Toric arrangements and equivariant matroid theory
Lecture Notes

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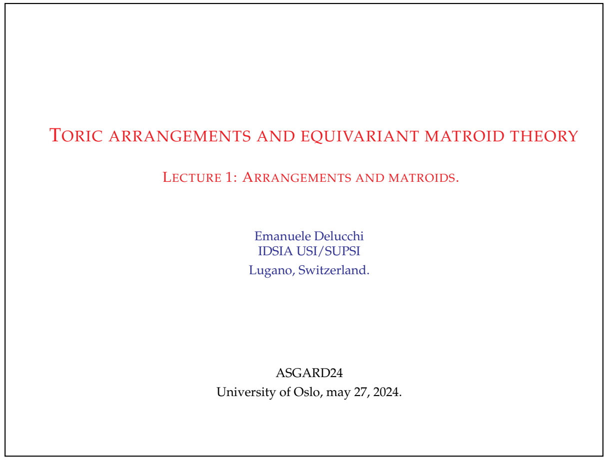


Throughout we will suppose that P be a chain-finite poset (all chains in P have finite cardinality), indeed all our posets will be of **finite length** (i.e., the cardinality of the chains P is bounded). Given $x \in P$ let $P_{\geq x} := \{x' \in P \mid x' \geq x\}$, $P_{\leq x} := \{x' \in P \mid x' \leq x\}$.

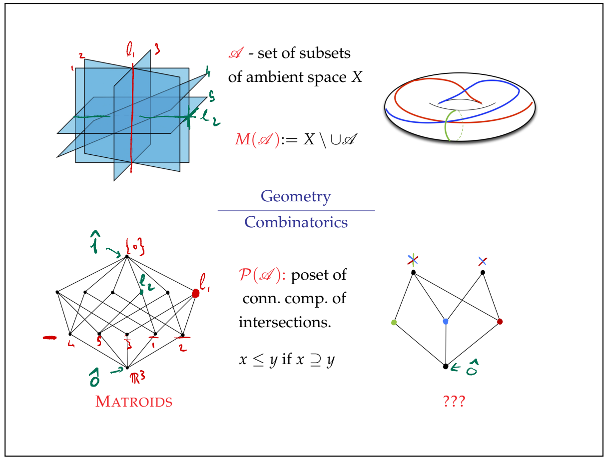
Define the set of joins, resp. of meets of a pair of elements $x, y \in P$ as

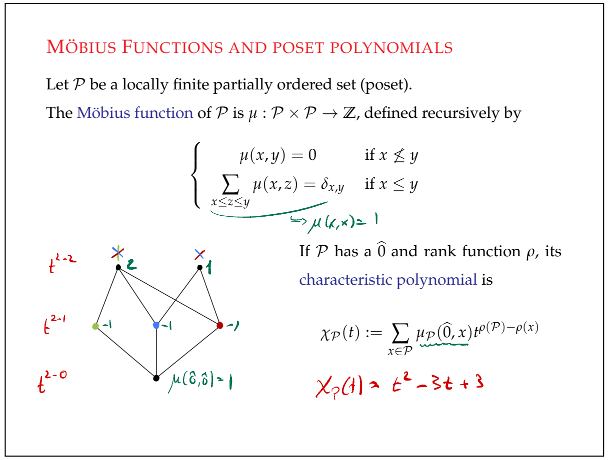
$$x \vee y := \min(P_{\geq x} \cap P_{\geq y}); \qquad x \wedge y := \max(P_{\leq x} \cap P_{\leq y}).$$

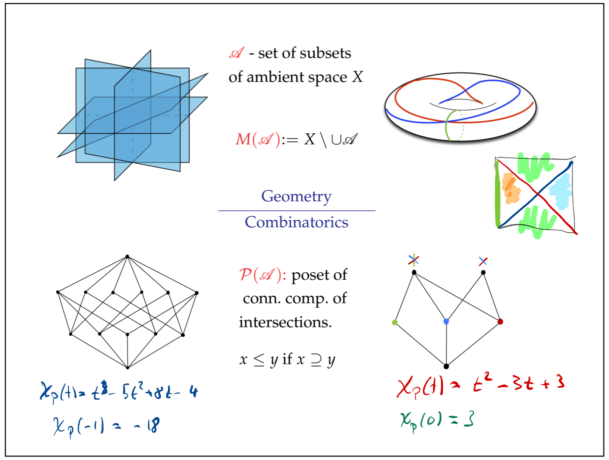
The poset P is a *meet-semilattice* if $|x \wedge y| = 1$ for all $x, y \in P$. If additionally $|x \vee y| = 1$ for all $x, y \in P$, then P is a *lattice*.

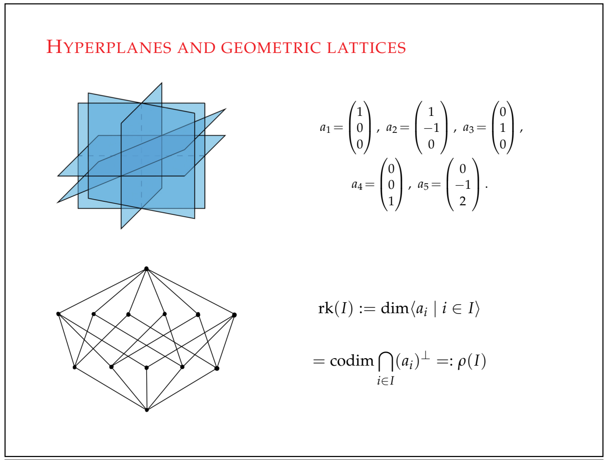
A poset P is **bounded below** if it has a unique minimal element, that is called $\hat{0}$. I.e., there is $\hat{0} \in P$ such that $P_{\geq \hat{0}} = P$. We will say that P is a "poset with $\hat{0}$".

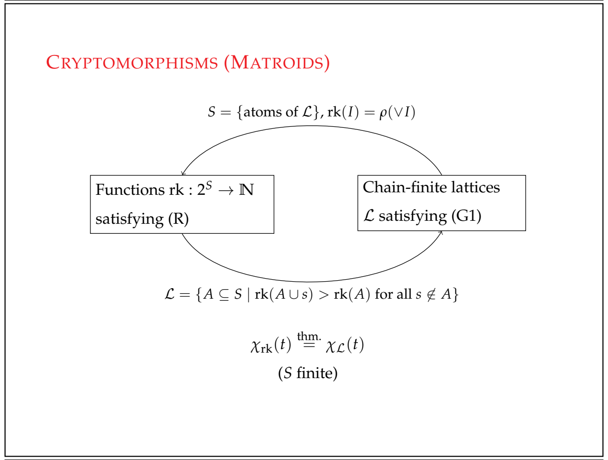
A **poset rank function** on a poset P with $\hat{0}$ is a function $\rho : P \to \mathbb{N}$ with $\rho(\hat{0}) = 0$ and such that $x \gtrdot y$ implies $\rho(x) = \rho(y) + 1$. (Recall that $x \gtrdot y$ means that $x > z \geq y$ implies $z = y$ for all z.) Note that if such a rank function exists, it is unique.

