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Representation stability for the cohomology of arrangements associated to root systems

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Abstract From a root system, one may consider the arrangement of reflecting hyperplanes, as well as its toric and elliptic analogues. The corresponding Weyl group acts on the complement of the arrangement and hence on its cohomology. We consider a sequence of linear, toric, or elliptic arrangements which arise from a family of root systems of type A, B, C, or D, and we show that the rational cohomology stabilizes as a sequence of Weyl group representations. Our techniques combine a Leray spectral sequence argument similar to that of Church in the type A case along with FI_W-module theory which Wilson developed and used in the linear case. A key to the proof relies on a combinatorial description, using labelled partitions, of the poset of connected components of intersections of subvarieties in the arrangement.

Keywords Hyperplane arrangement · Toric arrangement · Elliptic arrangement · Weyl group · Representation stability

Mathematics Subject Classification 52C35 · 05E18

1 Introduction

In this paper, we consider arrangements of codimension-one subvarieties in a complex vector space, torus, or abelian variety, determined by a root system of type A, B, C, or D. The subvarieties in such an arrangement are determined by realizing the root system as a set of characters on a torus. In each of these cases, the complement of the

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union of subvarieties comes with a natural action of the corresponding Weyl group. This action makes the rational cohomology into a representation over the Weyl group, which is the object we study.

These arrangements arising from root systems also have interesting combinatorics. In the type A case, taking all intersections of subvarieties in the arrangement forms a lattice which is isomorphic to the partition lattice. In the other linear cases, Barcelo and Ihrig [2] give a combinatorial description of the intersection lattice. However, in the toric and elliptic cases, intersections of hyperplanes need not be connected and form a partially ordered set which is not necessarily a lattice. In these cases, we consider the poset consisting of connected components of intersections, and in Theorem 1 we give a combinatorial description akin to that of Barcelo and Ihrig. Understanding the Weyl group action on the connected components of intersections is then equivalent to understanding its action on certain types of partitions, called labelled partitions.

We have already said that we are interested in the rational cohomology of the complement as a representation. But more specifically, we consider the sequence of representations arising from each family of root systems. We show in Theorem 3 that this sequence of representations stabilizes in the sense of Church–Farb [9]. That is, for n large enough, if we decompose the representations into irreducibles, the multiplicity of each irreducible representation does not depend on n. As a special case, by taking the trivial representation, the orbit space enjoys homological stability (Corollary 3).

In the case of the symmetric group, the complement is an ordered configuration space. Church [7] showed representation stability of the rational cohomology of ordered configuration spaces using a Leray spectral sequence and the partition lattice. We generalize his method of using this spectral sequence for other types of arrangements by combining it with our combinatorics and with $\mathrm{FI}_{\mathscr{W}}$-module theory developed by Wilson [16,17]. Wilson [17] also showed representation stability for each linear case.

We also give a slight improvement on Church's stable range for type A elliptic arrangements in Proposition 3. Recently, Hersh and Reiner [11] showed a better improvement for the type A linear case, and we wonder if their result or methods may also be applied to these other arrangements.

2 Arrangements

2.1 Linear, toric, and elliptic arrangements

The three types of arrangements which we consider in this paper are as follows. A *linear arrangement* is a set of hyperplanes in a complex vector space. A *toric arrangement* is a set of codimension-one subtori (possibly translated) in a complex torus. An *abelian arrangement* is a set of codimension-one abelian subvarieties (possibly translated) in a complex abelian variety. In the case of an abelian arrangement, all of our abelian varieties will be products of an elliptic curve and we call it an *elliptic arrangement*. We denote the complement of $\mathscr{A}$ in V by $M(\mathscr{A}) = V \setminus \cup_{H\in\mathscr{A}} H$.

A *layer* of an arrangement $\mathscr{A}$ is a connected component of an intersection $\bigcap_{H\in S} H$ for some subset $S \subseteq \mathscr{A}$. Note that the intersections themselves need not be connected. We say that the arrangement is *unimodular* if every intersection is connected. Note

that linear arrangements are always unimodular, but this is not true in general for toric or elliptic arrangements. The set of layers forms a ranked poset, ordered by reverse inclusion, with rank given by the complex codimension.

In the linear case, the cohomology of the complement $M(\mathscr{A})$ is isomorphic to the Orlik–Solomon algebra [14], which notably depends only on the combinatorics of the arrangement, its poset of layers. A toric or elliptic arrangement $\mathscr{A}$ is locally a linear arrangement, by which we mean the following. For every $x \in X^n$, the exponential map gives a homeomorphism between a small neighbourhood U of x and a small ball V around 0 in the tangent space $T_x X^n$. This homeomorphism restricts to homeomorphisms $U \cap H \cong V \cap T_x H$ whenever $H \ni x$. These hyperplanes $V \cap T_x H$ form a linear arrangement inside $V \cap T_x X^n \cong \mathbb{C}^n$ whose poset of layers is the subposet of the layers of $\mathscr{A}$ which contain x. Since $U \cap M(\mathscr{A})$ is homeomorphic to the complement of this linear arrangement, the cohomology of $U \cap M(\mathscr{A})$ is isomorphic to the Orlik–Solomon algebra of the subposet. We will make use of this in the proof of Lemma 2.

