# Toric arrangements and equivariant matroid theory

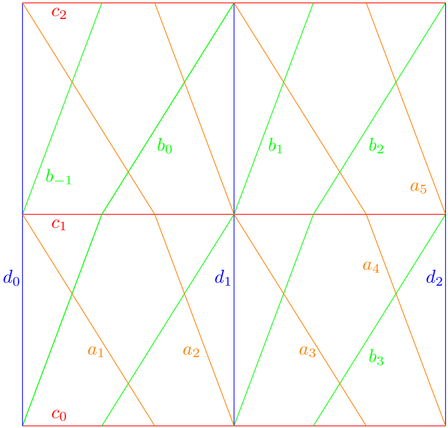
## Lecture 2: Polynomials and Stanley-Reisner rings

Emanuele Delucchi
IDSIA USI/SUPSI
Lugano, Switzerland.

ASGARD24
University of Oslo, may 28, 2024.

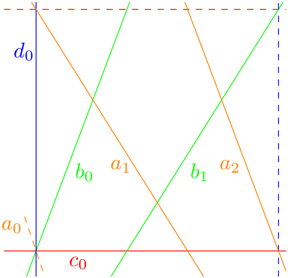
# Group actions on semimatroids

$\mathbb{Z}^2 \circlearrowright$

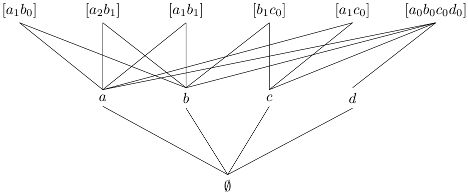



$S := \{a_i, b_j, c_k, d_l\}_{i,j,k,l \in \mathbb{Z}}$, $\quad \mathcal{L} :=$ poset of intersections

# EXAMPLE ($G = \mathbb{Z}^2$)




$\mathcal{P}_\alpha := \mathcal{L}/G$




$E_\alpha := S/G = \{a, b, c, d\}$

$$\mathcal{K} := \{I \subseteq E_\alpha \mid \vee I \neq \emptyset\} = \left\{ \begin{array}{cccccc} & & \{a,b,c,d\}^{(1)} & & & \\ \{a,b,c\}^{(1)} & \{a,b,d\}^{(1)} & & \{a,c,d\}^{(1)} & \{b,c,d\}^{(1)} & \\ \{a,b\}^{(4)} & \{a,c\}^{(2)} & \{b,c\}^{(2)} & \{a,d\}^{(1)} & \{b,c\}^{(1)} & \{b,d\}^{(1)} \\ a^{(1)} & b^{(1)} & & c^{(1)} & d^{(1)} & \\ & & \emptyset^{(1)} & & & \end{array} \right\}$$

$\mathrm{rk}_\alpha(A) := \max \rho(\vee A)$

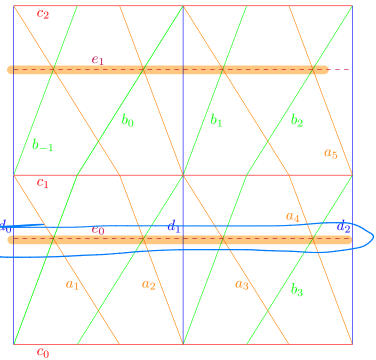
$m_\alpha(A) := |\vee A|$

$$T_\alpha(x,y) := \sum_{A \subseteq E_\alpha} m_\alpha(A)(x-1)^{\mathrm{rk}_\alpha(E_\alpha) - \mathrm{rk}_\alpha(A)}(y-1)^{|A| - \mathrm{rk}_\alpha(A)}$$

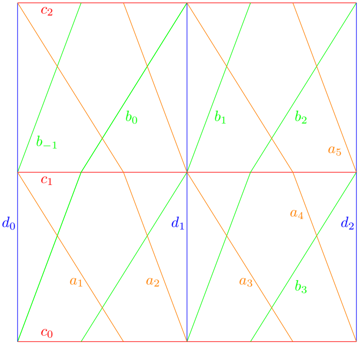
$T_\alpha(x,y) = y^2 + x^2 + 2y + 2x + 5$

# Deletion / Contraction

$\alpha$:

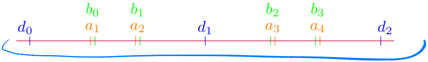



$\alpha \setminus e$:




$e := Ge_0$; $\mathrm{stab}(e) := \mathrm{stab}(e_i)$

$\alpha / e$: $\mathrm{stab}(e) \circlearrowright$



# Translative actions

[D.-Riedel '18]

Recall: if $\alpha$ is translative, then $(\underline{\mathcal{K}}, \mathrm{rk}_\alpha)$ is a semimatroid on $E_\alpha$.