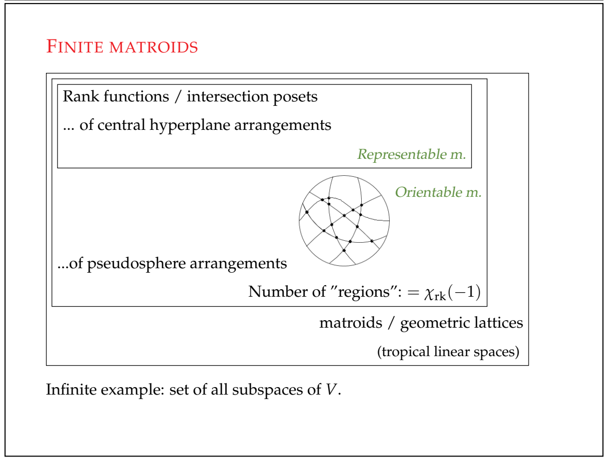












Let V be a vectorspace of (finite) dimension d over the field $\mathbb{K}$.

An **arrangement of hyperplanes** in V is any locally finite family $\mathscr{A} = \{H_i\}_{i\in S}$, where $H_i = \{v \in V \mid a_i(v) = b_i\}$ for some choice of tuples $(a_i)_{i\in S} \subseteq V^*$ and $(b_i)_{i\in S} \subseteq \mathbb{K}$.

The associated *rank function* is $\mathrm{rk}_{\mathscr{A}} : 2^S \to \mathbb{N}$, $\mathrm{rk}_{\mathscr{A}}(X) := \dim\langle a_i \mid i \in X\rangle$.

Call $\mathscr{A}$ **central** if $b_i = 0$ for all i (this implies $|S| < \infty$). In this case, $\mathcal{P}(\mathscr{A})$ satisfies the following.

Definition. A **geometric lattice** is a chain-finite lattice P such that

$$\text{(G1)}\ x \lessdot y \Leftrightarrow y \in x \vee a \text{ for some } a \gtrdot \hat{0}.$$

Notice that a chain-finite meet-semilattice is always bounded below. In any bounded-below poset P, the elements of $A(P) := \{a \in P \mid a \gtrdot \hat{0}\}$ are called **atoms**.

Remark. Every geometric lattice admits a poset rank function ρ. If $\mathscr{A}$ is a hyperplane arrangement, $\rho_{\mathscr{A}}(x) = \mathrm{codim}(X)$ for all $X \in \mathcal{P}(\mathscr{A})$.

In turn, the rank function $\mathrm{rk}_{\mathscr{A}}$ satisfies the following definition.

Definition. A **matroid rank function** on the ground set S is any monotone function $\mathrm{rk} : 2^S \to \mathbb{N}$ s.t.

$$(R) : \begin{cases} 0 \le \mathrm{rk}(X) \le |X| \quad \forall X \subseteq S \\ \mathrm{rk}(X\cup Y) + \mathrm{rk}(X \cap Y) \le \mathrm{rk}(X) + \mathrm{rk}(Y) \quad \forall X, Y \subseteq S \\ \forall X \subseteq S\ \exists Y \subseteq X,\ |Y| < \infty,\ \mathrm{rk}(X) = \mathrm{rk}(Y). \end{cases}$$

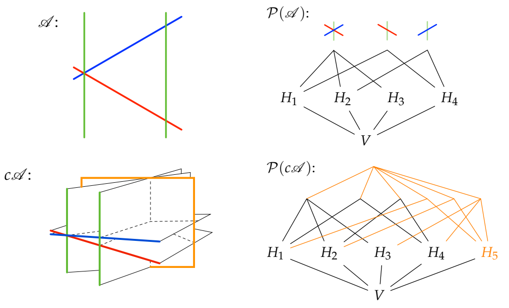
If $|S| < \infty$, define

$$\chi_{\mathrm{rk}}(t) := \sum_{X\subseteq S} (-1)^{|X|} t^{\mathrm{rk}(S)-\mathrm{rk}(X)}.$$

For matroids in this (finitary) setting see [**1**].

Affine hyperplane arrangement ($\mathbb{K} = \mathbb{R}$)

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad (b_1, b_2, b_3, b_4) = (0,0,0,1)$$



We now consider an example of an affine hyperplane arrangement. Then, the poset of intersections has the following structure.

Definition. A **geometric semilattice** is any poset of the form $L \setminus L_{\geq a}$ where L is a geometric lattice and $a \in A(L)$ is an atom of L.

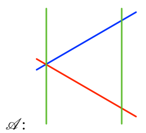
Equivalently, a geometric semilattice is any chain-finite meet-semilattice P that satisfies (G1) – thus admits a rank function ρ – and

(G2) for all $x, y \in P$ and for all $I \subseteq A(P)$ s.t. $y \in \vee I$ and $\rho(x) < \rho(y) = |I|$, there is $a \in I$ with $x \vee a \neq \emptyset$.

For this and more on geometric semilattices see [**19**].

Affine hyperplane arrangements ($\mathbb{K} = \mathbb{R}$)

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad (b_1, b_2, b_3, b_4) = (0,0,0,1)$$



I such that $\cap_{i \in I} H_i \neq \emptyset$

$\{\}, \{1\}, \{2\}, \{3\}, \{4\}$

$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}$

$\{1,2,3\}$

These are the *central sets*.

The family of central sets $\mathcal{K} \subseteq 2^S$ is an abstract simplicial complex.

(I.e.: if $I \in \mathcal{K}$ and $J \subseteq I$, then $J \in \mathcal{K}$)

For $I \in \mathcal{K}$ set $\mathrm{rk}(I) := \dim\langle a_i \mid i \in I \rangle$. This defines a semimatroid.

[Kawahara '04, Ardila '07]

Definition. A **semimatroid rank function** is any monotone function $\mathrm{rk} : \mathcal{K} \to \mathbb{N}$ from a finite-dimensional simplicial complex $\mathcal{K}$ to the natural numbers satisfying

$$(R^s): \begin{cases} 0 \leq \mathrm{rk}(X) \leq |X| & \forall X \in \mathcal{K} \\ \mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y) & \forall X, Y \subseteq S \text{ s.t.} X \cup Y \in \mathcal{K} \\ \text{for all } X, Y \in \mathcal{K}, \mathrm{rk}(X) = \mathrm{rk}(X \cap Y) \text{ implies } X \cup Y \in \mathcal{K} \\ \text{for all } X, Y \in \mathcal{K}, \mathrm{rk}(X) < \mathrm{rk}(Y) \text{ implies } X \cup a \in \mathcal{K} \text{ for some } a \in Y \setminus X \end{cases}$$

For finite semimatroids [**13, 2**]. The infinite case is in [**10**].

Cryptomorphisms (semimatroids)

$\mathcal{K} := \{I \subseteq A(\mathcal{L}) \mid \vee I \neq \emptyset\}, \quad \mathrm{rk}(I) = \rho(\vee I)$

Functions $\mathrm{rk} : \mathcal{K} \to \mathbb{N}$ satisfying (R+)	Chain-finite semilattices $\mathcal{L}$ satisfying (G1) & (G2)

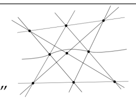
$\mathcal{L} = \{A \in \mathcal{K} \mid \mathrm{rk}(A') > \mathrm{rk}(A) \text{ for all } A' \in \mathcal{K}, A' \supsetneq A\}$

$\chi_{\mathrm{rk}}(t) \overset{\text{thm.}}{=} \chi_{\mathcal{L}}(t)$

(S finite)

