphi(t_k; v_0) = w_0$ for all k. Hence, the orbit through w_0

is periodic. By Lemma 8.14, $\omega(u_0)$ cannot have a periodic orbit as a proper subset, so $\omega(u_0)$ must itself be periodic. □

Proof of Corollary 8.6. Denote $\gamma := \omega(u_0)$. Since γ is a periodic orbit, it separates $\mathbb{R}^2$ into two open, connected components U, V, one of which is bounded, the other not. Let us assume that, say, $u_0 \in U$. We will prove that:

(i) For every $\varepsilon > 0$ there is some $\delta > 0$ such that if $v_0 \in U$ satisfies $\operatorname{dist}(v_0, \gamma) < \delta$ then $\operatorname{dist}(\varphi(t; v_0); \gamma) < \varepsilon$ for all $t \geqslant 0$.

(ii) There is some $\delta_0 > 0$ such that if $v_0 \in U$ satisfies $\operatorname{dist}(v_0, \gamma) < \delta_0$ then $\varphi(t; v_0) \to \gamma$ as $t \to \infty$.

Let $\bar{u} \in \gamma$, and let ℓ be a transversal of $\bar{u}$. Then the forward trajectory $\varphi(t; u_0)$ hits ℓ infinitely often, at times $t_1 < t_2 < \dots$. Let $\varepsilon > 0$, and let $T > 0$ be such that

$$\operatorname{dist}\big(\varphi(t; u_0), \gamma\big) < \varepsilon \qquad \forall\, t \geqslant T.$$

Let $N \in \mathbb{N}$ be such that $t_N \geqslant T$. As in the proof of Lemma 8.12(ii), let

$$S := \big\{\varphi(t; u_0) : t_N \leqslant t \leqslant t_{N+1}\big\} \cup S'$$

where $S' \subset \ell$ is the line segment between $\varphi(t_N; u_0)$ and $\varphi(t_{N+1}; v_0)$. Then γ and S "sandwich" a set $C \subset U$, i.e., $\partial C = \gamma \cup S$. This set is forward invariant, since its boundary consists of the orbits γ and $\big\{\varphi(t; u_0) : t_N \leqslant t \leqslant t_{N+1}\big\}$, as well as the line segment S', through which orbits can only pass *into* C. Defining now $\delta = \min\big(\operatorname{dist}(\varphi(t; u_0), \gamma\big)$, we see that if $v_0 \in U$ satisfies $\operatorname{dist}(v_0, \gamma) < \delta$, then automatically $v_0 \in C$, so $\operatorname{dist}(\varphi(t; v_0), \gamma) < \varepsilon$, by our choice of N. This proves Lyapunov stability.

To prove ω-attractiveness, it is enough to note that any orbit in the above set C must converge to γ, so we can choose δ_0 to be any of the numbers δ above. □

Proof of Lemma 8.7. Assume that neither (i), (ii) nor (iv) is the case. The Poincaré–Bendixson theorem then says that either $\alpha(u_0)$ or $\omega(u_0)$ must be periodic. We claim that these sets must be disjoint, so assume for the sake of contradiction that there is some $v_0 \in \alpha(u_0) \cap \omega(u_0)$. Then v_0 lies on a periodic orbit, so the periodic orbit $\Gamma(v_0)$ is a subset of both $\alpha(u_0)$ and $\omega(u_0)$ (since both sets are invariant).

Let ℓ be a transversal of v_0. Let $0 < t_1 < t_2 < \dots \to \infty$ and $0 > s_1 > s_2 > \dots \to -\infty$ be such that both $\varphi(t_k; u_0)$ and $\varphi(s_k; u_0)$ converge to v_0 as $k \to \infty$. By modifying t_k, s_k slightly, as in Lemma 8.11, we may assume that $\varphi(t_k; u_0)$ and $\varphi(s_k; u_0)$ both lie on ℓ for all k. The same type of argument as in the proof of Lemma 8.12(ii) shows that $\varphi(t_k; u_0), \varphi(t_{k+1}; u_0), \varphi(t_{k+2}; u_0), \dots$ must intersect ℓ in a monotone manner: Necessarily, $\varphi(t_{k+1}; u_0) \in (\!(\varphi(t_k; u_0), v_0)\!)$ for all k, where $(\!(u, v)\!) \subset \mathbb{R}^2$ is the open line segment between the points u and v. More generally, if $t \in \mathbb{R}$ is any time where $\varphi(t; u_0) \in \ell$, then $\varphi(t; u_0) \in (\!(\varphi(t_k; u_0), v_0)\!)$ if and only if $t > t_k$. The sequence $\varphi(s_j; u_0)$ lies on ℓ and converges to v_0, so for sufficiently large j, it lies on the line segment $(\!(\varphi(t_k; u_0), v_0)\!)$ for some k. But then $s_j > t_k > 0 > s_j$, a contradiction. □

The following result is needed in the proof of Corollary 8.8.

Lemma 8.15. *Let $n = 2$ and let $\Gamma(u_0)$ be a periodic orbit which encloses the open set U. Then at least one of the following is true:*

(i) U contains a fixed point

(ii) There is some $v_0 \in U$ such that $\Gamma(v_0)$ is periodic, and such that the set V which $\Gamma(v_0)$ encloses does not itself contain periodic orbits.

