Toric arrangements and equivariant matroid theory

Lecture 1: Arrangements and matroids.

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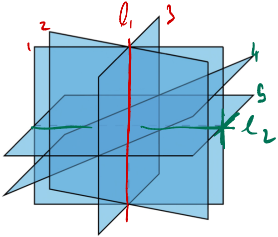
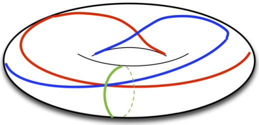
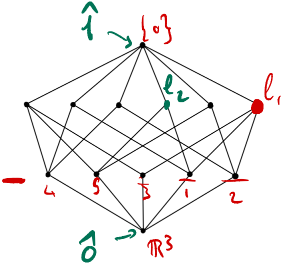


$\mathscr{A}$ - set of subsets of ambient space X



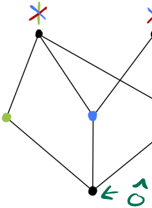
$M(\mathscr{A}) := X \setminus \cup \mathscr{A}$

$$\frac{\text{Geometry}}{\text{Combinatorics}}$$



$\mathcal{P}(\mathscr{A})$: poset of conn. comp. of intersections.

$x \leq y$ if $x \supseteq y$



MATROIDS

???

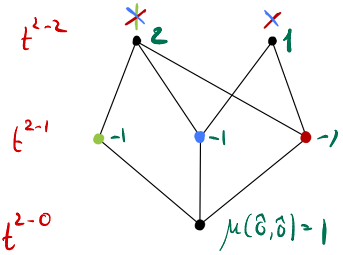
Möbius Functions and poset polynomials

Let $\mathcal{P}$ be a locally finite partially ordered set (poset).

The Möbius function of $\mathcal{P}$ is $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$, defined recursively by

$$\begin{cases} \mu(x,y) = 0 & \text{if } x \not\leq y \\ \displaystyle\sum_{x \leq z \leq y} \mu(x,z) = \delta_{x,y} & \text{if } x \leq y \end{cases}$$

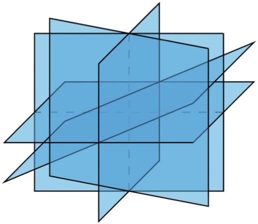
$\Rightarrow \mu(x,x) = 1$



If $\mathcal{P}$ has a $\hat{0}$ and rank function ρ, its characteristic polynomial is

$$\chi_{\mathcal{P}}(t) := \sum_{x \in \mathcal{P}} \mu_{\mathcal{P}}(\hat{0}, x) t^{\rho(\mathcal{P}) - \rho(x)}$$

$\chi_{\mathcal{P}}(t) = t^2 - 3t + 3$



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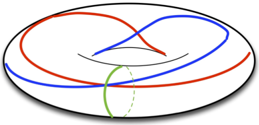

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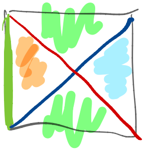
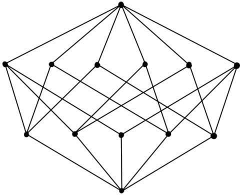


Geometry
<hr>
Combinatorics



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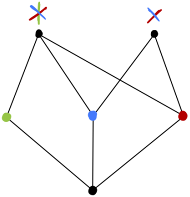
$x \leq y$ if $x \supseteq y$



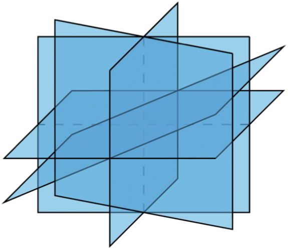
$$\chi_P(t) = t^3 - 5t^2 + 8t - 4$$

$$\chi_P(-1) = -18$$

$$\chi_P(t) = t^2 - 3t + 3$$

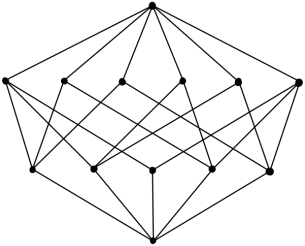
$$\chi_P(0) = 3$$

Hyperplanes and geometric lattices



$$a_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ a_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix},\ a_3 = \begin{pmatrix}0\\1\\0\end{pmatrix},$$