Abstract theory

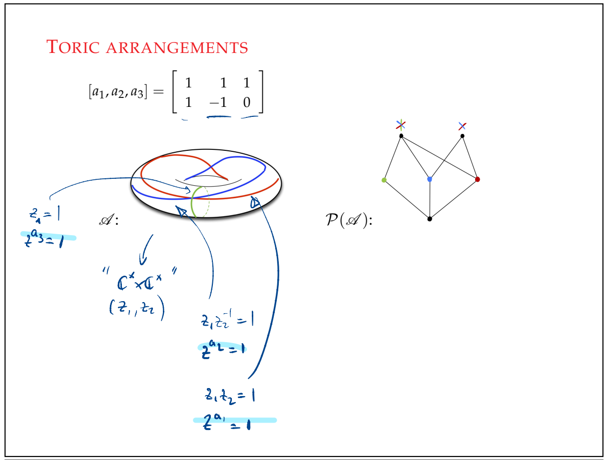
Semimatroid $(\mathcal{K}, \mathrm{rk})$ / intersection posets $\mathcal{L}$ of affine hyperplane arrangements

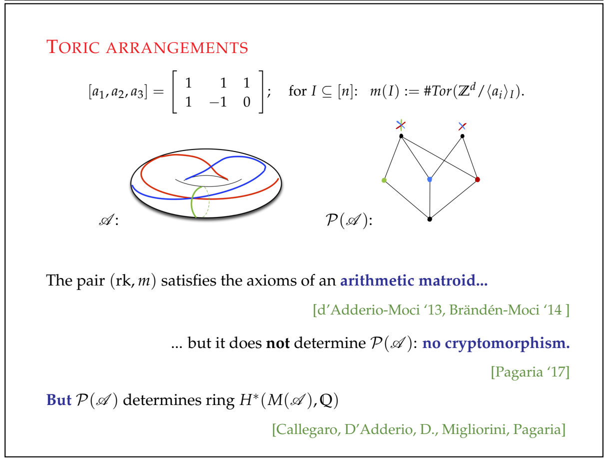


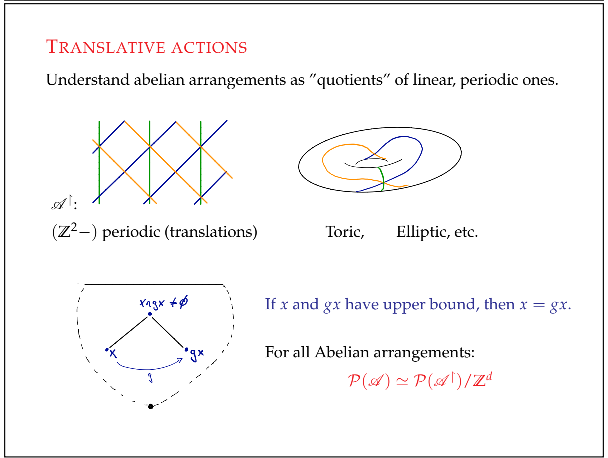
of "pseudoarrangements"

[Baum-Zhu '15, D.-Knauer DGC '24]

semimatroids / geometric semilattices







Let $a_1, \ldots, a_n \in \Gamma \simeq \mathbb{Z}^d$ be an n-tuple of nonzero, full-rank elements.

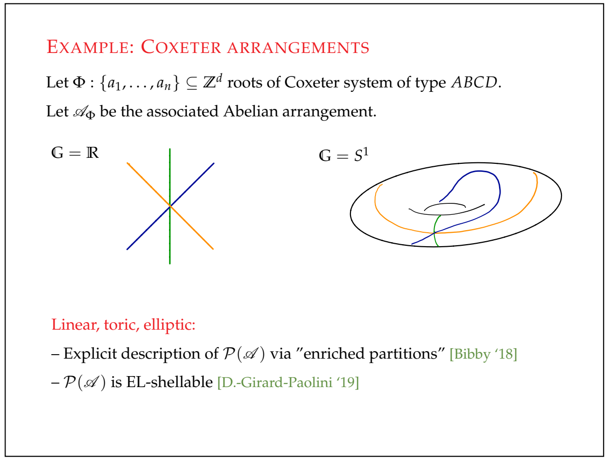
Definition. The associated **toric arrangement** is the set $\mathscr{A} = \{H_1, \ldots, H_n\}$ of subtori $H_i = \{z \in (\mathbb{C}^*)^d \mid z^{a_i} = 1\}$.

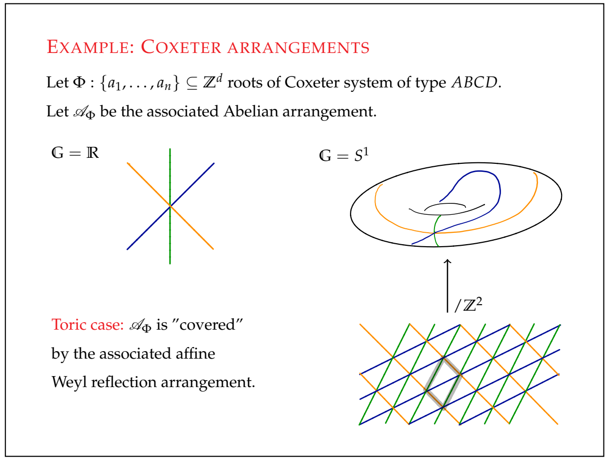
Fix $p, q \in \mathbb{N}$ and let $\mathbb{G} = \mathbb{R}^p \times (S^1)^q$.

Definition. The associated **abelian arrangement** in $\mathbb{G}^d$ is

$$\mathscr{A} = \{H_1, \ldots, H_n\}, \quad H_i = \ker\left(\mathrm{Hom}(\Gamma, \mathbb{G}) \to \mathbb{G}, \varphi \mapsto \varphi(a_i)\right)$$

If $(p, q) = (2, 0)$ call $\mathscr{A}$ "linear";
if $(p, q) = (1, 1)$ the arrangement $\mathscr{A}$ is toric;
if $(p, q) = (0, 2)$ call $\mathscr{A}$ "elliptic".





Given any abelian arrangement $\mathscr{A}$ let $\mathscr{A}^\dagger$ denote the lift of $\mathscr{A}$ to the universal cover of $\mathbb{G}^d$.

Note: in general, $\mathscr{A}^\dagger$ is an arrangement of subspaces (not necessarily of hyperplanes).
Yet, for every abelian $\mathscr{A}$ the poset $\mathcal{P}(\mathscr{A}^\dagger)$ is a geometric semilattice (see, e.g., [**7**]).

References.
For the Coxeter case see [**3, 8**].
For arithmetic matroids see [**6**].
Two toric arrangements with different posets but same arithmetic matroids [**14**].
For the cohomology computation in the toric case see [**5**].
For the "oriented matroidal counterpart" of semimatroids and group actions see [**9**].

Group actions on geometric semilattices

From now:
$\mathcal{L}$: a chain-finite geometric semilattice;
G: a group acting on $\mathcal{L}$ by poset automorphisms. $\Big\}\ \alpha : G \circlearrowleft \mathcal{L}$

$\mathcal{P}_\alpha = \mathcal{L}/G := \{Gx \mid x \in \mathcal{L}\}$; $\quad Gx \leq Gy$ iff $x \leq gy$ for some $g \in G$.

This is a poset (eg., since $\mathcal{L}$ has a rank function)

Definition. A c.-f. poset P with $\hat{0}$ is geometric if it satisfies (G1), (G2).

Theorem. Let $G \circlearrowleft P$ be a translative action. If P is geometric, so is P/G.

Corollary. If the action α is translative, then $\mathcal{P}_\alpha$ is a geometric poset.
So $\mathcal{P}(\mathscr{A})$ is geometric for every abelian arrangement $\mathscr{A}$.