Proof. Assume that U does not contain a fixed point. We claim that we can find a "smallest" periodic orbit in U, to which we can apply the above argument. To this end, let P be the set of all periodic orbits in U, and assume that P contains at least one element (otherwise we are already done). For an orbit $\gamma \in P$, we let U_γ be the "inside" of γ, and let $u_\gamma = \big(u_\gamma^{(1)}, u_\gamma^{(2)}\big)$ be the point on γ with the smallest x-coordinate (and smallest y-coordinate, if several points have the same x-coordinate). Note that

$$\text{if } \gamma_1, \gamma_2 \in P \text{ and } \gamma_2 \subset U_{\gamma_1}, \text{ then } u_{\gamma_1}^{(1)} < u_{\gamma_2}^{(1)}. \tag{8.1}$$

Let $w^{(1)} = \sup_{\gamma \in P} u_\gamma^{(1)}$. Since $\{u_\gamma\}_{\gamma \in P}$ is a bounded set in U, there is some sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ in P such that $u_{\gamma_k} \to (w^{(1)}, w^{(2)})$ as $k \to \infty$ for some $w^{(2)} \in \mathbb{R}$. Necessarily, $w \in U$. The set $\omega(w)$ cannot contain a fixed point, by assumption, so by Poincaré–Bendixson, it must be a periodic orbit, i.e. $\omega(w) \in P$. Moreover, $w^{(1)} = u_{\omega(w)}^{(1)}$, and $w^{(1)} \geqslant u_\gamma^{(1)}$ for all $\gamma \in P$, so $U_{\omega(w)}$ does not contain any periodic orbits, because of (8.1). Setting $v_0 = w$ and $V = U_{\omega(v)}$ completes the proof. □

8.4 Problems

Problem 8.1. Consider the system

$$\begin{aligned} \dot{x} &= -y + x\big(r^4 - 3r^2 + 1\big) \\ \dot{y} &= x + y\big(r^4 - 3r^2 + 1\big) \end{aligned}$$

where $r = \sqrt{x^2 + y^2}$.

(**a**) Show that $\dot{r} = r\big(r^4 - 3r^2 + 1\big)$.

(**b**) Show that the origin is an unstable focus, and that this is the only fixed point.

(**c**) Show that $\dot{r} < 0$ on the circle $r = 1$ and that $\dot{r} > 0$ on the circle $r = 2$. Deduce from the Poincaré–Bendixson theorem that there is a periodic orbit in the annulus $A := \{(x, y) \in \mathbb{R}^2 : 1 < r < 2\}$.

(**d**) Use the Poincaré–Bendixson theorem to show that there is a periodic orbit in the annulus $B := \{(x, y) \in \mathbb{R}^2 : 0 < r < 1\}$.

(**e**) Explain why there are no other periodic orbits for this system. Determine the stability of both orbits according to the classification in Definition 8.1.

(**f**) Find $\omega(u_0)$ and $\alpha(u_0)$ for all $u_0 \in \mathbb{R}^2$.

Problem 8.2. Consider the system

$$\begin{aligned} \dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3. \end{aligned}$$

It is a fact that this system has a single fixed point, at the origin.

(**a**) Show that the origin is an unstable focus.

(**b**) Show that $\dot{r} = r - \frac{x^4+y^4}{r}$.

(**c**) Use the Poincaré–Bendixson theorem to show that the system has at least one periodic orbit in the annulus $A := \{(x, y) \in \mathbb{R}^2 : 1/2 < r < 2\}$.

(**d**) Use a computer to draw a phase portrait in the domain $-2 \leqslant x, y \leqslant 2$.

Chapter 9

Bifurcations

Most differential equations coming from real-world applications have one or more *parameters* $a_1, \dots, a_m \in \mathbb{R}$. These might represent, say, the stiffness of a spring, or heat conductivity of a material, or gravitational force, etc. Thus, we want to solve

$$\dot{u} = F_a(u) \tag{9.1}$$

where we now explicitly express the dependence of F on the parameters a. In this chapter we will mostly assume that $a \in \mathbb{R}$. We will also assume that F is continuous with respect to both u and a (and in many places also several times differentiable), so that a small change in a will lead to a small change in F.

Model parameters are rarely known exactly, so it is important to know how sensitive the solution is to the choice of a — will a small change in a lead to a correspondingly small change in u? Generally, a *bifurcation* is when a small change in a leads to a *qualitative* change in the solution, such as

- fixed points being created or destroyed
- periodic orbits being created or destroyed
- the type of stability of a fixed point or periodic orbit changing.

Example 9.1. Consider the scalar equation

$$\dot{u} = u^2 + a$$

for some parameter $a \in \mathbb{R}$. If $a > 0$ there are no fixed points; if $a = 0$ there is one fixed point, namely $u^* = 0$; and if $a < 0$ then there are two fixed points, namely $u^*_- = -\sqrt{-a}$ and $u^*_+ = \sqrt{-a}$. Using the theory from Section 6.2, we see that u^*_- is repelling while u^*_+ is asymptotically stable.

We conclude that increasing a from negative to positive values, we see that the two fixed points $u^*_\pm$ approach one another ($a < 0$), merge ($a = 0$), and disappear ($a > 0$). It is clear that a big change in the qualitative behaviour of solutions occurs near the value $a^* = 0$. This type of bifurcation is called a *saddle-node bifurcation* — more on this later. We can summarize our analysis in the *bifurcation diagram* in Figure 9.1, which displays the fixed points for different values of a. The curve u^*_- is dotted to indicate that these fixed points are unstable.

