Toric arrangements and equivariant matroid theory

Lecture 3: Supersolvability and applications

Emanuele Delucchi
IDSIA USI/SUPSI
Lugano, Switzerland.

ASGARD24
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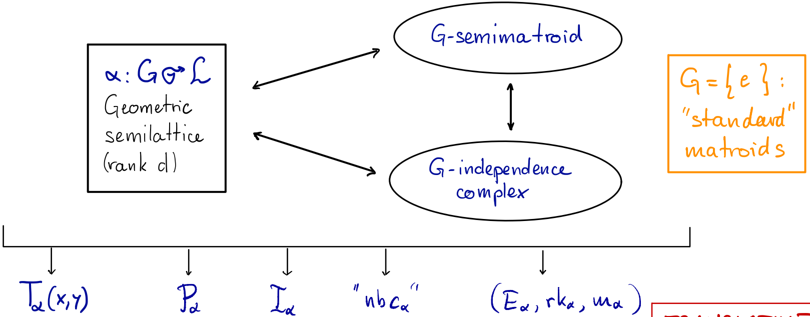
SUMMARY



TRANSLATIVE

Deletion/contraction

$$\chi_{P_\alpha}(t) = (-1)^d T_\alpha(1-t,0)$$

$$\mathrm{Hilb}(R(I)^G) = \frac{t^d T_\alpha\left(\frac{1}{t},1\right)}{(1-t)^{d-1}}$$

$$\tilde{\chi}(\mathrm{nbc}_\alpha) = (-1)^d T_\alpha(0,0)$$

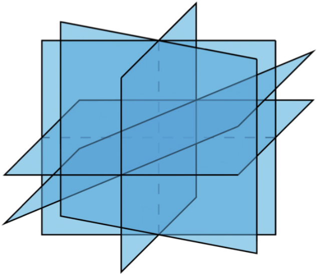
matroid & (P)

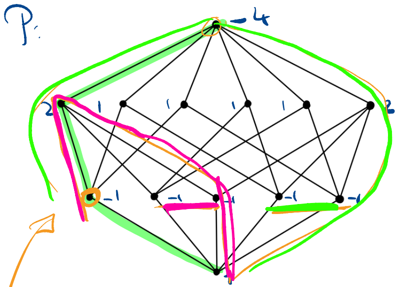
"NORMAL" &(A1.2) &(A2)

"REFINED" P_α, I_α CM(S)

"ARITHMETIC" Arithm. matroid

REPRESENTABLE

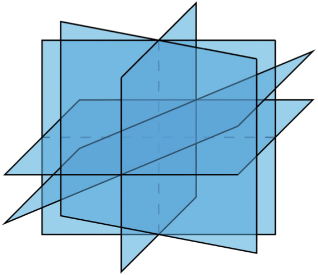


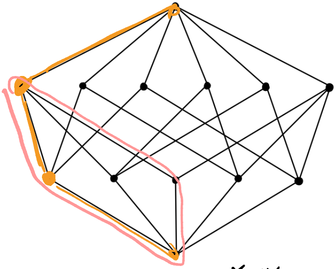


$\chi_p(1) = 0$

$$\chi_p(t) = t^3 - 5t^2 + 8t - 4$$
$$= (t-1)(t^2 - 4t + 4)$$
$$= (t-1)(t-2)(t-2)$$