A natural way in which arrangements may arise is from a set of characters on a complex torus T, say $\Psi \subseteq \mathrm{Hom}(T, \mathbb{C}^\times)$. Here, for each $\chi \in \Psi$, we take the set of connected components of $\ker \chi \subseteq T$. This collection of subvarieties defines a toric arrangement $\mathscr{A}(\mathbb{C}^\times, \Psi)$ in T, in view of DeConcini and Procesi [10]. Noting that the Lie algebra $\mathrm{Lie}(T)$ is a complex vector space, we may take the kernel of each $d\chi$ and get a linear arrangement $\mathscr{A}(\mathbb{C}, \Psi)$ in $\mathrm{Lie}(T)$. Moreover, for a complex elliptic curve E, there is an embedding $\mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \hookrightarrow \mathrm{Hom}(E, E)$ which sends the identity map on $\mathbb{C}^\times$ to the identity map on E. This then extends to an embedding

$$\iota : \mathrm{Hom}((\mathbb{C}^\times)^n, \mathbb{C}^\times) \hookrightarrow \mathrm{Hom}(E^n, E)$$

so that given $\Psi \subseteq \mathrm{Hom}(T, \mathbb{C}^\times)$, the collection of connected components of $\ker \iota\chi$ in E^n for $\chi \in \Psi$ gives an elliptic arrangement $\mathscr{A}(E, \Psi)$.

2.2 Arrangements from root systems

By taking the perspective of arrangements arising from characters on a torus, we will now consider the case in which the set of characters is the set of positive roots in a root system of type A, B, C, or D. Here, if T is an n-dimensional torus, we will denote the root system by $\Phi_n \subseteq \mathrm{Hom}(T, \mathbb{C}^\times)$. Letting X be $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, the collection of connected components of the kernel of $d\chi$, χ, or $\iota\chi$, respectively, for $\chi \in \Phi_n^+$ gives an arrangement which we denote by $\mathscr{A}(X, \Phi_n^+)$.

Using the standard basis $v_1, \ldots, v_n$ for the integer lattice $\mathrm{Hom}(T, \mathbb{C}^\times)$, the type C_n root system consists of:

$$\Phi_n = \{\pm(v_i \pm v_j) \mid 1 \le i < j \le n\} \cup \{\pm 2v_i \mid 1 \le i \le n\}$$

and hence the positive roots are:

$$\Phi_n^+ = \{v_i \pm v_j \mid 1 \le i < j \le n\} \cup \{2v_i \mid 1 \le i \le n\}.$$

For $\chi \in \Phi_n^+$, the kernel in X^n is not necessarily connected. It is if $\chi = v_i \pm v_j$, giving the subvarieties:

$$H_{ij} := \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j\}$$

and

$$H'_{ij} := \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j^{-1}\}$$

(writing the group operation on X multiplicatively). However, if $\chi = 2v_i$, the connected components depend on X. Let $X[2]$ denote the two-torsion points of X. For $X = \mathbb{C}$, this consists only of the origin, but $\mathbb{C}^\times$ has two two-torsion points and a complex elliptic curve has four. The connected components of the kernel of $d\chi$, χ, and $\iota\chi$ are then indexed by $X[2]$:

$$H_i^z := \{(x_1, \ldots, x_n) \in X^n \mid x_i = z\} \text{ for } z \in X[2]$$

In summary, the type C_n arrangement in X^n, denoted by $\mathscr{A}(X, \Phi_n^+)$, is defined as the collection of the above subvarieties: H_{ij} (for $1 \le i < j \le n$), H'_{ij} (for $1 \le i < j \le n$), and H_i^z (for $1 \le i \le n$, $z \in X[2]$).

The type B_n root system consists of:

$$\Phi_n = \{\pm(v_i \pm v_j) \mid 1 \le i < j \le n\} \cup \{\pm v_i \mid 1 \le i \le n\}.$$

Now, the kernel of v_i (or similarly for dv_i or ιv_i) is the identity component of the kernel of $2v_i$. Hence, the type B_n arrangement in X^n consists of H_{ij} ($1 \le i < j \le n$), H'_{ij} ($1 \le i < j \le n$), and H_i^e ($1 \le i \le n$, with e the identity of X). Note that in the linear case, the type B and C arrangements are equal.