△

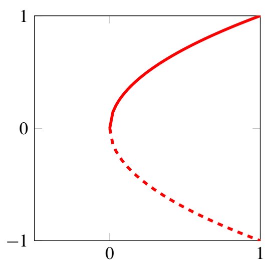


Figure 9.1: Bifurcation diagram for the system in Example 9.1. Horizontal axis a, vertical axis u.

Let us develop some heuristics for the conditions that must be in place for a bifurcation to occur. Assume that (9.1) has a fixed point u^* for values of a in some interval. Since the location of the fixed point might change as a is altered, the fixed point is a function of the parameter: $u^* = u^*(a)$. Assume now that the stability of u^* changes as the parameter crosses some threshold a^*, so e.g.

- $u^*(a)$ is asymptotically stable for $a < a^*$
- $u^*(a)$ is unstable for $a > a^*$.

We know from Chapter 6 that the stability of hyperbolic fixed points can be characterized in terms of the sign of the real parts of the eigenvalues of $DF_a(u^*)$, so it seems necessary that

$$\textit{the fixed point } u^* \textit{ is non-hyperbolic at the bifurcation value } a = a^*. \tag{9.2}$$

In particular, if the equation is scalar (i.e., $n = 1$) then necessarily $\frac{\partial F_{a^*}}{\partial u}(u^*) = 0$. (Check this!)

It turns out that a similar property holds at a bifurcation where a fixed point is created or destroyed. Again, let u^* be a fixed point for (9.1) for some value of $a = a^* \in \mathbb{R}$, so that $F_{a^*}(u^*) = 0$. Let us assume for the moment that the Jacobian matrix $D_u F_{a^*}(u^*)$ is non-singular. We can then apply the implicit function theorem (cf. Theorem A.3) to deduce the existence of a neighbourhood U of a^* and a function $g\colon U \to \mathbb{R}^n$ such that

$$g(a^*) = u^* \qquad \text{and} \qquad F_a(g(a)) = 0 \qquad \text{for all } a \in U.$$

In other words: For all values of a close to a^*, there is a single fixed point $g(a)$ close to u^*. Thus, under our assumption that $D_u F_{a^*}(u^*)$ is non-singular, there can be no creation or destruction of fixed points close to u^*! Since being singular is the same as having 0 as an eigenvalue, and a fixed point with 0 as an eigenvalue is in particular non-hyperbolic, we deduce again that a bifurcation where a fixed point is created or destroyed must satisfy (9.2).

9.1 Saddle-node bifurcations

We have already seen an example of a saddle-node bifurcation in Example 9.1. More concretely, we say that (9.1) *undergoes a saddle-node bifurcation at $a = a^*$ near the point u^** if u^* is a fixed point for (9.1) with $a = a^*$, and both of the following are satisfied:

- There are two fixed points close to u^* for values $a < a^*$.
- There are no fixed points close to u^* for values $a > a^*$.

(If the above holds with the inequalities $a < a^*$ and $a > a^*$ switched around, then we still call it a saddle-node bifurcation.)

The following theorem gives a sufficient condition for the existence of a saddle-node bifurcation, and is typical of the types of results that can be found in bifurcation theory.

Theorem 9.2. *Let $n = 1$ and assume that $(a^*, u^*) \in \mathbb{R} \times \mathbb{R}$ is a point where* *(i)* $F_{a^*}(u^*) = 0$

(ii) $\frac{\partial F_{a^*}}{\partial u}(u^*) = 0$

(iii) $\frac{\partial^2 F_{a^*}}{\partial u^2}(u^*) \neq 0$

(iv) $\frac{\partial F_{a^*}}{\partial a}(u^*) \neq 0.$

Then the system (9.1) *undergoes a saddle-node bifurcation at a^* near u^*.*

Proof. Let us denote $G(a,u) := F_a(u)$, so that our conditions can be written

(i) $G(a^*, u^*) = 0$

(ii) $\frac{\partial G}{\partial u}(a^*, u^*) = 0$

(iii) $\frac{\partial^2 G}{\partial u^2}(a^*, u^*) \neq 0$

(iv) $\frac{\partial G}{\partial a}(a^*, u^*) \neq 0.$

Using properties (i) and (iv) in the implicit function theorem, we deduce the existence of a neighbourhood U of u^* and a function $a \colon U \to \mathbb{R}$ so that $a(u^*) = a^*$ and

$$G(a(u), u) = 0 \qquad \text{for all } u \in U. \tag{9.3}$$

In other words, u is a fixed point for (9.1) for the parameter $a = a(u)$. Differentiating (9.3) and inserting $u = u^*$ yields

$$0 = \frac{d}{du}\Big(G(a(u), u)\Big)\Big|_{u=u^*} = \frac{\partial G}{\partial a}(a^*, u^*)a'(u^*) + \frac{\partial G}{\partial u}(a^*, u^*).$$

Conditions (ii) and (iv) say that $\frac{\partial G}{\partial a}$ is nonzero and $\frac{\partial G}{\partial u}$ is zero at (a^*, u^*), so necessarily $a'(u^*) = 0$. Differentiating (9.3) a second time and inserting $u = u^*$ yields

$$\begin{aligned}
0 &= \frac{d^2}{du^2}\Big(G(a(u), u)\Big)\Big|_{u=u^*} \\
&= \frac{\partial^2 G}{\partial a^2}(a^*, u^*)a'(u^*)^2 + \frac{\partial^2 G}{\partial a \partial u}(a^*, u^*)a'(u^*) + \frac{\partial G}{\partial a}(a^*, u^*)a''(u^*) \\
&\quad + \frac{\partial^2 G}{\partial u \partial a}(a^*, u^*)a'(u^*) + \frac{\partial^2 G}{\partial u^2}(a^*, u^*).
\end{aligned}$$