Group actions on geometric semilattices

From now:
$\mathcal{L}$: a chain-finite geometric semilattice;
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$\mathcal{P}_\alpha = \mathcal{L}/G := \{Gx \mid x \in \mathcal{L}\}$; $\quad Gx \leq Gy$ iff $x \leq gy$ for some $g \in G$.

This is a poset with rank function ρ_α and set of atoms E_α. For $I \subseteq E_\alpha$ let

$$\mathrm{rk}_\alpha(I) := \max \rho_\alpha(\vee I), \qquad m_\alpha(I) := \# \vee I.$$

Theorem. $(E_\alpha, \mathrm{rk}_\alpha)$ is a (semi)matroid if and only if α is translative.

Definition.

$$T_\alpha(x,y) := \sum_{I \subseteq E_\alpha} m_\alpha(I)(x-1)^{\mathrm{rk}_\alpha(E)-\mathrm{rk}_\alpha(I)}(y-1)^{|I|-\mathrm{rk}_\alpha(I)}$$

Group actions on semimatroids

Let G be a group

A G-semimatroid	A G-geometric semilattice
$\alpha : G \circlearrowleft (\mathcal{K}, \mathrm{rk})$	$\alpha : G \circlearrowleft \mathcal{L}$
is an action of G on a semimatroid $(\mathcal{K}, \mathrm{rk})$ by automorphisms of $\mathcal{K}$.	is an action of G on a geometric semilattice $\mathcal{L}$ by poset-automorphisms.

Cryptomorphism! ✓

If α is translative: (1) $T_\alpha(x,y)$ has positive coefficients
(2) $\chi_{\mathcal{P}_\alpha}(t) = (-1)^{\mathrm{rk}(E_\alpha)} T_\alpha(1-t, 0)$ [D.-Riedel '18]

Toric arrangements and equivariant matroid theory

Lecture 2: Polynomials and Stanley-Reisner rings

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We call a poset **pure** if all maximal chains have the same (finite) length.

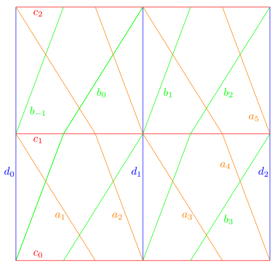
Recall. A **geometric poset** is any bounded-below, pure poset satisfying (G1) and (G2).

An action of a group G on a chain-finite poset P is **translative** if, for all $g \in G$, $x \vee gx \neq \emptyset$ implies $x = gx$.

Main references for this lecture: For the Tutte polynomial part: [**10**], for the remainder [**7**].

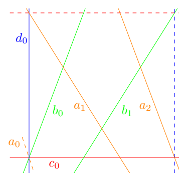
Group actions on semimatroids

$\mathbb{Z}^2 \circlearrowleft$

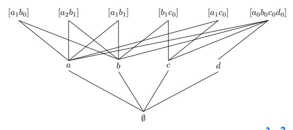


$S := \{a_i, b_j, c_k, d_l\}_{i,j,k,l \in \mathbb{Z}}$, $\quad \mathcal{L} :=$ poset of intersections

Example ($G = \mathbb{Z}^2$)



$\mathcal{P}_\alpha := \mathcal{L}/G$



$E_\alpha := S/G = \{a, b, c, d\}$

$\mathcal{K} := \{I \subseteq E_\alpha \mid \vee I \neq \emptyset\} = \{ \{a,b,c,d\}^{(1)};\ \{a,b,c\}^{(1)}, \{a,b,d\}^{(1)}, \{a,c,d\}^{(1)}, \{b,c,d\}^{(1)};\ \{a,b\}^{(4)}, \{a,c\}^{(2)}, \{b,c\}^{(2)}, \{a,d\}^{(1)}, \{b,c\}^{(1)}, \{b,d\}^{(1)};\ a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)};\ \emptyset^{(1)} \}$

$\mathrm{rk}_\alpha(A) := \max \rho(\vee A)$

$m_\alpha(A) := |\vee A|$

$T_\alpha(x,y) := \sum_{A \subseteq E_\alpha} m_\alpha(A)(x-1)^{\mathrm{rk}_\alpha(E_\alpha)-\mathrm{rk}_\alpha(A)}(y-1)^{|A|-\mathrm{rk}_\alpha(A)}$

$1\cdot(x-1)^{2-2}(y-1)^{4-2}$
$4\,(x-1)^{2-2}(y-1)^{3-2}$
$11\,(x-1)^{2-2}(y-1)^{2-2}$
$4\cdot(x-1)^{2-1}$
$1\cdot(x-1)^{2}\cdot 1$

$y^2-2y+1+4y-4+11+4x-4+x^2-2x+1$

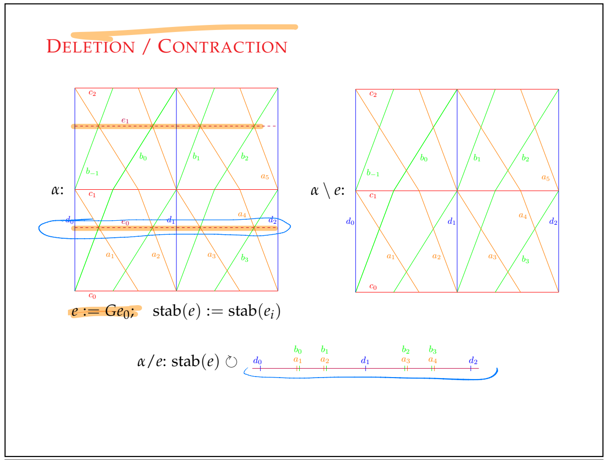
$T_\alpha(x,y) = y^2+x^2+2y+2x+5$

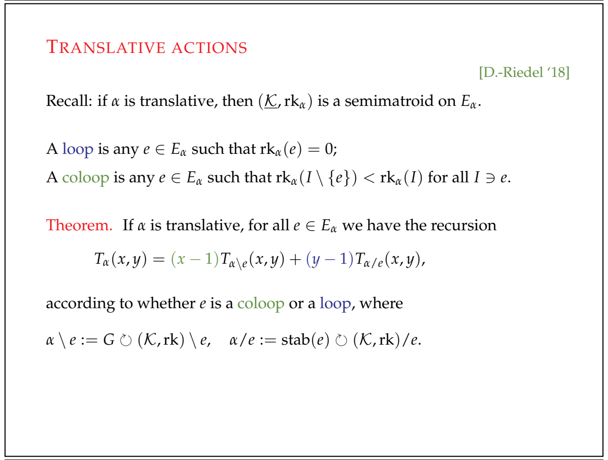
We write $\alpha : G \circlearrowleft \mathcal{L}$ for the data of a group action on a geometric semilattice $\mathcal{L}$.

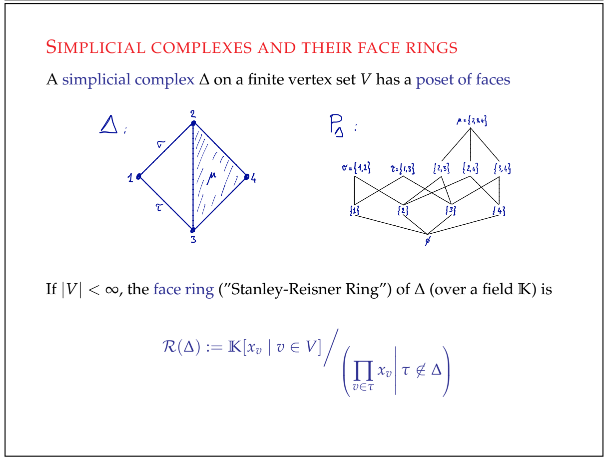
We have seen that if α is translative, then

(1) $\mathcal{P}_\alpha := \mathcal{L}/G$ is a geometric poset.
(2) $(E_\alpha, \mathrm{rk}_\alpha)$ is a (semi)matroid.
(3) $\chi_{\mathcal{P}_\alpha}(t) = (-1)^{\mathrm{rk}_\alpha(E_\alpha)} T_\alpha(x-1) = \chi_{\mathrm{rk}_\alpha}(t)$
(4) $T_\alpha(x,y)$ has positive coefficients.