The type D_n root system consists of:

$$\Phi_n = \{\pm(v_i \pm v_j) \mid 1 \le i < j \le n\}.$$

Hence the type D_n arrangement consists of the subvarieties H_{ij} and H'_{ij} for $1 \le i < j \le n$.

The type A_{n-1} root system consists of:

$$\Phi_n = \{\pm(v_i - v_j) \mid 1 \le i < j \le n\}.$$

Hence the type A_{n-1} arrangement consists of the subvarieties H_{ij} for $1 \le i < j \le n$.

Note that toric and elliptic arrangements of types B, C, and D, are not unimodular, as $H_{ij} \cap H'_{ij}$ has connected components indexed by $X[2]$. More specifically, $H_{ij} \cap H'_{ij}$ is the collection of points whose i'th and j'th coordinates are both equal to each other and their inverse, hence equal to a two-torsion point.

Example 1 The best pictures we have for these arrangements are in $n = 2$ with the real version of linear and toric arrangements. We draw here the pictures of the toric

case, in $S^1 \times S^1$; the subtori of the arrangement are the thickened lines. We warn the reader that, while the combinatorics of the real versus complex pictures agree, the topology is very different. For example, the complement of the complex arrangement is connected.

A_1 B_2 C_2 D_2

2.3 Weyl group action

Let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve. Considering our root system as a subset $\Phi \subseteq \mathrm{Hom}(X^n, X)$, the action of the corresponding Weyl group $\mathscr{W}_n$ on Φ gives rise to a natural action of $\mathscr{W}_n$ on both the poset of layers and on the complement of the arrangement. If H_χ is the kernel of $\chi \in \Phi$ and $w \in \mathscr{W}_n$, then $w \cdot H_\chi = H_{w\cdot\chi}$. We will describe this explicitly in type C.

Consider the hyperoctahedral group $W_n = (\mathbb{Z}/2) \wr S_n = (\mathbb{Z}/2)^n \rtimes S_n$, the Weyl group in types B_n and C_n. The group W_n acts on X^n via a combination of permuting the coordinates and inverting some. More specifically, given $w = (\sigma, (\epsilon_1, \ldots, \epsilon_n)) \in S_n \ltimes (\mathbb{Z}/2)^n$ and $x = (x_1, \ldots, x_n) \in X^n$, $w \cdot x$ has $\epsilon_i(x_i)$ as its $\sigma(i)$-th coordinate. Here, we are considering $\mathbb{Z}/2 = \{\pm 1\}$ so that $\epsilon_i(x_i) = x_i$ if $\epsilon_i = 1$, and $\epsilon_i(x_i) = x_i^{-1}$ if $\epsilon_i = -1$. This gives us the following action on our set of subvarieties:

- $w \cdot H_{ij} = H_{\sigma(i)\sigma(j)}$ if $\epsilon_i\epsilon_j = 1$,
- $w \cdot H_{ij} = H'_{\sigma(i)\sigma(j)}$ if $\epsilon_i\epsilon_j = -1$,
- $w \cdot H'_{ij} = H'_{\sigma(i)\sigma(j)}$ if $\epsilon_i\epsilon_j = 1$,
- $w \cdot H'_{ij} = H_{\sigma(i)\sigma(j)}$ if $\epsilon_i\epsilon_j = -1$,
- $w \cdot H^z_i = H^z_{\sigma(i)}$.

For this last subvariety, we note that since the two-torsion points are fixed by the action of W_n on X, this gives us the action on the connected components of the kernel of the character $2v_i, \cup_{z\in X[2]} H^z_i$. Denoting the kernel of $\chi \in \Phi$ by H_χ, we also have the action on an intersection given by $w \cdot \cap_{\chi\in S} H_\chi = \cap_{\chi\in S} w \cdot H_\chi = \cap_{\chi\in S} H_{w\cdot\chi}$. For a connected component of an intersection (i.e., a layer), we see that if the component has the point $z \in X[2]$ as the i-th coordinate, then the action will send it to a layer whose $\sigma(i)$-th coordinate is z. This gives an action of W_n on the poset of layers.

Since the type B and D arrangements are subarrangements of the type C arrangement, this also describes the action in these cases. For type A, we note that we do not have an action of $\mathscr{W}_n$, but this description restricts to an action of S_n.

Remark 1 While we naturally have an action of the corresponding Weyl group on our arrangements, we emphasize that in the type D case, we actually have an action of the hyperoctahedral group W_n. By considering this action, rather than the type D Weyl group, on the type D_n arrangements, we can get even stronger results. By [16, Prop. 3.22], our stability result in Theorem 3 stated for W_n will imply stability for the type D Weyl group, as stated in Corollary 1. Moreover, by working with a single group,

W_n, to get our results in the three new cases (B, C, and D), this simplifies the exposition as well. Thus, except for our Corollaries 1, 2, and 3, we will work exclusively with W_n for the type B, C, and D arrangements, and with S_n for the type A arrangement.