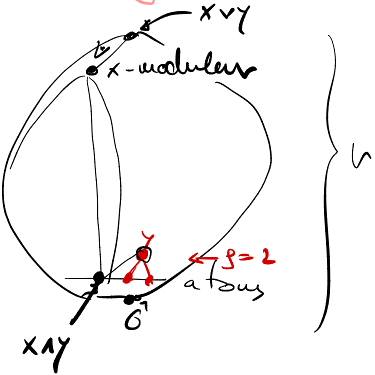
supersolvable

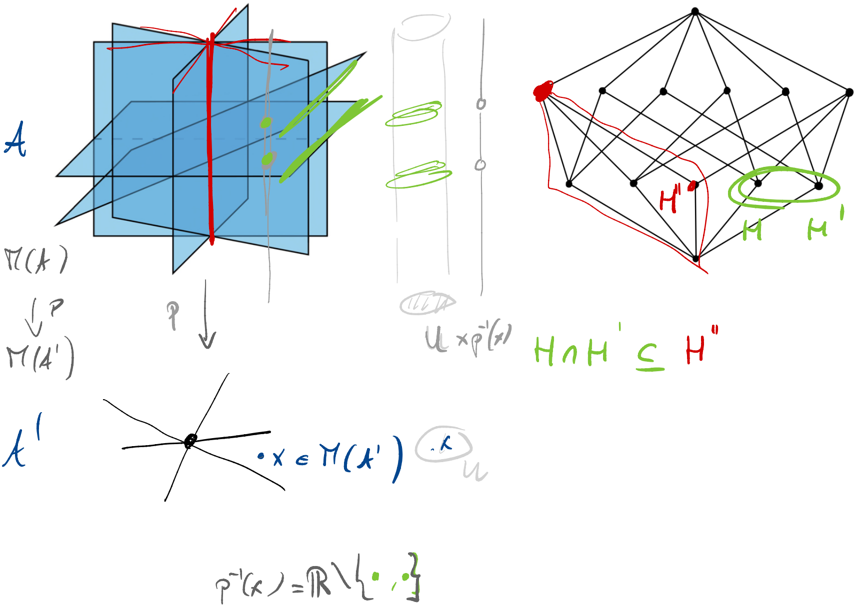




$$\rho(x \vee y) + \rho(x \wedge y) \qquad \rho(x) + \rho(y)$$

$$r + 1 = r-1 + 2$$

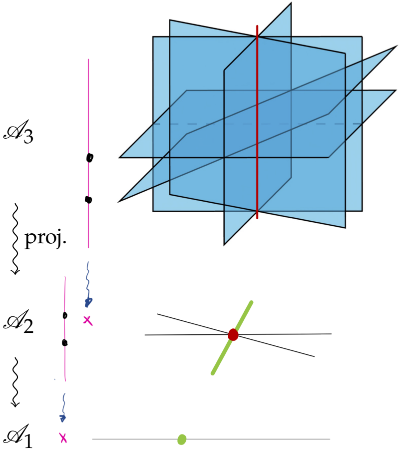


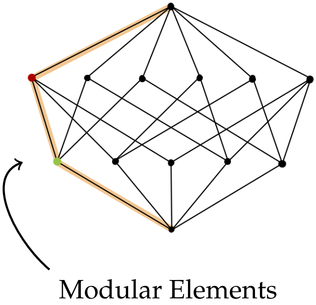


A
M(A)
p
M(A')
p
U × p⁻¹(x)
H''
H
H'
H ∩ H' ⊆ H''
A'
x ∈ M(A')
x
U
p⁻¹(x) = ℝ \ {• , •}

Fibrations of hyperplane arrangements

[Falk-Randell '85; Terao '86]





Fiber-type $\Leftrightarrow$ Supersolvable

The $K(\pi, 1)$-problem

An arrangement $\mathscr{A}$ is called $K(\pi, 1)$ if $\pi_i(M(\mathscr{A}))$ is trivial for $i > 1$.

$K(\pi, 1)$ problem: does $\mathcal{P}(\mathscr{A})$ know whether $\mathscr{A}$ is $K(\pi, 1)$?

For hyperplane arrangements this is a classical and storied problem.

E.g., finite real [Deligne '72] and complex [Bessis '12] reflection arrangements, as well as fiber-tipe arrangements [Falk-Randell '85] are $K(\pi, 1)$.

The following classes of non-linear arrangements are $K(\pi, 1)$.

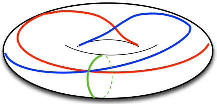
Toric **Coxeter** arrangements (via [Paolini-Salvetti '21])

Large type toric arrangements (via [Hendriks '85])

Fiber-type toric and elliptic arrangements [Bibby-D. '20]

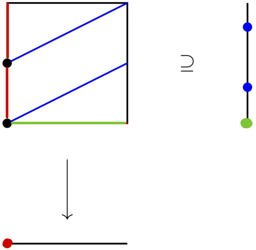
Fiber-type abelian arrangements

Let $\mathscr{A}$ be an abelian arrangement in $\mathrm{Hom}(\Gamma, \mathbb{G}) \simeq \mathbb{G}^d$.

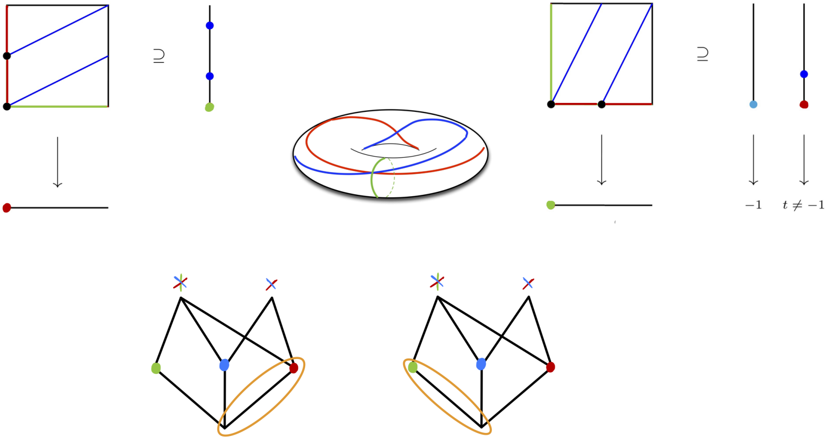


$\mathscr{A}$ is fiber-type if $d = 1$, or if there exists a rank-one, split-direct summand $N \subseteq \mathbb{Z}^d$ and an arrangement $\mathscr{B}$ in $\mathrm{Hom}(\Gamma/N, \mathbb{G}) \simeq \mathbb{G}^{d-1}$, such that:

- $\mathscr{B}$ is fiber-type
- The natural projection $\mathbb{G}^d \to \mathbb{G}^{d-1}$ restricts to a fibration $M(\mathscr{A}) \to M(\mathscr{B})$ with fiber homeomorphic to $\mathbb{G} \setminus \{\text{points}\}$.