Using (iii), (iv) and the fact that $a'(u^*) = 0$, we can solve for $a''(u^*)$ to find that

$$a''(u^*) = -\frac{\frac{\partial^2 G}{\partial u^2}(a^*, u^*)}{\frac{\partial G}{\partial a}(a^*, u^*)} \neq 0.$$

Thus, we have shown that $a(u^*) = a^*$, $a'(u^*) = 0$, and $a''(u^*) \neq 0$. Let us assume that, say, $a''(u^*) > 0$; a similar analysis follows in the negative case. Then a is convex near u^*, so for values $a_0 > a^*$ there are *two* points u^*_-, u^*_+ on either side of u^* such that $a(u^*_-) = a(u^*_+) = a_0$ — in other words, there are two fixed points u^*_-, u^*_+ for the value $a = a_0$. On the other hand, when $a_0 < a^*$ there are *no* points u^* such that $a(u^*) = a_0$ — in other words, there are no fixed points for the value $a = a_0$. This proves that the system undergoes a saddle-node bifurcation at a^*. □

Remark 9.3. Condition (i) of Theorem 9.2 simply says that u^* is a fixed point. As we have seen from the discussion in the introduction of this chapter (cf. (9.2)), condition (ii) is necessary for the appearance of a bifurcations. Conditions (iii) and (iv) require a bit more work to interpret, but they both say that some numbers must be nonzero. Since most numbers *are* nonzero, we can infer that (iii) and (iv) are false only in very special cases. Thus, we can interpret Theorem 9.2 as saying that for scalar equations, *the typical form of bifurcations is a saddle-node bifurcation.* △

Example 9.4. Here is an example of a bifurcation in a scalar equation which is not a saddle-node bifurcation. Consider the scalar equation

$$\dot{u} = u^3 - au$$

for $a \in \mathbb{R}$. The value $u^* = 0$ is always a fixed point; if $a \leqslant 0$ then it is the only fixed point, while if $a > 0$ there are two more fixed points, $u^*_\pm = \pm\sqrt{a}$. Using the theory from Section 6.2, we see that $u^* = 0$ is repelling for $a < 0$ and a sink for $a > 0$, while $u^*_\pm$ are repelling.

Thus, the system undergoes a bifurcation at $a = 0$ where three fixed points ($a > 0$) emerge from one fixed point ($a < 0$). Such a bifurcation is called a *pitchfork bifurcation*; Figure 9.2 shows a bifurcation diagram. The reader should check that the conditions of Theorem 9.2 are not satisfied (Problem 9.2). △

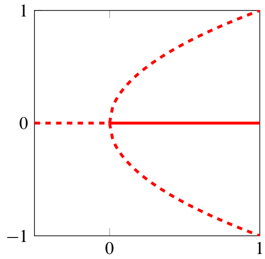


Figure 9.2: Bifurcation diagram for the system in Example 9.4. Horizontal axis a, vertical axis u.

Example 9.5. Here is an example of a saddle-node bifurcation in two dimensions. Consider the system given in polar coordinates by

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = \sin(\theta)^2 + a \end{cases}$$

for a parameter $a \in \mathbb{R}$. It is clear that the origin ($r = 0$) is always a fixed point, and by noting that $\dot{r} > 0$ when r is small but positive, we see that the origin is repelling.

Since $r(1 - r^2) = 0$ only when $r = 0$ or $r = 1$, the only other fixed points must lie on the unit circle $r = 1$. Such a fixed point would need to satisfy $\sin(\theta)^2 + a = 0$, which is only possible when $a \in [-1, 0]$. (The reader should check this.) At $a = 0$ there are two fixed points at angles $\theta = 0$ and $\theta = \pi$. Similarly, at $a = -1$ there are two fixed points at angles $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. When $a \in (-1, 0)$, a fixed point would have to satisfy $\sin(\theta)^2 = -a$, i.e. $\sin(\theta) = \pm\sqrt{-a}$, which has four solutions:

$$\theta = \arcsin\big(\pm\sqrt{-a}\big), \qquad \theta = \arcsin\big(\pm\sqrt{-a}\big) + \pi.$$

Thus, apart from the origin, the system has the following fixed points:

- none when $a < -1$
- two when $a = -1$
- four when $-1 < a < 0$
- two when $a = 0$
- none when $a > 0$.

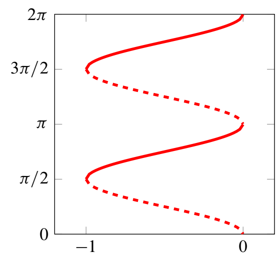


Figure 9.3: Bifurcation diagram for the system in Example 9.5. Horizontal axis a, vertical axis θ.

Clearly, a bifurcation occurs at $a = -1$ and at $a = 0$.