We now explain the deletion-contraction recursion in the case α is translative.







Let $(\mathcal{K}, \mathrm{rk})$ be a semimatroid. Call S the set vertices of $\mathcal{K}$ and let $A \subseteq S$.

Set

$\mathcal{K}_{\setminus A} := \{X \setminus A \mid X \in \mathcal{K}\}$, $\mathrm{rk}_{\setminus A} := \mathrm{rk}_{|\mathcal{K}_{\setminus A}}$;

$\mathcal{K}_{/A} := \{X \in \mathcal{K}_{\setminus A} \mid X \cup A \in \mathcal{K}\}$; $\mathrm{rk}_{/A} : \mathcal{K}_{/A} \to \mathbb{N}$, $\mathrm{rk}_{/A}(X) := \mathrm{rk}(X \cup A) - \mathrm{rk}(A)$

Definition.
The **deletion** of A from $\mathcal{K}$ is the semimatroid $(\mathcal{K}, \mathrm{rk}) \setminus A := (\mathcal{K}_{\setminus A}, \mathrm{rk}_{\setminus A})$.
If $A \in \mathcal{K}$, the **contraction** of A in $(\mathcal{K}, \mathrm{rk})$ is $(\mathcal{K}, \mathrm{rk})/A := (\mathcal{K}_{/A}, \mathrm{rk}_{/A})$.

Note. The geometric semilattices of deletion and contraction of the semimatroids are as follows:

$$\mathcal{L}((\mathcal{K}, \mathrm{rk})/A) = \bigcup_{a \in A} \mathcal{L}(\mathcal{K}, \mathrm{rk})_{\geq a}; \qquad \mathcal{L}((\mathcal{K}, \mathrm{rk}) \setminus A) = \{\vee X \mid X \subseteq A(\mathcal{L}) \setminus A\} \subseteq \mathcal{L}$$

Now let $\alpha : G \circlearrowleft (\mathcal{K}, \mathrm{rk})$ denote a simplicial, rk-preserving action of a group G on $\mathcal{K}$, and let $e \in E_\alpha$.
Recall that then $e = Gs$ is the orbit of some $s \in S$.

Definition.
The **deletion** of e from α is the semimatroid $\alpha \setminus e : G \circlearrowleft (\mathcal{K}, \mathrm{rk}) \setminus e$.
The **contraction** of e in α is $\alpha/e : \mathrm{stab}(s) \circlearrowleft (\mathcal{K}, \mathrm{rk})/s$.

Note. If α is translative, then so are α/e and $\alpha \setminus e$.

Note. We have $\mathcal{P}_{\alpha/e} = \mathcal{P}_{\alpha \geq e}$. Moreover, $\mathcal{P}_{\alpha \setminus e}$ is the poset obtained as all elements of $\mathcal{P}_\alpha$ that can be obtained as joins of atoms different than e; more precisely: $\mathcal{P}_{\alpha \setminus e} = \bigcup_{A \subseteq A(\mathcal{P}_\alpha) \setminus \{e\}} \vee A$.

Warning. In the deletion-contraction formula given in the slide, the factor $(x-1)$ appears *only* if e is a coloop; the factore $(y-1)$ *only* if e is a loop.

Let $(\mathcal{K}, \mathrm{rk})$ be a semimatroid. The associated **independence complex** is the simplicial complex

$$\mathcal{I}(\mathcal{K}, \mathrm{rk}) := \{I \in \mathcal{K} \mid \mathrm{rk}(I) = |I|\}.$$

Note. The **poset of faces** of this simplicial complex, $P_{\mathcal{I}(\mathcal{K}, \mathrm{rk})}$ is a geometric semilattice, and all its lower intervals are boolean.
Note. If the semimatroid has no loops and has geometric semilattice $\mathcal{L}$, this is isomorphic to the abstract simplicial complex of independent sets of atoms $\mathcal{I}(\mathcal{L}) = \{I \subseteq A(\mathcal{L}) \mid |I| = \rho(\vee I)\}$.

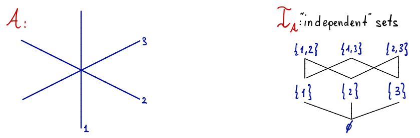
Definition. Let $\alpha : G \circlearrowleft (\mathcal{K}, \mathrm{rk})$ be a group action on a semimatroid. Define the **independence poset**

$$\mathcal{I}_\alpha := P_{\mathcal{I}(\mathcal{K}, \mathrm{rk})}/G$$

Note. If the action on $\mathcal{K}$ is translative (equivalently: the action on the associated semilattice is translative), then so is the induced action on $P_{\mathcal{I}(\mathcal{K}, \mathrm{rk})}$.

Arrangements and matroids

Let $\mathscr{A}$ be a linear arrangement; $\mathcal{I}_{\mathscr{A}}$ complex of "independent sets"



Algebraic interpretation of some number sequences / polynomials. E.g.,

$$\mathrm{Hilb}(\mathcal{R}(\mathcal{I}_{\mathscr{A}}), t) = \frac{t^d T(1/t, 1)}{(1-t)^d}$$

(This really is a "matroid" statement)

Stanley-Reisner rings

[...à la Yuzvinsky]

Let P be a simplicial poset. For $\emptyset \neq M \subseteq \max P$ set $M^\cap := \bigcap_{m \in M} P_{\leq m}$.

$M^\cap$ is the poset of faces of a simplicial complex Δ_M

$X(P) := \{\Delta_M \mid \emptyset \neq M \subseteq P\}$, $\Delta_{M_1} \leq \Delta_{M_2}$ if $M_1 \supseteq M_2$

$Y(P)$: sheaf on $X(P)$. $Y(P)(\Delta_M) := \mathcal{R}(\Delta_M)$, natural projections.

Definition. The Stanley-Reisner ring of P is the ring of global sections

$$\boxed{\mathcal{R}(P) := \Gamma Y(P)}$$

Theorem If $|P| < \infty$, then $\mathcal{R}(P)$ is Stanley's "face ring" of P.
(In particular, $\mathcal{R}(P_\Delta) \simeq \mathcal{R}(\Delta)$ for every finite simplicial complex Δ.)

[D.-D'Alì '21, See also: Lü-Panov '11, Brun-Römer '08]

Rings of invariants

Let P be a simplicial poset.
Every action $G \circlearrowleft P$ induces an action $G \circlearrowleft \mathcal{R}(P)$.

Theorem. If the action $G \circlearrowleft P$ is translative, then

ring of invariant

$$\mathcal{R}(P)^G \simeq \mathcal{R}(P/G)$$

[Garsia-Stanton '86: finite Coxeter complexes]
[Reiner '92: finite balanced complexes]

Δ finite s.c. $G \circlearrowleft \Delta$ simplicial action induces $G \circlearrowleft P_\Delta$

- $G \circlearrowleft P_\Delta$ is translative $\Leftrightarrow$ G preserves proper coloring of Δ
  $\Leftrightarrow$ "Bredon's condition (A)"
- $G \circlearrowleft \Delta(P_\Delta)$ Always translative

Group actions on simplicial posets

[D.-d'Alì '21]

Call a chain-finite poset P simplicial if (1) P is bounded below
(2) $P_{\leq x}$ is Boolean for all x.