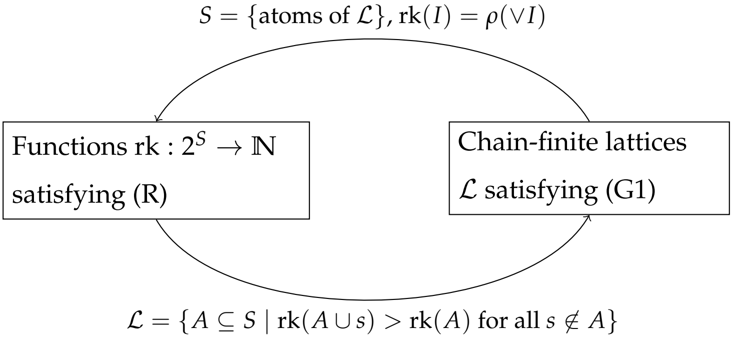
$$a_4 = \begin{pmatrix}0\\0\\1\end{pmatrix},\ a_5 = \begin{pmatrix}0\\-1\\2\end{pmatrix}.$$



$$\mathrm{rk}(I) := \dim\langle a_i \mid i \in I\rangle$$

$$= \operatorname{codim} \bigcap_{i\in I} (a_i)^\perp =: \rho(I)$$

CRYPTOMORPHISMS (MATROIDS)



$$\chi_{\text{rk}}(t) \overset{\text{thm.}}{=} \chi_{\mathcal{L}}(t)$$

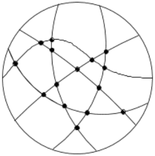
(S finite)

Finite matroids

Rank functions / intersection posets

... of central hyperplane arrangements

Representable m.



Orientable m.

...of pseudosphere arrangements

Number of "regions": $= \chi_{\mathrm{rk}}(-1)$

matroids / geometric lattices

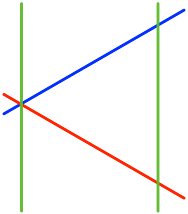
(tropical linear spaces)

Infinite example: set of all subspaces of V.

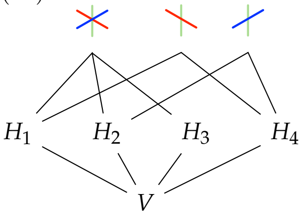
Affine hyperplane arrangement ($\mathbb{K} = \mathbb{R}$)

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad (b_1, b_2, b_3, b_4) = (0, 0, 0, 1)$$

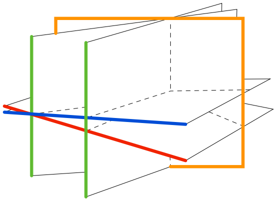
$\mathscr{A}$:



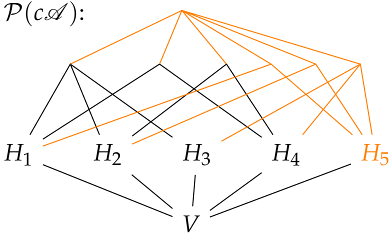
$\mathcal{P}(\mathscr{A})$:



$c\mathscr{A}$:



$\mathcal{P}(c\mathscr{A})$:



Affine hyperplane arrangements ($\mathbb{K} = \mathbb{R}$)

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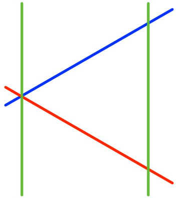


I such that $\cap_{i \in I} H_i \neq \emptyset$

$\{\}, \{1\}, \{2\}, \{3\}, \{4\}$

$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}$

$\{1,2,3\}$

These are the *central sets.*

The family of central sets $\mathcal{K} \subseteq 2^S$ is an abstract simplicial complex.

(I.e.: if $I \in \mathcal{K}$ and $J \subseteq I$, then $J \in \mathcal{K}$)

For $I \in \mathcal{K}$ set $\mathrm{rk}(I) := \dim\langle a_i \mid i \in I \rangle$. This defines a semimatroid.