Checking the stability of each fixed point using the theory in Section 6.2, we find that the fixed points alternate between being stable and unstable; a bifurcation diagram is shown in Figure 9.3. (Since the system is two-dimensional, a full bifurcation diagram should be three-dimensional, but this would be difficult to visualize. However, we know that $r = 1$ for all fixed points of interest, so we plot only the θ component.) From this diagram we clearly see that saddle-node bifurcations occur at $a = -1$ near $\theta = \pi/2$ and $\theta = 3\pi/2$, and at $a = 0$ near $\theta = 0$ (which is the same as $\theta = 2\pi$) and $\theta = \pi$. All points lie on the unit circle $r = 1$. △

9.2 Hopf bifurcations

So far we have seen bifurcations where two or more fixed points merge. Since fixed points are usually easy to find, such bifurcations are easy to identify. We will now study a more complex case, where a fixed point and a periodic orbit merge.

Definition 9.6. We say that (9.1) *undergoes a Hopf bifurcation at a^* near u^** if

Also called an *Andronov–Hopf bifurcation*.

- for $a < a^*$ there is a single fixed point near u^*
- for $a > a^*$ there is a single fixed point and a periodic orbit near u^*.

Although the above definition is somewhat informal, the reader should be able to understand it using the intuition from the previous section. Note that a Hopf bifurcation can only happen in dimensions $n \geqslant 2$, since there are no periodic solutions in scalar equations.

Example 9.7. Let $a \in \mathbb{R}$ be a parameter and consider the planar system

$$\begin{cases} \dot{x} = ax - y - x(x^2 + y^2) \\ \dot{y} = x + ay - y(x^2 + y^2), \end{cases}$$

which can be written in polar coordinates as

$$\begin{cases} \dot{r} = r(a - r^2) \\ \dot{\theta} = 1. \end{cases}$$

The only fixed point for the system is the origin. When $a < 0$ we have $\dot{r} < 0$ for all $r > 0$, while when $a > 0$ we have $\dot{r} > 0$ for small $r > 0$. Thus, the origin is attracting for $a < 0$ and repelling for $a > 0$.

When $a > 0$ there is also a periodic orbit along the circle $r = r^* := \sqrt{a}$. Since $\dot{r} > 0$ for $r < r^*$ and $\dot{r} < 0$ for $r > r^*$, the periodic orbit is attracting. Figure 9.4 shows a bifurcation diagram for this system, where fixed points are shown in red and periodic orbits are outlined in blue. As in Example 9.5, we only show one of the two coordinates, for the sake of clarity. △

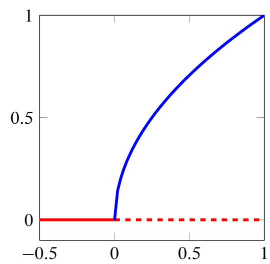


Figure 9.4: Bifurcation diagram for the system in Example 9.7. Horizontal axis a, vertical axis r.

The previous example was rather easy to analyse since its formulation in polar coordinates was very simple. The system in the next example does not admit such a simple reformulation.

Example 9.8. Consider the Lotka–Volterra model

$$\begin{cases} \dot{x} = x\left(1 - \dfrac{x}{K} - \dfrac{y}{1 + x}\right) \\ \dot{y} = y\left(\delta\dfrac{x}{1 + x} - \gamma\right) \end{cases} \tag{9.4}$$

where $K, \delta, \gamma > 0$ are parameters satisfying $\gamma < \delta$. The unknowns x and y represent the number of species in populations of prey and predators, respectively. In the absence of predators, the growth factor $1 - \frac{x}{K}$ will move the prey population towards the *carrying capacity* K of the environment. The decay factor $-\frac{y}{1+x}$ is the rate at which the predators kill off prey; it increases linearly with the size of the predator population, but it decays as x increases. The latter effect is reflected in the natural fact that there are limits to how much each predator can eat each day. Turning to the second equation, the predator population increases at the rate $\delta\frac{x}{1+x}$, where δ is the predator's *benefit of consumption*. Note that the growth factor $\frac{x}{1+x}$ is upper bounded as x increases, again reflecting the natural effect that there are bounds to how much a predator can eat. Last, γ is the death rate of predators.

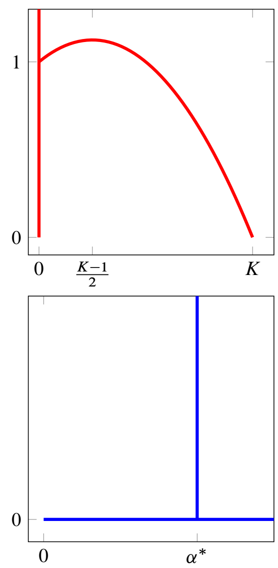


Figure 9.5: Nullclines N_x (top) and N_y (bottom) for the system in Example 9.8.

The nullclines are

$$N_x = \{x = 0\} \cup \{y = (1+x)(1 - x/K)\},$$
$$N_y = \{y = 0\} \cup \{x = \alpha^*\}$$

where we have defined $\alpha^* := \frac{\alpha}{1-\alpha}$ and $\alpha := \frac{\gamma}{\delta} < 1$. The nullclines are depicted in Figure 9.5. Taking the intersection of the nullclines, we find the following fixed points:

Extinction: $(x, y) = (0, 0)$, exists for all parameter values

Prey-only: $(x, y) = (K, 0)$, exists for all parameter values

Coexistence: $(x_c, y_c) = \Big(\alpha^*,\ (1+\alpha^*)(1 - \alpha^*/K)\Big)$, exists only when $\alpha^* < K$.

Recalling that K represents the amount of food available to the prey population, we see that a scarcity of food for the prey (specifically, $K < \alpha^*$) will eliminate coexistence as a steady state, leading to the possible extinction of the predator.