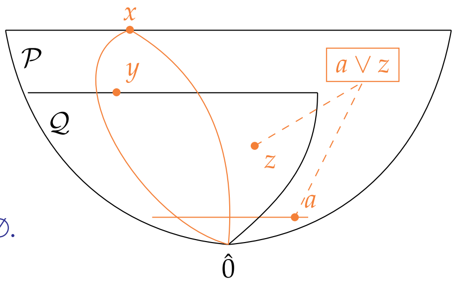


Note. Fiber-type linear, toric, elliptic arrangements are $K(\pi, 1)$.


⊇
⊇
−1
t ≠ −1

Supersolvable posets

Let $\mathcal{P}$ be a locally geometric poset.



An M-ideal of $\mathcal{P}$ is a pure,

join-closed order ideal $\mathcal{Q} \subseteq \mathcal{P}$ s.t.

$a, b \in \mathcal{Q} \rightsquigarrow a \vee b \in \mathcal{Q}$

- For $z \in \mathcal{Q}$ any atom $a \notin \mathcal{Q}$: $z \vee a \neq \emptyset$.

- For every $x \in \max \mathcal{P}$ there is $y \in \max \mathcal{Q}$ s.t. y is modular in $\mathcal{P}_{\leq x}$

Definition. $\mathcal{P}$ is supersolvable if there is a sequence of M-ideals

$$\{\hat{0}\} = \mathcal{Q}_0 \subseteq \mathcal{Q}_1 \subseteq \ldots \subseteq \mathcal{Q}_k = \mathcal{P} \quad \text{with } \mathcal{Q}_i \text{ of height } h(\mathcal{Q}_i) = i.$$

SUPERSOLVABLE POSETS

Let $\mathcal{P}$ be a locally geometric poset.

TM-ideal

An M-ideal of $\mathcal{P}$ is a pure,

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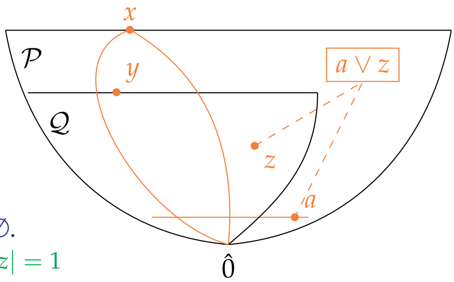

- For $z \in \mathcal{Q}$ any atom $a \notin \mathcal{Q}$: $z \vee a \neq \emptyset$.

$|a \vee z| = 1$

- For every $x \in \max \mathcal{P}$ there is $y \in \max \mathcal{Q}$ s.t. y is modular in $\mathcal{P}_{\leq x}$

strictly

Definition. $\mathcal{P}$ is supersolvable if there is a sequence of M-ideals

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Factorizations

[Bibby-D. '21]

Let $\mathcal{P}$ be a finite locally geometric poset.

Lemma. If $\mathcal{Q}$ is a TM-ideal of P with $h(\mathcal{Q}) = h(\mathcal{P}) - 1$, then

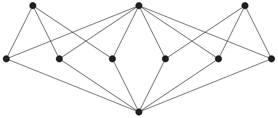
$$\chi_{\mathcal{P}}(t) = \big(t - |A(\mathcal{P}) \setminus A(\mathcal{Q})|\big)\chi_{\mathcal{Q}}(t).$$

Theorem. If $\mathcal{P}$ is strictly supersolvable via $Q_0 \subseteq \ldots \subseteq \mathcal{Q}_k$, then

$$\chi_{\mathcal{P}}(t) = (t - a_1)\cdots(t - a_k)$$

where $a_i := |A(\mathcal{Q}_i) \setminus A(\mathcal{Q}_{i-1}|$

Note. This is not a necessary condition, see [Pagaria-Pismataro-Tran-Vecchi]



$$\chi(t) = t^2 - 6t + 9 = (t-3)^2$$

Fibration theorem

Theorem. [Bibby-D. '21] Let $\mathscr{A}$ be an abelian arrangement

$$\mathscr{A} \text{ is fiber-type if and only if } \mathcal{P}(\mathscr{A}) \text{ is supersolvable.}$$

In particular, if $\mathscr{A}$ is linear, toric or elliptic, then $\mathscr{A}$ is $K(\pi, 1)$.

Theorem. [Bibby-Cohen-D. '24+]

If $\mathscr{A}$ is a supersolvable toric arrangement, then $\pi_1(M(\mathscr{A}))$ is an iterated semidirect product of free groups. (Almost direct if *strictly* supersolvable.)

Lemma. [Bibby-D. '21] If $\mathcal{P}$ is a geometric poset, $G \circlearrowright \mathcal{P}$ is translative and $\mathcal{Q} \subseteq \mathcal{P}$ is G-invariant, then $\mathcal{Q}$ is an M-ideal **if and only if** $\mathcal{Q}/G \subseteq \mathcal{P}/G$ is.

Application. Bloch-Kato property of $\pi_1(M(\mathscr{A}))$. [D.-Marmo '24+]

Thank you!

"Takk skal du ha"