[Kawahara '04, Ardila '07]

Cryptomorphisms (semimatroids)

$$\mathcal{K} := \{I \subseteq A(\mathcal{L}) \mid \vee I \neq \emptyset\}, \quad \mathrm{rk}(I) = \rho(\vee I)$$

Functions $\mathrm{rk} : \mathcal{K} \to \mathbb{N}$ satisfying (R+)

Chain-finite semilattices $\mathcal{L}$ satisfying (G1) & (G2)

$$\mathcal{L} = \{A \in \mathcal{K} \mid \mathrm{rk}(A') > \mathrm{rk}(A) \text{ for all } A' \in \mathcal{K}, A' \supsetneq A\}$$

$$\chi_{\mathrm{rk}}(t) \overset{\text{thm.}}{=} \chi_{\mathcal{L}}(t)$$

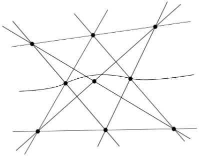
(S finite)

ABSTRACT THEORY

Semimatroid $(\mathcal{K}, \mathrm{rk})$ / intersection posets $\mathcal{L}$

of affine hyperplane arrangements

of "pseudoarrangements"



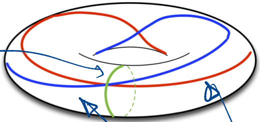
[Baum-Zhu '15, D.-Knauer DGC '24]

semimatroids / geometric semilattices

Toric arrangements

$$[a_1, a_2, a_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$\mathscr{A}$:



$z_1 = 1$

$z^{a_3} = 1$

"$\mathbb{C}^\times \times \mathbb{C}^\times$"

(z_1, z_2)

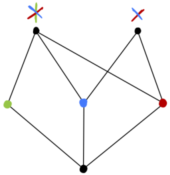
$z_1 z_2^{-1} = 1$

$z^{a_2} = 1$

$z_1 z_2 = 1$

$z^{a_1} = 1$

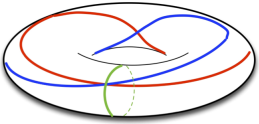
$\mathcal{P}(\mathscr{A})$:



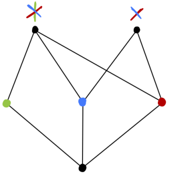
Toric arrangements

$$[a_1, a_2, a_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}; \quad \text{for } I \subseteq [n]: \quad m(I) := \#Tor(\mathbb{Z}^d / \langle a_i \rangle_I).$$

$\mathscr{A}$:



$\mathcal{P}(\mathscr{A})$:



The pair (rk, m) satisfies the axioms of an **arithmetic matroid...**

[d'Adderio-Moci '13, Brändén-Moci '14]

... but it does **not** determine $\mathcal{P}(\mathscr{A})$: **no cryptomorphism.**

[Pagaria '17]

But $\mathcal{P}(\mathscr{A})$ determines ring $H^*(M(\mathscr{A}), \mathbb{Q})$

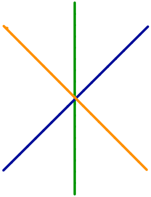
[Callegaro, D'Adderio, D., Migliorini, Pagaria]

Example: Coxeter arrangements

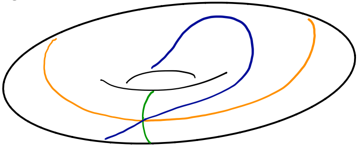
Let $\Phi : \{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^d$ roots of Coxeter system of type $ABCD$.

Let $\mathscr{A}_\Phi$ be the associated Abelian arrangement.

$\mathbb{G} = \mathbb{R}$



$\mathbb{G} = S^1$



Linear, toric, elliptic:

– Explicit description of $\mathcal{P}(\mathscr{A})$ via "enriched partitions" [Bibby '18]

– $\mathcal{P}(\mathscr{A})$ is EL-shellable [D.-Girard-Paolini '19]

Example: Coxeter arrangements

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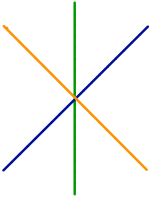

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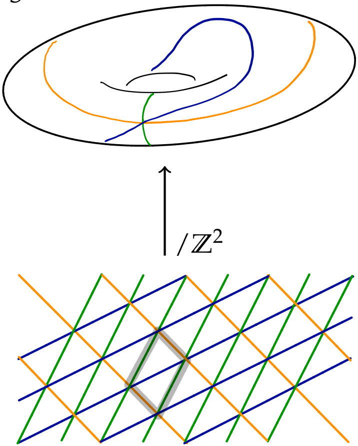
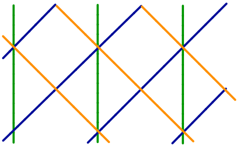
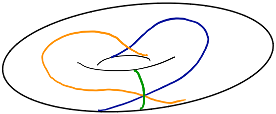


Toric case: $\mathscr{A}_\Phi$ is "covered" by the associated affine Weyl reflection arrangement.