Examples. The face poset P_Δ of a simplicial complex Δ.
The independence poset $\mathcal{I}(\mathcal{L})$ of a geometric semilattice.

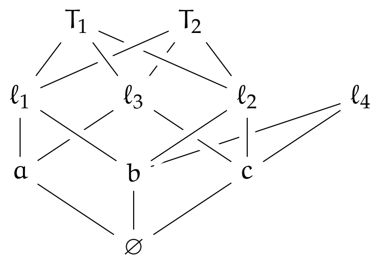
Theorem. Let G act by automorphisms on a simplicial poset P. Then

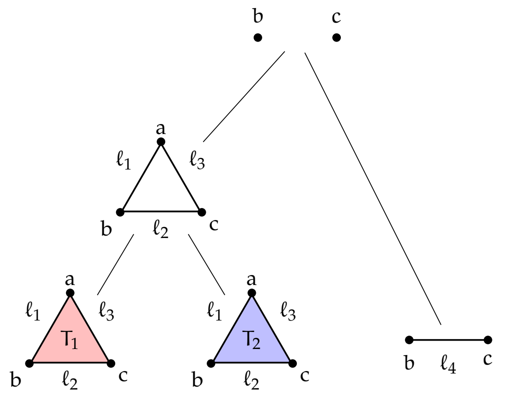
P/G is a simplicial poset $\Leftrightarrow$ the action is translative.

Corollary. If $\alpha : G \circlearrowleft \mathcal{L}$ translative, then $\mathcal{I}_\alpha$ is a simplicial poset.
For toric arrangements see also [Martino/Lenz]

Note. For every group action on a semimatroid α, the poset $\mathcal{I}_\alpha$ is simplicial.

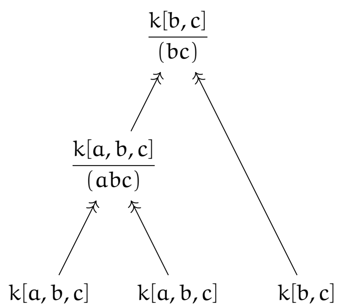
Note. General for this and the remainder of this lecture [7].

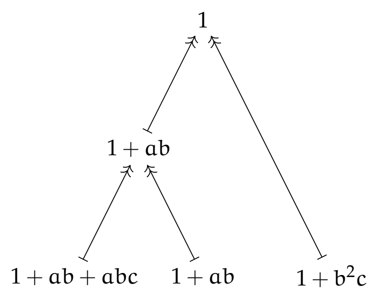




Example. A simplicial poset P and its associated $X(P)$.

Below is the sheaf $Y(P)$ and one of its global sections.





The Cohen-Macaulay property

A f.d. simplicial complex Δ is Cohen-Macaulay if, for every $\sigma \in \Delta$, the link of σ in Δ, $\mathrm{lk}(\sigma)$, is connected through codimension 1. $\leftarrow \widetilde{H}(\mathrm{lk}\,\sigma, \mathbb{Z})$

A f.l. (simplicial) poset P is Cohen-Macaulay if the simplicial complex $\Delta(P)$ of all chains in P is Cohen-Macaulay.

Note When saying "Cohen-Macaulay in characteristic κ" replace by:

$$\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma), \mathbb{K}) = 0 \text{ for all } i < \dim(\mathrm{lk}_\Delta(\sigma)) \text{ and } \forall \mathbb{K}\ \mathrm{char}(\mathbb{K}) = \kappa.$$

Back to geometric semilattices

Let $\mathcal{L}$ be a geometric semilattice, $\alpha : G \circlearrowright \mathcal{L}$ a translative action.

Observation. $\mathrm{Hilb}(\mathcal{R}(\mathcal{I}_\alpha)^G, t) = \dfrac{t^d T_\alpha(\frac{1}{t}, 1)}{(1-t)^{d-1}}$.

Definition an action $\alpha : G \circlearrowright \mathcal{L}$ is refined if it is translative, G is free abelian, and there is $k \in \mathbb{N}$ such that, for every $x \in \mathcal{L}$:

$\mathrm{stab}(x)$ is a direct summand of G, free of rank $k \cdot (\rho(\mathcal{L}) - \rho(x))$

Theorem. If α is refined, then $\mathcal{P}_\alpha$ and $\mathcal{I}_\alpha$ are Cohen-Macaulay in characteristic 0 and in every characteristic not dividing an explicitly computable δ_α.

Note. Top Betti numbers: $T_\alpha(0,0)$ and $-T_\alpha(0,1)$.

"Explicitly computable"...

Let $\alpha : G \circlearrowright \mathcal{L}$ be a refined action on a geometric semilattice with associated underlying matroid $(E_\alpha, \mathrm{rk}_\alpha)$.

For every $I \subseteq E_\alpha$ let

$$G^{(I)} := G/\,\mathrm{stab}_G(I).$$

Then:

$$\delta_\alpha := \mathrm{lcm}\{\delta_\alpha(B) \mid B \text{ basis of } (E_\alpha, \mathrm{rk}_\alpha)\},$$

where, for every basis B,

$$\delta_\alpha(B) := \left[G^{(B)} : \bigoplus_{b \in B} \mathrm{stab}_{G^{(I)}}(B \setminus b) \right]$$

Toric arrangements and equivariant matroid theory

Lecture 3: Supersolvability and applications

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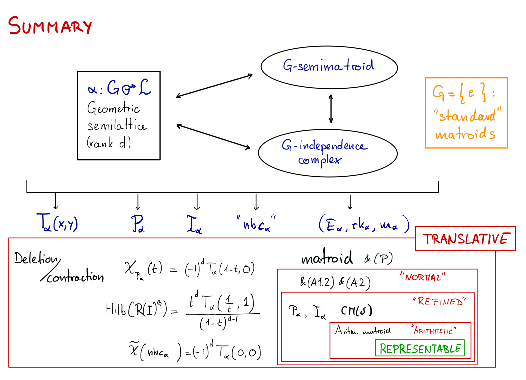
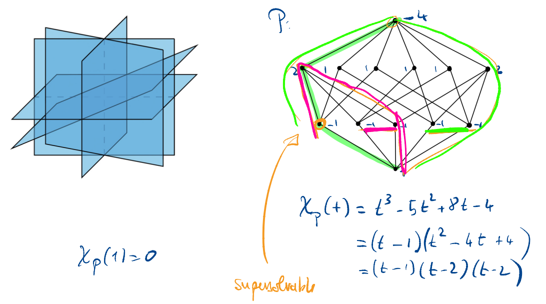




Let P be a geometric lattice with rank function ρ. An element $x \in P$ is **modular in** P if

$$\rho(x \vee x) + \rho(x \wedge y) = \rho(x) + \rho(y) \text{ for all } y \in P.$$

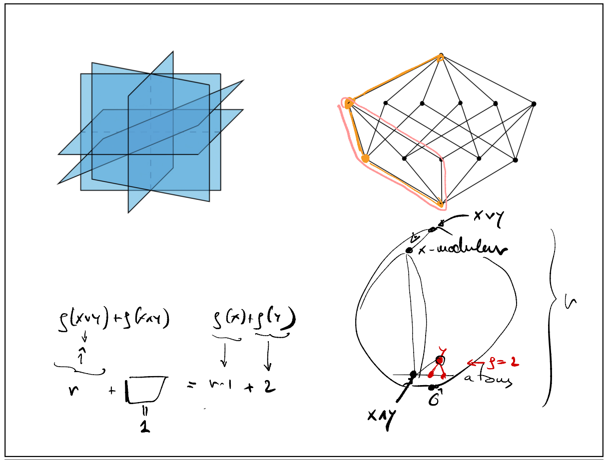
Theorem. [17] Suppose that P is finite. If x is modular in P, then $\chi_{P_{\leq x}}(t)$ divides $\chi_P(t)$. Moreover, if $x \lessdot \hat{1}$, then

$$\chi_P(t) = \chi_{P_{\leq x}}(t)(t-a), \text{ where } a = |A(P) \setminus A(P_{\leq x})|.$$

Definition. A geometric lattice P is **supersolvable** if it possesses a maximal chain consisting of modular elements.