3 Combinatorial description of layers

The goal of this section is to provide a combinatorial description of layers, which allows us to better understand and handle the Weyl group action on layers. It will also help us to break down representations which appear in Sect. 4 into simpler building blocks, which we can then use to show stability. We start by describing the combinatorial objects needed and then show their relationship to layers in Theorem 1.

3.1 Labelled partitions

Let S and L be sets. We say that a *partition of S labelled by L*, or an L-labelled partition of S, is a partition Σ of S, together with a subset $T \subseteq \Sigma$ and injection $f : T \to L$. We say that the parts in T are the *labelled parts* of Σ, while $\Sigma \backslash T$ consists of the *unlabelled parts*. For $p \in T$ corresponding to $z \in L$, we say that p is *labelled by* z and use the notation $p = \Sigma_z$. We use the convention that if z is not in the image of f, then $\Sigma_z = \emptyset$. One may similarly define a partition of a number k labelled by L.

Given one of our root systems, we will define a particular set of labelled partitions, which in Theorem 1 will be shown to describe the layers of the corresponding arrangement. First, we introduce some notation. Again let X be $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve with 2-torsion points $X[2]$. Let $[n] = \{1, 2, \ldots, n\}$ and $\mathbf{n} = \{1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$. For $S \subseteq \mathbf{n}$, let $\bar{S} = \{\bar{x} \mid x \in S\}$, taking $\bar{\bar{k}} = k$. We say that a set $S \subseteq \mathbf{n}$ is *bar-invariant* (or self-barred as in [2]) if $\bar{S} = S$.

Let $\mathscr{P}_n(X)$ be the set of partitions Σ of $\mathbf{n}$ labelled by $X[2]$ such that

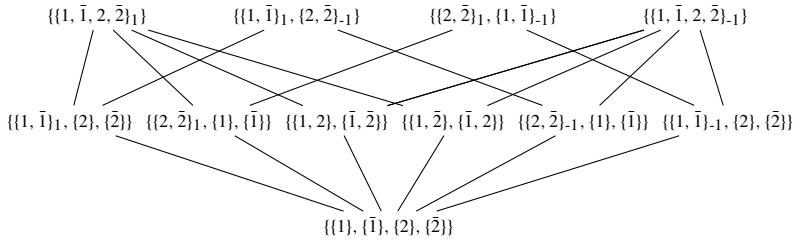
(i) For every $S \in \Sigma$, $\bar{S} \in \Sigma$, and
(ii) $S = \bar{S}$ if and only if S is labelled.

Each type of root system Φ_n will correspond to a subset $\mathscr{C}(X, \Phi_n) \subseteq \mathscr{P}_n(X)$ as follows:

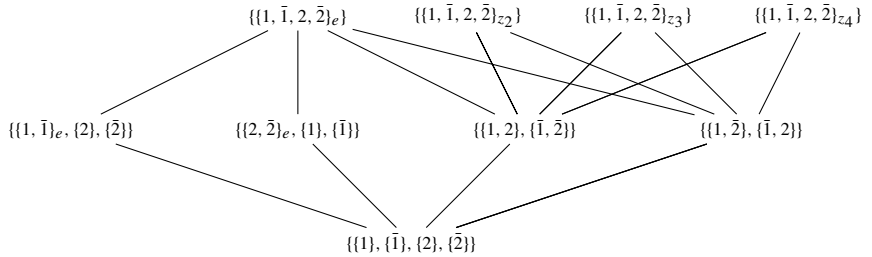
- If Φ_n is type C, let $\mathscr{C}(X, \Phi_n) = \mathscr{P}_n(X)$.
- If Φ_n is type B, take the subset of all $\Sigma \in \mathscr{P}_n(X)$ such that if $|\Sigma_z| = 2$ then $z = e$ (the identity of X).
- If Φ_n is type D, take the subset of all $\Sigma \in \mathscr{P}_n(X)$ such that $|\Sigma_z| \neq 2$ for any $z \in X[2]$.
- If Φ_n is type A, take the subset of all $\Sigma = \Sigma_+ \cup \overline{\Sigma_+}$ where $\Sigma_+ \vdash [n]$.

In each type, $\mathscr{C}(X, \Phi_n)$ is a partially ordered set, with $\Sigma < \Sigma'$ if Σ is a refinement of Σ' such that $\Sigma_z \subseteq \Sigma'_z$ for each $z \in X[2]$. That is, we order it by refinements which respect the labelling. Moreover, $\mathscr{C}(X, \Phi_n)$ is a ranked poset with $\mathrm{rk}(\Sigma) = n - \frac{\ell}{2}$, where ℓ is the number of unlabelled parts of Σ.