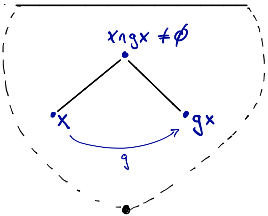
Translative actions

Understand abelian arrangements as ”quotients” of linear, periodic ones.

$\mathscr{A}^{\upharpoonright}$: 



$(\mathbb{Z}^2-)$ periodic (translations) Toric, Elliptic, etc.



If x and gx have upper bound, then $x = gx$.

For all Abelian arrangements:

$$\mathcal{P}(\mathscr{A}) \simeq \mathcal{P}(\mathscr{A}^{\upharpoonright})/\mathbb{Z}^d$$

Group actions on geometric semilattices

From now:

$\left.\begin{array}{l}\mathcal{L}\text{: a chain-finite geometric semilattice;}\\ G\text{: a group acting on } \mathcal{L} \text{ by poset automorphisms.}\end{array}\right\} \alpha : G \circlearrowright \mathcal{L}$

$$\mathcal{P}_\alpha = \mathcal{L}/G := \{Gx \mid x \in \mathcal{L}\}; \qquad Gx \leq Gy \quad \text{iff} \quad x \leq gy \quad \text{for some } g \in G.$$

This is a poset (eg., since $\mathcal{L}$ has a rank function)

Definition. A c.-f. poset P with $\hat{0}$ is geometric if it satisfies (G1), (G2).

Theorem. Let $G \circlearrowright P$ be a translative action. If P is geometric, so is P/G.

Corollary. If the action α is translative, then $\mathcal{P}_\alpha$ is a geometric poset.

So $\mathcal{P}(\mathscr{A})$ is geometric for every abelian arrangement $\mathscr{A}$.

Group actions on geometric semilattices

From now:

$\mathcal{L}$: a chain-finite geometric semilattice;
G: a group acting on $\mathcal{L}$ by poset automorphisms.
$\Big\}\ \alpha : G \circlearrowright \mathcal{L}$

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This is a poset with rank function ρ_α and set of atoms E_α. For $I \subseteq E_\alpha$ let

$$\mathrm{rk}_\alpha(I) := \max \rho_\alpha(\vee I), \qquad m_\alpha(I) := \# \vee I.$$

Theorem. $(E_\alpha, \mathrm{rk}_\alpha)$ is a (semi)matroid if and only if α is translative.

Definition.

$$T_\alpha(x,y) := \sum_{I \subseteq E_\alpha} m_\alpha(I)(x-1)^{\mathrm{rk}_\alpha(E)-\mathrm{rk}_\alpha(I)}(y-1)^{|I|-\mathrm{rk}_\alpha(I)}$$

Group actions on semimatroids

Let G be a group

A G-semimatroid

$$\alpha : G \circlearrowright (\mathcal{K}, \mathrm{rk})$$

is an action of G
on a semimatroid $(\mathcal{K}, \mathrm{rk})$
by automorphisms of $\mathcal{K}$.

A G-geometric semilattice

$$\alpha : G \circlearrowright \mathcal{L}$$

is an action of G
on a geometric semilattice $\mathcal{L}$
by poset-automorphisms.

Cryptomorphism! ✓

If α is translative: (1) $T_\alpha(x, y)$ has positive coefficients

(2) $\chi_{\mathcal{P}_\alpha}(t) = (-1)^{\mathrm{rk}(E_\alpha)} T_\alpha(1-t, 0)$ [D.-Riedel '18]

COMING SOON....

Consider an action $\alpha : G \circlearrowright \mathcal{L}$ and the associated

$$T_\alpha(x,y) = \sum_{I \subseteq E} m_\alpha(I)(x-1)^{\mathrm{rk}_\alpha(E)-\mathrm{rk}_\alpha(I)}(y-1)^{|I|-\mathrm{rk}_\alpha(I)}$$

If α is translative then:

– $T_\alpha(x,y)$ satisfies deletion-contraction recursion

– Yet, m_α is not necessarily arithmetic

– More structure of $\mathcal{L}/G$

– Stanley-Reisner rings.