Letting F denote the right-hand side of (9.4), we compute the Jacobian

$$\nabla F(x, y) = \begin{pmatrix} 1 - \frac{2x}{K} - \frac{y}{(1+x)^2} & -\frac{x}{1+x} \\ \frac{\delta y}{(1+x)^2} & \delta\left(\frac{x}{1+x} - \alpha\right) \end{pmatrix}.$$

We use the linearized system to analyse the stability of each fixed point:

Extinction: We have

$$\nabla F(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma \end{pmatrix},$$

so the extinction state is a saddle, with stable manifold parallel to $r_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and unstable manifold parallel to $r_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (see Section 6.4). These can be interpreted as signifying the death of predators in the absence of prey, and the increase of prey in the absence of predators, respectively.

Prey-only: We have

$$\nabla F(K, 0) = \begin{pmatrix} -1 & -\frac{K}{1+K} \\ 0 & \delta\left(\frac{K}{1+K} - \alpha\right) \end{pmatrix},$$

whose eigenvalues are -1 and $\delta\left(\frac{K}{1+K} - \alpha\right)$. The second eigenvalue is negative if and only if $\frac{K}{1+K} < \alpha$, which is equivalent to $K < \alpha^*$. Thus, the prey-only state is attracting when food is scarce ($K < \alpha^*$), and a saddle when food is abundant ($K > \alpha^*$). We also note that when $K > \alpha^*$, the stable manifold $W^s(K, 0)$ is parallel to $r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the unstable manifold is tangent to r_2 which, although having a messy expression, points in a direction where y grows and x shrinks.

Coexistence: Recall that this state only exists when $K > \alpha^*$. We have

$$\nabla F(x_c, y_c) = \begin{pmatrix} \alpha\Big(1 - \big(\frac{1+\alpha}{1-\alpha}\big)/K\Big) & -\alpha \\ 1 - \alpha\frac{1+K}{K} & 0 \end{pmatrix}.$$

The expression for the corresponding eigenvalues λ_1, λ_2 is big and ugly, but since we are only interested in the stability of the fixed point, we proceed somewhat indirectly to determine the sign of $\mathrm{Re}(\lambda)$.

From linear algebra we know that $\lambda_1\lambda_2 = \det(\nabla F)$ and $\lambda_1 + \lambda_2 = \mathrm{tr}(\nabla F)$, so

$$\lambda_1\lambda_2 = \alpha\Big(1 - \alpha\frac{1+K}{K}\Big), \qquad \lambda_1 + \lambda_2 = \alpha\Big(1 - \Big(\frac{1+\alpha}{1-\alpha}\Big)/K\Big).$$

We have assumed that $K > \alpha^*$, which is equivalent to $\alpha < \frac{K}{1+K}$, so $\lambda_1\lambda_2 > 0$. Thus, either λ_1, λ_2 is a nonzero complex conjugate pair, or they are real, nonzero numbers of the same sign. Next, denoting $K^* := \frac{1+\alpha}{1-\alpha}$ and noting that $K^* > \alpha^*$, we have the following possibilities for $\lambda_1 + \lambda_2$:

$$\lambda_1 + \lambda_2 \begin{cases} < 0 & \text{if } K < K^* \\ = 0 & \text{if } K = K^* \\ > 0 & \text{if } K > K^*. \end{cases}$$

Combined with the fact that $\lambda_1\lambda_2 > 0$, we deduce the following regimes:

- $K < K^*$: $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) < 0$
- $K = K^*$: $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = 0$ and $\mathrm{Im}(\lambda_1), \mathrm{Im}(\lambda_2) \neq 0$
- $K > K^*$: $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) > 0$.

We conclude that (x_c, y_c) is attracting when $\alpha^* < K < K^*$ and repelling when $K > K^*$.

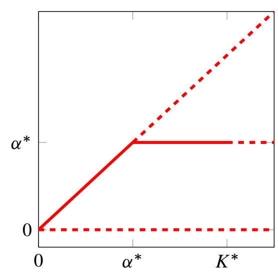


Figure 9.6: Bifurcation diagram for the fixed points of the system in Example 9.8. Horizontal axis K, vertical axis x.

Figure 9.6 shows a bifurcation diagram of the fixed points of our system, where K is the bifurcation parameter, and the fixed points are represented by their x coordinate. From the diagram it is clear that the system has at least three bifurcations: The first at $K = 0$ and $x = 0$, where the extinction and prey-only states merge; the second at $K = \alpha^*$ and $x = \alpha^*$, where the prey-only and coexistence states merge; and a third one at $K = K^*$ and $x = \alpha^*$ where the coexistence state loses its stability. As it turns out, this bifurcation is a Hopf bifurcation, but it requires some work to prove this. We proceed with an outline of a proof.