Corollary. If P is a finite geometric lattice that is supersolvable via a chain $\hat{0} = x_0 \lessdot x_1 \lessdot \ldots \lessdot x_d = \hat{1}$, then

$$\chi_P(t) = \prod_{i=1}^{d} \left(t - |A(P_{\leq x_i}) \setminus A(P_{\leq x_{i-1}})|\right)$$



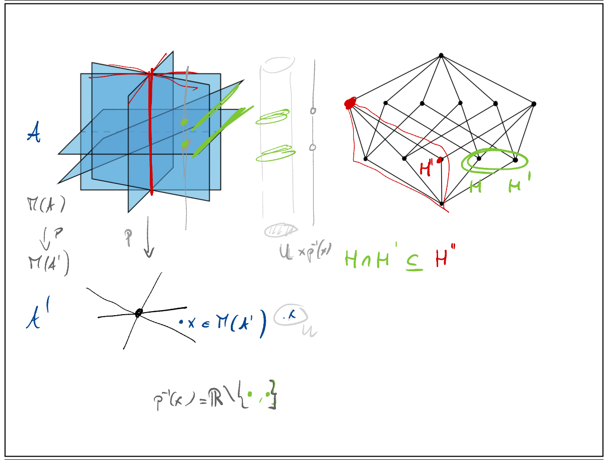
Let P be a geometric lattice and x a modular element with $x \lessdot \hat{1}$.

Consider $a, a' \in A(P) \setminus A(P_{\leq x})$ and let $y := a \vee a'$. Then $\rho(y) = 2$ and $x \vee y = \hat{1}$. Modularity of x implies

$$\rho(\hat{1}) + \rho(x \wedge y) = \rho(x \vee y) + \rho(x \wedge y) = \rho(x) + \rho(y) = (\rho(\hat{1}) - 1) + 2$$

whence $\rho(x \wedge y) = 1$. This means that *there is* $b \in A(P_{\leq x})$ *with* $y > b$.

This fact has a geometric meaning: if P is the lattice of intersections of a central hyperplane arrangement $\mathscr{A}$ and if ℓ is a modular intersection of dimension 1, then the intersection of any two hyperplanes not containing ℓ will be contained in some hyperplane that does contain ℓ. This means that the projection along ℓ defines a fibration of the complement of $\mathscr{A}$.

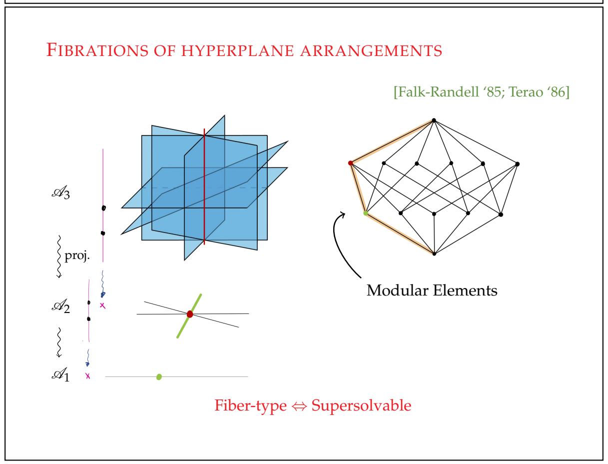


Definition. Let E, B be topological spaces. A continuous surjection $p : E \to B$ is a **bundle map** (or "has the bundle property" as in [**12**, III.4, p. 65]) if there exist a topological space F (called the "fiber space") such that for every $b \in B$ there is an open neighborhood U and a homeomorphism $\phi_U : U \times F \to p^{-1}(U)$ such that $p\phi_U$ is the canonical projection $U \times F \to U$.

Theorem 4.1 of [**12**] proves that every bundle map satisfies the homotopy lifting property (called "CHP" for "Covering Homotopy Property" in [**12**]) for every CW-complex, and thus is a fibration in the sense of Serre. In particular, if $p : E \to B$ is a bundle map with fiber F, $e_0 \in E$ is any basepoint, $b_0 := p(e_0)$ and $x_0 \in F$ is such that $\phi(b_0, x_0) = e_0$, there is a long exact sequence of homotopy groups

$$\ldots \to \pi_n(F, x_0) \to \pi_n(E, e_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \ldots \quad \ldots \to \pi_0(B, b_0) \to 0.$$

In particular, if $\pi_i(F, x_0) = 0$ for $i > 1$, we can conclude $\pi_i(B, b_0) \simeq \pi_i(E, e_0)$ for all $i > 1$.



Falk and Randell [**11**] studied fiber-type arrangements of hyperplanes. These are central hyperplane arrangements $\mathscr{A}$ in $\mathbb{C}^d$ that possess an intersection ℓ of dimension 1 such that the projection p of $\mathbb{C}^d$ along ℓ restricts to a bundle map $M(\mathscr{A}) \to M(\mathscr{A}')$, where $\mathscr{A}' := \{p(H) \mid H \in \mathscr{A} \text{ s.t. } \ell \subseteq H\}$ consists of the projections of all $H \in \mathscr{A}$ that contain ℓ is recursively required to be fiber-type, and the fiber then is homeomorphic to $\mathbb{C}$ minus a finite set of points.

Repeated application of the Long Exact Sequence argument above shows that a fiber-type arrangement $\mathscr{A}$ has $\pi(M(\mathscr{A})) = 0$ for all $i > 1$.

Terao's theorem [**18**] shows that a hyperplane arrangement $\mathscr{A}$ is fiber-type if and only if $\mathcal{P}(\mathscr{A})$ is a supersolvable geometric lattice. In fact, $\ell \lessdot \hat{1}$ is a suitable intersection of dimension 1 if and only if it is a modular element in $\mathcal{P}(\mathscr{A})$.

The $K(\pi,1)$-problem

An arrangement $\mathscr{A}$ is called $K(\pi,1)$ if $\pi_i(M(\mathscr{A}))$ is trivial for $i>1$.

$K(\pi,1)$ problem: does $\mathcal{P}(\mathscr{A})$ know whether $\mathscr{A}$ is $K(\pi,1)$?

For hyperplane arrangements this is a classical and storied problem.

E.g., finite real [Deligne '72] and complex [Bessis '12] reflection arrangements, as well as fiber-tipe arrangements [Falk-Randell '85] are $K(\pi,1)$.

The following classes of non-linear arrangements are $K(\pi,1)$.