Example 2 Below is the Hasse diagram of $\mathscr{C}(\mathbb{C}^\times, \Phi_2)$ when Φ_2 is a type C root system. Note that the subscripts on some blocks in the partitions denote the labelling; here, our two-torsion points are $X[2] = \{\pm 1\}$.



Example 3 Below is the Hasse diagram of $\mathscr{C}(E, \Phi_2)$ where Φ_2 is a type B root system and E is a complex elliptic curve. Denote the two-torsion points of E by $z_1 = e$ (the identity), z_2, z_3, and z_4.



3.2 Weyl group action

In type A, the set $\mathscr{C}(X, \Phi_n)$ is really just the partition lattice, which has an action of S_n. In the other types, we have an action of the hyperoctahedral group $W_n = (\mathbb{Z}/2) \wr S_n$. This action is induced by its action on $\mathbf{n}$, but we will describe it more explicitly. Let $w = (\sigma, \epsilon) \in W_n$ with $\sigma \in S_n$ and $\epsilon \in (\mathbb{Z}/2)^n$. Then for $k \in [n]$, we have $w \cdot k = \overline{\sigma(k)}$ if $\epsilon_k = -1$ and $w \cdot k = \sigma(k)$ if $\epsilon_k = 1$. This extends to $\mathbf{n}$ by $w \cdot \bar{k} = \overline{w \cdot k}$. Then for $S \subseteq \mathbf{n}$, we have $w \cdot S = \{w \cdot x \mid x \in S\}$, and for a partition $\Sigma \vdash \mathbf{n}$, we have $w \cdot \Sigma = \{w \cdot S \mid S \in \Sigma\}$, If Σ is a labelled partition, then $w \cdot \Sigma$ is labelled so that $(w \cdot \Sigma)_z = w \cdot (\Sigma_z)$ for each label z.

Given a partition $\Sigma \in \mathscr{P}_n(X)$, we can define a partition $\widehat{\Sigma}$ of n labelled by $X[2]$, where $\widehat{\Sigma}_z = \frac{|\Sigma_z|}{2}$ and the unlabelled parts are given by $|S|$ for each pair of unlabelled parts $S, \bar{S} \in \Sigma$. For example, if Σ is the labelled partition

$$\{\{1, \bar{1}\}_e, \{2, \bar{4}\}, \{\bar{2}, 4\}, \{3\}, \{\bar{3}\}\}$$

of $\mathbf{4}$, then we have $\widehat{\Sigma} = (1_e, 2, 1)$, a labelled partition of 4.

Lemma 1 *Let $\mathscr{P}_n(X)$ be the set of partitions of $\mathbf{n}$ labelled by $X[2]$, and let $\mathscr{Q}_n(X)$ be the set of partitions of n labelled by $X[2]$. The nonempty fibres of the function $\mathscr{P}_n(X) \to \mathscr{Q}_n(X)$ defined by $\Sigma \mapsto \widehat{\Sigma}$ are W_n-orbits.*

Proof The W_n-action preserves the size of the parts in the partition, which means that for each $w \in W_n$ and $\Sigma \in \mathscr{P}_n(X)$, we have $\widehat{w\Sigma} = \widehat{\Sigma}$. Moreover, if we have $\widehat{\Sigma} = \widehat{\Sigma}'$, then we could find some $w \in W_n$ with $w\Sigma = \Sigma'$, constructed as follows.

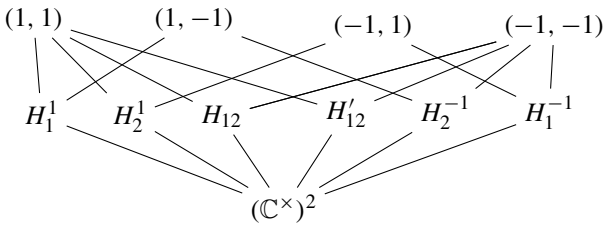
Suppose that $\widehat{\Sigma} = (\lambda_{z_1}, \ldots, \lambda_{z_m}, \lambda_1, \ldots, \lambda_\ell)$, where $X[2] = \{z_1, \ldots, z_m\}$. So we have $|\Sigma_{z_i}| = |\Sigma'_{z_i}|$ for each i; let σ_i be a permutation in S_n which gives a one-to-one correspondence between these two parts, and let $w_i = (\sigma_i, 1)$. Now, since $\lambda_1, \ldots, \lambda_\ell$ denotes the sizes of (pairs of) unlabelled blocks, we can index the unlabelled parts of Σ and Σ' as $S_1, \bar{S}_1, \ldots, S_\ell, \bar{S}_\ell$ and $S'_1, \bar{S}'_1, \ldots, S'_\ell, \bar{S}'_\ell$ so that we have one-to-one correspondences between S_i and S'_i. Ignoring the bars, this one-to-one correspondence determines a permutation $\sigma_i \in S_n$. To get $w_i = (\sigma_i, \epsilon_i)$, we let the j-th coordinate of ϵ_i be -1 if the one-to-one correspondence either sends j to $\bar{k}$ or $\bar{j}$ to k for some k; otherwise, the j-th coordinate of ϵ_i will be 1. With this $w = w_{z_1} \cdots w_{z_m} w_1 \cdots w_\ell$, we have $\Sigma' = w\Sigma$. □