Consider the fixed point $(K, 0)$, and assume that $K > K^*$, so that (x_c, y_c) is repelling. The nullclines are depicted in Figure 9.7. It is straightforward to check that the eigenvector r_2 corresponding to the unstable subspace points in a direction *above* the parabola defining the N_x nullcline. Thus, by the stable manifold theorem, there is an orbit Γ emanating from $(K, 0)$, moving up above N_x. Necessarily, Γ continues through the vertical segment in N_y above (x_c, y_c), down through N_x again, right through N_y,

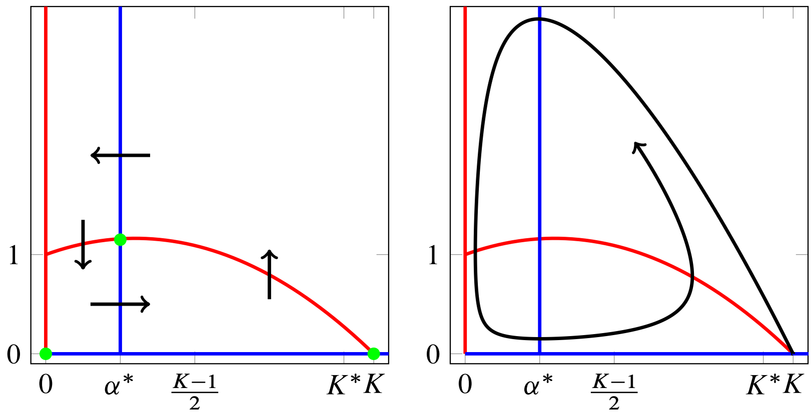


Figure 9.7: Nullclines N_x (red) and N_y (blue) when $K > K^*$. Fixed points are marked in green.

and up through N_x again. The loop just described defines the boundary of a forward invariant domain K containing (x_c, y_c). There is only one fixed point in K, and that point is repelling, so $(x_c, y_c) \notin \omega(\Gamma)$. Hence, using the Poincaré–Bendixson theorem, we can be sure that there is a periodic orbit somewhere in K. This proves the existence of a periodic orbit near (x_c, y_c) when $K > K^*$. The proof that there is no periodic orbit when $K < K^*$ is more intricate, and is skipped here. △

9.3 Problems

Problem 9.1. Use Theorem 9.2 to verify that the system in Example 9.1 undergoes a saddle-node bifurcation at $a^* = 0$ near $u^* = 0$.

Problem 9.2. Check that the conditions of Theorem 9.2 are *not* satisfied at $a^* = 0$ and $u^* = 0$ in the system in Example 9.1.

Problem 9.3. Consider the planar system in Example 9.7. We analysed the system by converting to polar coordinates, but the same analysis can be done directly to the system expressed in Cartesian coordinates, as follows.

(**a**) Show that the origin $(x, y) = (0, 0)$ is a fixed point, and that there are no other fixed points.

(**b**) Linearize the system around the origin and show that it is attracting for $a < 0$ and repelling for $a > 0$.

(**c**) When $a > 0$, show that there is a periodic orbit along the circle C with radius $\sqrt{a}$ centred at the origin.

Hint: Show first that C is invariant.

(**d**) When $a > 0$, show that the periodic orbit is attracting.

Hint: Let $L(x, y) = x^2 + y^2$. Show that $\dot{L} > 0$ when (x, y) lies on the inside of C and $\dot{L} < 0$ when it lies outside of C.

Appendix A

Facts from linear algebra and analysis

TODO: Definition of matrix norm. Basic properties. Gronwall's inequality, both differential and integral form.

We say that a function *is C^k* (for some $k \geqslant 1$) if it is k times differentiable, and all derivatives are continuous.

Definition A.1. If $u \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, we define

$$\operatorname{dist}(u, A) := \inf\big\{\|u - v\| : v \in A\big\}. \tag{A.1}$$

$S \subset \mathbb{R}^2$ is a *Jordan curve* if it is closed and does not self-intersect. More precisely, there is a continuous, surjective function $\gamma\colon [0,1] \to S$ so that $\gamma(0) = \gamma(1)$ and γ is injective on $[0, 1)$.

Theorem A.2 (The Jordan Curve Theorem)**.** *Let S be a Jordan curve. Then its complement $\mathbb{R}^2 \setminus S$ consists of precisely two components $U, V \subset \mathbb{R}^2$, where U is bounded and V is unbounded.*

Theorem A.3 (The implicit function theorem)**.** *Let $f \in \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a C^k function and let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ be a point where $f(x_0, y_0) = 0$, and where the matrix $D_x f(x_0, y_0)$ is non-singular. Then there is a neighbourhood U of y_0 and a C^k function $g\colon U \to \mathbb{R}^n$ such that $g(y_0) = x_0$ and*

$$f(g(y), y) = 0 \qquad \forall\ y \in U.$$

Appendix B

Solutions to selected problems

Problem 4.2 We have

$$\begin{aligned}\Gamma(\varphi(t;u_0)) &= \{\varphi(s;\varphi(t;u_0)) : s \in \mathbb{R}\} = \{\varphi(s+t;u_0) : s \in \mathbb{R}\} \\ &= \{\varphi(\tau;u_0) : \tau \in \mathbb{R}\} = \Gamma(u_0)\end{aligned}$$

which is (i). For (ii), if $v_0 \in \Gamma(u_0)$ then $v_0 = \varphi(t;u_0)$ for some $t \in \mathbb{R}$, so the claim follows from (i). For (iii), assume conversely that $\Gamma(u_0) \neq \Gamma(v_0)$ but there exist $u \in \Gamma(u_0) \cap \Gamma(v_0)$. We will show that $v_0 \in \Gamma(u_0)$, which by (ii) implies that $\Gamma(u_0) = \Gamma(v_0)$, which contradicts our assumption. Let $t, s \in \mathbb{R}$ be such that $\varphi(t;u_0) = \varphi(s;v_0) = u$. Then

$$\varphi(t-s;u_0) = \varphi(-s;\varphi(t;u_0)) = \varphi(-s;u) = \varphi(-s;\varphi(s;v_0)) = \varphi(-s+s;v_0) = v_0$$

so $v_0 \in \Gamma(u_0)$. This concludes the proof.