A loop is any $e \in E_\alpha$ such that $\mathrm{rk}_\alpha(e) = 0$;

A coloop is any $e \in E_\alpha$ such that $\mathrm{rk}_\alpha(I \setminus \{e\}) < \mathrm{rk}_\alpha(I)$ for all $I \ni e$.

Theorem. If $\alpha$ is translative, for all $e \in E_\alpha$ we have the recursion

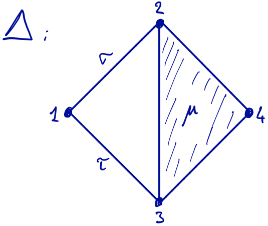
$$T_\alpha(x,y) = (x-1)T_{\alpha\setminus e}(x,y) + (y-1)T_{\alpha/e}(x,y),$$

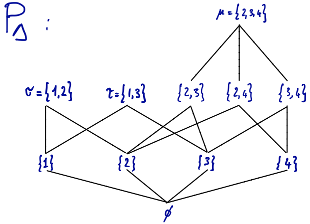
according to whether $e$ is a coloop or a loop, where

$\alpha \setminus e := G \circlearrowright (\mathcal{K}, \mathrm{rk}) \setminus e, \quad \alpha/e := \mathrm{stab}(e) \circlearrowright (\mathcal{K}, \mathrm{rk})/e.$

# Simplicial complexes and their face rings

A simplicial complex $\Delta$ on a finite vertex set $V$ has a poset of faces



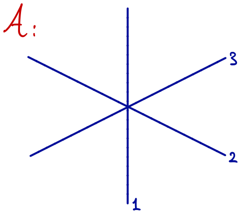



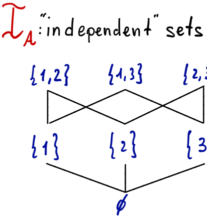

If $|V| < \infty$, the face ring ("Stanley-Reisner Ring") of $\Delta$ (over a field $\mathbb{K}$) is

$$\mathcal{R}(\Delta) := \mathbb{K}[x_v \mid v \in V] \Big/ \left( \prod_{v \in \tau} x_v \,\middle|\, \tau \notin \Delta \right)$$

# Arrangements and matroids

Let $\mathscr{A}$ be a linear arrangement; $\mathcal{I}_{\mathscr{A}}$ complex of "independent sets"




Algebraic interpretation of some number sequences / polynomials. E.g.,

$$\mathrm{Hilb}(\mathcal{R}(\mathcal{I}_{\mathscr{A}}), t) = \frac{t^d T(1/t, 1)}{(1-t)^d}$$

(This really is a "matroid" statement)

# Group actions on simplicial posets

[D.-d'Alì '21]

Call a chain-finite poset $P$ simplicial if

(1) $P$ is bounded below

(2) $P_{\leq x}$ is Boolean for all $x$.

Examples. The face poset $P_\Delta$ of a simplicial complex $\Delta$.

The independence poset $\mathcal{I}(\mathcal{L})$ of a geometric semilattice.

Theorem. Let $G$ act by automorphisms on a simplicial poset $P$. Then

$$P/G \text{ is a simplicial poset} \Leftrightarrow \text{the action is translative.}$$

Corollary. If $\alpha : G \circlearrowright \mathcal{L}$ translative, then $\mathcal{I}_\alpha$ is a simplicial poset.

For toric arrangements see also [Martino/Lenz]

# Stanley-Reisner rings

[...à la Yuzvinsky]

Let $P$ be a simplicial poset. For $\emptyset \neq M \subseteq \max P$ set $M^{\cap} := \bigcap_{m \in M} P_{\leq m}$.