Toric **Coxeter** arrangements (via [Paolini-Salvetti '21])

Large type toric arrangements (via [Hendriks '85])

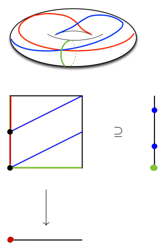
Fiber-type toric and elliptic arrangements [Bibby-D. '20]

Fiber-type abelian arrangements

Let $\mathscr{A}$ be an abelian arrangement in $\mathrm{Hom}(\Gamma,\mathbb{G})\simeq\mathbb{G}^d$.

$\mathscr{A}$ is fiber-type if $d=1$, or if there exists a rank-one, split-direct summand $N\subseteq\mathbb{Z}^d$ and an arrangement $\mathscr{B}$ in $\mathrm{Hom}(\Gamma/N,\mathbb{G})\simeq\mathbb{G}^{d-1}$, such that:

- $\mathscr{B}$ is fiber-type
- The natural projection $\mathbb{G}^d\to\mathbb{G}^{d-1}$ restricts to a fibration $M(\mathscr{A})\to M(\mathscr{B})$ with fiber homeomorphic to $\mathbb{G}\setminus\{\text{points}\}$.

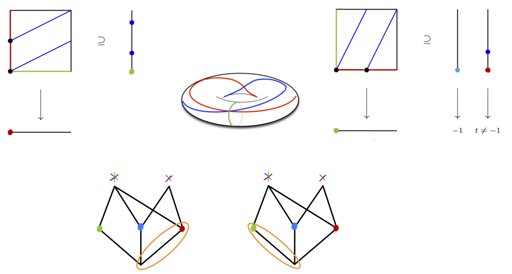


Note. Fiber-type linear, toric, elliptic arrangements are $K(\pi,1)$.

Main reference for what follows: [**4**]

For linear, resp. toric or elliptic arrangements the fiber is a punctured $\mathbb{R}^2$, resp. punctured $\mathbb{C}^*$ or $(\mathbb{S}^1)^2$.

In any case the fiber is homotopy equivalent to a wedge of circles, and so the above LES shows that the complement of a fiber-type linear, toric or elliptic arrangement is $K(\pi,1)$.



The projection illustrated in the left-hand side restricts to a fibration on the complement. The one on the right-hand side does not. Our goal is to characterize the subposets (circled orange in the pitcure) that do correspond to fibrations.

Supersolvable posets

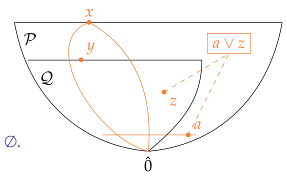
Let $\mathcal{P}$ be a locally geometric poset.

An M-ideal of $\mathcal{P}$ is a pure,

join-closed order ideal $\mathcal{Q}\subseteq\mathcal{P}$ s.t.

$a,b\in\mathcal{Q}\to a\vee b\in\mathcal{Q}$

- For $z\in\mathcal{Q}$ any atom $a\notin\mathcal{Q}$: $z\vee a\neq\emptyset$.
- For every $x\in\max\mathcal{P}$ there is $y\in\max\mathcal{Q}$ s.t. y is modular in $\mathcal{P}_{\le x}$



Definition. $\mathcal{P}$ is supersolvable if there is a sequence of M-ideals

$\{\hat{0}\}=\mathcal{Q}_0\subseteq\mathcal{Q}_1\subseteq\ldots\subseteq\mathcal{Q}_k=\mathcal{P}$ with $\mathcal{Q}_i$ of height $h(\mathcal{Q}_i)=i$.

Supersolvable posets

Let $\mathcal{P}$ be a locally geometric poset.

TM-ideal

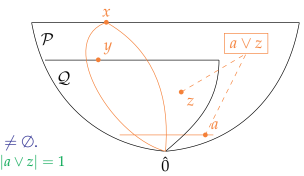
An M-ideal of $\mathcal{P}$ is a pure,

join-closed order ideal $\mathcal{Q}\subseteq\mathcal{P}$ s.t.

- For $z\in\mathcal{Q}$ any atom $a\notin\mathcal{Q}$: $z\vee a\neq\emptyset$.

$|a\vee z|=1$

- For every $x\in\max\mathcal{P}$ there is $y\in\max\mathcal{Q}$ s.t. y is modular in $\mathcal{P}_{\le x}$



strictly

Definition. $\mathcal{P}$ is supersolvable if there is a sequence of M-ideals

$\{\hat{0}\}=\mathcal{Q}_0\subseteq\mathcal{Q}_1\subseteq\ldots\subseteq\mathcal{Q}_k=\mathcal{P}$ with $\mathcal{Q}_i$ of height $h(\mathcal{Q}_i)=i$.

Factorizations

[Bibby-D. '21]

Let $\mathcal{P}$ be a finite locally geometric poset.

Lemma. If $\mathcal{Q}$ is a TM-ideal of P with $h(\mathcal{Q}) = h(\mathcal{P}) - 1$, then

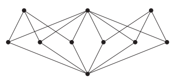
$$\chi_{\mathcal{P}}(t) = (t - |A(\mathcal{P}) \setminus A(\mathcal{Q})|)\chi_{\mathcal{Q}}(t).$$

Theorem. If $\mathcal{P}$ is strictly supersolvable via $\mathcal{Q}_0 \subseteq \ldots \subseteq \mathcal{Q}_k$, then

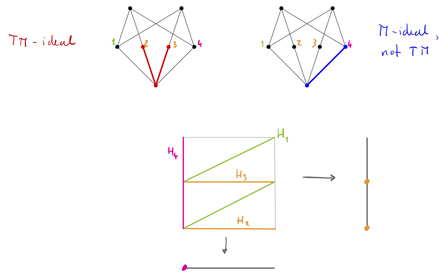
$$\chi_{\mathcal{P}}(t) = (t - a_1) \cdots (t - a_k)$$

where $a_i := |A(\mathcal{Q}_i) \setminus A(\mathcal{Q}_{i-1}|$

Note. This is not a necessary condition, see [Pagaria-Pismataro-Tran-Vecchi]



$$\chi(t) = t^2 - 6t + 9 = (t-3)^2$$



The characteristic polynomial of the poset in the example is $\chi(t) = t^2 - 4t + 4 = (t-2)^2$. This polynomial is divisible by $(t-2)$ (i.e., the characteristic polynomial of the TM-ideal depicted in red) but not by $(t-1)$ (i.e., the characteristic polynomial of the M-ideal depicted in blue, which is not a TM-ideal).

Further reference for factorizations of characteristic polynomials of geometric posets: [**15**]. See also [**16**].

Fibration theorem

Theorem. [Bibby-D. '21] Let $\mathscr{A}$ be an abelian arrangement

$\mathscr{A}$ is fiber-type if and only if $\mathcal{P}(\mathscr{A})$ is supersolvable.

In particular, if $\mathscr{A}$ is linear, toric or elliptic, then $\mathscr{A}$ is $K(\pi, 1)$.

Theorem. [Bibby-Cohen-D. '24+]
If $\mathscr{A}$ is a supersolvable toric arrangement, then $\pi_1(M(\mathscr{A}))$ is an iterated semidirect product of free groups. (Almost direct if *strictly* supersolvable.)

Lemma. [Bibby-D. '21] If $\mathcal{P}$ is a geometric poset, $G \circlearrowleft \mathcal{P}$ is translative and $\mathcal{Q} \subseteq \mathcal{P}$ is G-invariant, then $\mathcal{Q}$ is an M-ideal **if and only if** $\mathcal{Q}/G \subseteq \mathcal{P}/G$ is.

Application. Bloch-Kato property of $\pi_1(M(\mathscr{A}))$. [D.-Marmo '24+]

Thank you!

"Takk skal du ha"

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