This lemma means that for each $\lambda \in \mathscr{Q}_n(X)$, the set $\{\Sigma \in \mathscr{C}(X, \Phi_n) \mid \widehat{\Sigma} = \lambda\}$ is either a W_n-orbit or is empty. Example 9 depicts the orbits of $\mathscr{C}(\mathbb{C}^\times, \Phi_2)$ when Φ_2 is a type B root system.

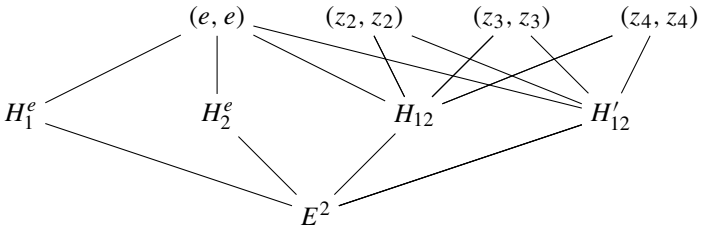
3.3 Layers as labelled partitions

The goal of this section is to prove that these labelled partitions give a combinatorial description of layers of the arrangement. This description will help us to get a handle on certain representations in Lemma 3. But before proving our claim in Theorem 1, we consider some examples to demonstrate it.

Example 4 Below is the Hasse diagram for the poset of layers in the case that Φ_2 is a type C root system and $X = \mathbb{C}^\times$. The bijection with the poset in Example 2 should be visible here.



Example 5 Below is the Hasse diagram of the poset of layers in the case that Φ_2 is a type B root system and $X = E$ is a complex elliptic curve. Denote the two-torsion points of E by $z_1 = e$ (the identity), z_2, z_3, and z_4. The bijection with the poset in Example 3 should be visible.



We finally describe how the poset of labelled partitions $\mathscr{C}(X, \Phi_n)$ corresponds to the poset of layers of the arrangement $\mathscr{A}(X, \Phi_n^+)$. Let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, let Φ_n be a root system of type B, C, or D, and let W_n be the hyperoctahedral group.

Given a partition $\Sigma \in \mathscr{C}(X, \Phi_n)$, we will define a layer F_Σ of the arrangement $\mathscr{A}(X, \Phi_n^+)$ as follows. For $S \in \Sigma$, take the collection of subvarieties H_{ij} if either $i, j \in S$ or $\bar{i}, \bar{j} \in S$; H'_{ij} if either $i, \bar{j} \in S$ or $\bar{i}, j \in S$; and H_i^z if $S = \Sigma_z$ and $i \in S$. Denote the intersection of these subvarieties by F_S, so that we define:

$$F_\Sigma = \bigcap_{S \in \Sigma} F_S$$

Our claim, in Theorem 1, is that F_Σ is indeed a layer of the arrangement and that this gives a bijection between the two posets. Since F_Σ makes an appearance later in the paper, we will also give a more explicit description of it now.

Write $\Sigma = \{S_1, \bar{S}_1, \ldots, S_\ell, \bar{S}_\ell, \Sigma_{z_1}, \ldots, \Sigma_{z_m}\}$ where $\{z_1, \ldots, z_m\} \subseteq X[2]$. For $S \in \Sigma$, let X_S denote the factor of X^n corresponding to indices i with either $i \in S$ or $\bar{i} \in S$, so that we can write:

$$X^n = X_{S_1} \times \cdots \times X_{S_\ell} \times X_{\Sigma_{z_1}} \times \cdots \times X_{\Sigma_{z_m}}$$

We have inclusions $\iota_k : X \hookrightarrow X_{S_k}$ such that the i-th coordinate of $\iota_k(x)$ is x when $i \in S_k$ and x^{-1} when $\bar{i} \in S_k$. We also have the inclusion $\iota_{z_j} : \{z_j\} \hookrightarrow X_{\Sigma_{z_j}}$ whose image is the point $(z_j, \ldots, z_j)$. For a single S_k, we have an inclusion $X \times X^{n-|S_k|} \to X_{S_k} \times X^{n-|S_k|}$ given by $\iota_k \times \mathrm{id}$ whose image is F_{S_k}. Similarly, for $S = \Sigma_{z_j}$, $\iota_{z_j} \times \mathrm{id}$ gives an inclusion $\{z_j\} \times X^{n-|S|} \to X_{\Sigma_{z_j}} \times X^{n-|S|}$ whose image is $F_{\Sigma_{z_j}}$.