Problem 5.2 Let A and B denote the left-hand and right-hand side sets, respectively. If $k \in \mathbb{Z}$ then $\varphi(kT;u_0) = \varphi(T;\dots\varphi(T;u_0)\dots) = u_0$, so $kT \in A$, and therefore $A \supseteq B$. Conversely, if $t \in A$, let $k \in \mathbb{Z}$ be the largest number such that $kT \leqslant t$, and let $s = t - kT \in [0, T)$. Then

$$\varphi(s;u_0) = \varphi(t-kT;u_0) = \varphi(-kT;\varphi(t;u_0)) = \varphi(-kT;u_0) = u_0.$$

Since, by definition, T is the smallest positive number with the property $\varphi(T;u_0) = u_0$, and $s < T$, we must have $s = 0$. Then $t = kT$, so $t \in B$. Hence, $A \subseteq B$.

Problem 5.3 Since $\Gamma(u_0)$ is a closed curve, it is bounded, so $\Gamma(u_0) \subset B_M(0)$ for some $M > 0$. The orbit of v_0 cannot cross the Jordan curve $\Gamma(u_0)$, so $\Gamma(v_0)$ also lies in $B_M(0)$. In particular, the orbit through v_0 is bounded, so by Theorem 3.11, the solution through v_0 exists for all times.

Problem 6.1

(**a**) Eigenvalues are 1 and 6, so this is an unstable node. Solutions are of the form $u(t) = \alpha e^t r_1 + \beta e^{6t} r_2$ for $\alpha, \beta \in \mathbb{R}$. From this we see that the origin is repelling.

(**b**) Eigenvalues are 0 and 7, so this neither of the categories in (i). The solution is of the form $u(t) = \alpha r_1 + \beta e^{7t} r_2$ for $\alpha, \beta \in \mathbb{R}$, so whenever $\beta \neq 0$, the solution diverges. Therefore, u^* is neither L-stable nor ω-attracting. Backwards in time the origin is L-stable, but not ω-attracting, since initial data along r_1 stay constant.

(**c**) Eigenvalues are -1 and 2, so this is a saddle. Solutions are of the form $u(t) = \alpha e^{-t} r_1 + \beta e^{2t} r_2$. Since there are solutions starting close to the origin which diverge when either $t \to \infty$ or $t \to -\infty$, the origin is neither L-stable, ω-attracting or repelling.

(**d**) Eigenvalues are $\frac{1}{2} \pm i\frac{1}{2}\sqrt{15}$, so this is an unstable focus. Since all solutions have a coefficient $e^t/2$, the origin is repelling.

(**e**) Eigenvalues are $-\frac{1}{2} \pm i\frac{1}{2}\sqrt{15}$, so this is a stable focus. Since all solutions have a coefficient $e^{-t/2}$, the origin is asymptotically stable.

(**f**) Eigenvalues are $\pm 2i$, so this is a centre. Since all solutions are linear combinations of the functions $\cos(2t)$ and $\sin(2t)$, the origin is L-stable, but not ω-attracting.

Problem 7.3 Any conserved variable G of (1) automatically satisfies (7.5). If u^* is a strict local minimum for G — that is, there is some $\delta > 0$ such that $G(u^*) < G(u)$ for all $u \in B_\delta(u^*) \setminus \{u^*\}$ — then it is necessarily a critical point for (1), since if $\varphi(t; u^*)$ ever moves away from u^* then G would have to increase. Condition (7.5) is therefore satisfied e.g. in $U := B_\delta(u^*)$.

Problem 7.4 Denote $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. To be a gradient system there would have to be some $G = G(x, y)$ for which $-\frac{\partial G}{\partial x}(x, y) = ax + by$ and $-\frac{\partial G}{\partial y}(x, y) = cx + dy$, so

$$G(x, y) = -\frac{a}{2}x^2 - bxy + f(y), \qquad G(x, y) = -cxy + \frac{d}{2}y^2 + g(x)$$

for functions $f, g\colon \mathbb{R} \to \mathbb{R}$. Matching terms yields $f(y) = -\frac{d}{2}y^2$, $g(x) = -\frac{a}{2}x^2$ and $b = c$. Hence, the matrix A must be symmetric in order for the system to be a gradient flow.

A similar computation shows that A must satisfy $a = -d$ for the system to be Hamiltonian. In order to be both, A would have to be of the form

$$A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$$

for $a, b \in \mathbb{R}$. This would yield $H(x, y) = axy + \frac{b}{2}y^2 + \frac{b}{2}x^2$

Problem 7.5 TODO: Draw phase portraits

(**a**) Gradient flow with $G(x, y) = x^2 y^2$

(**b**) Hamiltonian system with $H(x, y) = (x - 2)^2 + (y - 3)^2$

(**c**) Neither a gradient flow nor a Hamiltonian system

(**d**) Hamiltonian system with $H(x, y) = x^2 y^2$

(**e**) Gradient flow with $G(x) = x - x^3$

(**f**) Hamiltonian system with $H(x, y) = \cos x \sin y$.

Bibliography

[Per01] L. Perko. *Differential Equations and Dynamical Systems*. Springer-Verlag Berlin Heidelberg, 3rd edition, 2001.

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