$M^{\cap}$ is the poset of faces of a simplicial complex $\Delta_M$

$X(P) := \{\Delta_M \mid \emptyset \neq M \subseteq P\}, \quad \Delta_{M_1} \leq \Delta_{M_2}$ if $M_1 \supseteq M_2$

$Y(P)$: sheaf on $X(P)$. $\quad Y(P)(\Delta_M) := \mathcal{R}(\Delta_M)$, natural projections.

Definition. The Stanley-Reisner ring of $P$ is the ring of global sections

$$\boxed{\mathcal{R}(P) := \Gamma Y(P)}$$

Theorem If $|P| < \infty$, then $\mathcal{R}(P)$ is Stanley's "face ring" of $P$.

(In particular, $\mathcal{R}(P_\Delta) \simeq \mathcal{R}(\Delta)$ for every finite simplicial complex $\Delta$.)

[D.-D'Alì '21, See also: Lü-Panov '11, Brun-Römer '08]

## RINGS OF INVARIANTS

Let $P$ be a simplicial poset.

Every action $G \circlearrowright P$ induces an action $G \circlearrowright \mathcal{R}(P)$.

**Theorem.** If the action $G \circlearrowright P$ is translative, then

ring of invariants

$$\mathcal{R}(P)^G \simeq \mathcal{R}(P/G)$$

[Garsia-Stanton '86: finite Coxeter complexes]

[Reiner '92: finite balanced complexes]

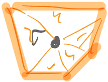
$\Delta$ finite s.c. $G \circlearrowright \Delta$ simplicial action induces $G \circlearrowright P_\Delta$

- $G \circlearrowright P_\Delta$ is translative $\iff$ G preserves proper coloring of $\Delta$
  $\iff$ "Bredon's condition (A)"

- $G \circlearrowright \Delta(P_\Delta)$ Always translative

# The Cohen-Macaulay property

A f.d. simplicial complex $\Delta$ is Cohen-Macaulay if, for every $\sigma \in \Delta$, the link of $\sigma$ in $\Delta$, $\mathrm{lk}(\sigma)$, is connected through codimension 1.



A f.l. (simplicial) poset $P$ is Cohen-Macaulay if the simplicial complex $\Delta(P)$ of all chains in $P$ is Cohen-Macaulay.

Note When saying "Cohen-Macaulay in characteristic $\kappa$" replace by:

$$\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma), \mathbb{K}) = 0 \text{ for all } i < \dim(\mathrm{lk}_\Delta(\sigma)) \text{ and } \mathrm{char}(\mathbb{K}) = \kappa.$$

# Back to geometric semilattices

Let $\mathcal{L}$ be a geometric semilattice, $\alpha : G \circlearrowright \mathcal{L}$ a translative action.

Observation. $\mathrm{Hilb}(\mathcal{R}(\mathcal{I}_\alpha)^G, t) = \dfrac{t^d T_\alpha(\frac{1}{t}, 1)}{(1-t)^{d-1}}$.

Definition an action $\alpha : G \circlearrowright \mathcal{L}$ is refined if it is translative, $G$ is free abelian, and there is $k \in \mathbb{N}$ such that, for every $x \in \mathcal{L}$:

$$\mathrm{stab}(x) \text{ is a direct summand of } G\text{, free of rank } k \cdot (\rho(\mathcal{L}) - \rho(x))$$

Theorem. If $\alpha$ is refined, then $\mathcal{P}_\alpha$ and $\mathcal{I}_\alpha$ are Cohen-Macaulay in characteristic 0 and in every characteristic not dividing an explicitly computable $\delta_\alpha$.

Note. Top Betti numbers: $T_\alpha(0,0)$ and $-T_\alpha(0,1)$.

## "EXPLICITLY COMPUTABLE"...

Let $\alpha : G \circlearrowright \mathcal{L}$ be a refined action on a geometric semilattice with associated underlying matroid $(E_\alpha, \mathrm{rk}_\alpha)$.

For every $I \subseteq E_\alpha$ let

$$G^{(I)} := G/\operatorname{stab}_G(I).$$

Then:

$$\delta_\alpha := \operatorname{lcm}\{\delta_\alpha(B) \mid B \text{ basis of } (E_\alpha, \mathrm{rk}_\alpha)\},$$

where, for every basis $B$,

$$\delta_\alpha(B) := \left[ G^{(B)} : \bigoplus_{b \in B} \operatorname{stab}_{G^{(I)}}(B \setminus b) \right]$$