Together, $\iota_1 \times \cdots \times \iota_\ell \times \iota_{z_1} \times \cdots \times \iota_{z_m}$ gives an inclusion

$$X \times \cdots \times X \times \{z_1\} \times \cdots \times \{z_m\} \to X_{S_1} \times \cdots \times X_{S_\ell} \times X_{\Sigma_{z_1}} \times \cdots \times X_{\Sigma_{z_m}}$$

with image F_Σ. In particular, observe that F_Σ is connected and has codimension $n - \ell$, where ℓ is the number of pairs of unlabelled parts in Σ.

We provide one more example before proving that this assignment $\Sigma \mapsto F_\Sigma$ does indeed give a bijection.

Example 6 Consider a triple intersection of hyperplanes H_{ij}, H'_{ij} and H_{ik} (i, j, k distinct) in X^n (for $n \geq 3$). We have

$$H_{ij} \cap H'_{ij} \cap H_{ik} = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j, x_i = x_j^{-1}, x_i = x_k\}.$$

So for a point x in the intersection, the i-th, j-th, and k-th coordinates must all be equal and be in $X[2]$. So we can write:

$$H_{ij} \cap H'_{ij} \cap H_{ik} = \bigcup_{z \in X[2]} \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j = x_k = z\}.$$

If $\{i, j, k\} = \{1, 2, 3\}$, the connected components will correspond to the labelled partitions

$$\Sigma = \{\{1, \bar{1}, 2, \bar{2}, 3, \bar{3}\}_z, \{4\}, \{\bar{4}\}, \ldots, \{n\}, \{\bar{n}\}\}$$

with $z \in X[2]$. Note in this example, setting $\{i, j, k\} = \{1, 2, 3\}$ and $z \in X[2]$, we have F_Σ as a connected component of $H_{ij} \cap H'_{ij} \cap H_{ik}$, but this can also be written as the intersection of all of these subvarieties: $H_{12}, H_{13}, H_{23}, H'_{12}, H'_{13}, H'_{23}, H^z_1, H^z_2, H^z_3$.

Theorem 1 *Let Φ_n be a root system of type B, C, or D, and let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve. Denote the corresponding arrangement by $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$ and the corresponding set of labelled partitions by $\mathscr{C}_n = \mathscr{C}(X, \Phi_n)$. Then, there is a W_n-equivariant isomorphism of ranked posets between $\mathscr{C}_n$ and the layers of $\mathscr{A}_n$.*

Proof We will first describe the bijection and prove the theorem for type C, and then we restrict to our other types.

The bijection in type C:

Let $\Sigma \in \mathscr{P}_n(X)$, our poset of type C, and consider F_Σ as described above. It is an intersection of subvarieties in $\mathscr{A}_n$ and it is connected; thus, it is a layer of the arrangement. Moreover, if Σ and Σ' are distinct elements of $\mathscr{P}_n(X)$, then F_Σ and $F_{\Sigma'}$ are distinct subvarieties in X^n. To see this, suppose that we have $F_\Sigma = F_{\Sigma'}$. Suppose we had $\Sigma_z \neq \Sigma'_z$ for some $z \in X[2]$. If, say, $i \in \Sigma_z \backslash \Sigma'_z$, then every point of F_Σ would have the i-th coordinate equal to z, while this is not the case in $F_{\Sigma'}$. So we must have $\Sigma_z = \Sigma'_z$ for each z. If the unlabelled parts differed, then we would have some i, j such that either i and j or i and $\bar{j}$ are in the same unlabelled part in Σ but not in Σ' (or vice versa). This means that for every $x \in F_\Sigma$ we have $x_i = x_j$, or $x_i = x_j^{-1}$ respectively, while this is not true of every point in $F_{\Sigma'}$.

Now let F be a layer of $\mathscr{A}_n$, and we will define a labelled partition Σ in $\mathscr{P}_n(X)$ such that $F = F_\Sigma$. From F, we can define an equivalence relation on $\mathbf{n}$ as follows: $i \sim j$ iff $\bar{i} \sim \bar{j}$ iff $H_{ij} \supseteq F$, $i \sim \bar{j}$ iff $\bar{i} \sim j$ iff $H'_{ij} \supseteq F$, and $i \sim \bar{i}$ iff $H^z_i \supseteq F$ for some z. This gives (by taking equivalence classes) a partition of $\mathbf{n}$ where some parts satisfy $S = \bar{S}$ and the others come in pairs $(S, \bar{S})$. We will label the bar-invariant parts so that we get an element of $\mathscr{P}_n(X)$. If $S = \bar{S}$, then for each $i \in S$, there exists a $z \in X[2]$ with $H^z_i \supseteq F$. Moreover, this z is the same no matter which $i \in S$ we consider: if i and j are both in S, then $i \sim j$ and $i \sim \bar{j}$, which means F is contained in a connected component of $H_{ij} \cap H'_{ij}$ (the one corresponding to our z). Thus, we may label S by z. Moreover, if S and T are distinct bar-invariant parts, they correspond to different elements of $X[2]$; otherwise, we would have for $i, \bar{i} \in S$ and $j, \bar{j} \in T$ such that $H_{ij} \supseteq F$ and hence $i \sim j$.

Compatibility with order and rank: (Type C)

Assume that Σ is a refinement of Σ' such that $\Sigma_z \subseteq \Sigma'_z$ for all $z \in X[2]$. For all $S \in \Sigma$, there exists $T \in \Sigma'$ such that $S \subseteq T$, and all $T \in \Sigma'$ have such an S. Moreover, if $S = \Sigma_z$, then $T = \Sigma'_z$. Since $S \subseteq T$ implies $F_S \supseteq F_T$, we have

$$F_\Sigma = \bigcap_{S\in\Sigma} F_S \supseteq \bigcap_{T\in\Sigma'} F_T = F_{\Sigma'}.$$

As for rank, recall from our construction that the codimension of F_Σ is equal to $n - \ell$, when Σ has 2ℓ unlabelled parts. This is equal to the rank of Σ.

Compatibility with Weyl group action: (Type C)

Let $w = (\sigma, \epsilon) \in W_n$ and F a layer of $\mathscr{A}_n$. If $H_{ij} \supseteq F$, then

$$wF \subseteq wH_{ij} = \begin{cases} H_{\sigma(i)\sigma(j)} & \text{if } \epsilon_i\epsilon_j = 1 \\ H'_{\sigma(i)\sigma(j)} & \text{if } \epsilon_i\epsilon_j = -1. \end{cases}$$

Similarly, if $H'_{ij} \supseteq F$, then

$$wF \subseteq wH'_{ij} = \begin{cases} H'_{\sigma(i)\sigma(j)} & \text{if } \epsilon_i\epsilon_j = 1 \\ H_{\sigma(i)\sigma(j)} & \text{if } \epsilon_i\epsilon_j = -1 \end{cases}$$

Finally, if $H^z_i \supseteq F$, then $H^z_{\sigma(i)} \supseteq wF$. The first two pieces imply that if $S \in \Sigma$ is unlabelled then wS is an unlabelled part of $w\Sigma$, and the last one implies that $(w\Sigma)_z = w(\Sigma_z)$.

Type B_n:

We now have $\mathscr{C}_n$ the set of $\Sigma \in \mathscr{P}_n(X)$ such that if $|\Sigma_z| = 2$ then $z = e$, and $\mathscr{A}_n$ now denotes the type B_n arrangement in X^n. Given $\Sigma \in \mathscr{C}_n$, we may construct F_Σ as above, but we need to show that F_Σ is a layer of $\mathscr{A}_n$. The type B_n arrangement is a subarrangement of type C_n, where we exclude H^z_i for $z \neq e$. It is clear that if $S \in \Sigma$ is unlabelled, then F_S is a layer; we need only worry about F_{Σ_z}. If $|\Sigma_z| = 2$, then $z = e$, and we have $F_{\Sigma_z} = H^e_i$. If $|\Sigma_z| \neq 2$, then consider the intersection H_{Σ_z} of the subvarieties H_{ij} and H'_{ij} for $i, j \in \Sigma_z$. This intersection is not connected, but its connected components are indexed by $X[2]$, and F_{Σ_z} is the connected component indexed by z.

We also need to show that in the inverse map, if we are restricting ourselves to layers of the type B_n arrangement, the partition we get will not have $|\Sigma_z| = 2$ for $z \neq e$. Suppose that $|\Sigma_z| = 2$, then there exists i such that $i \sim \bar{i}$ but no j with $i \sim j$ or $i \sim \bar{j}$. This implies that $H^z_i \supseteq F$ but no H_{ij} or H'_{ij} contains F. The only way this can be a layer in type B is if $z = e$.

Type D_n:

Now let $\mathscr{C}_n$ be the set of $\Sigma \in \mathscr{P}_n(X)$ such that $|\Sigma_z| \neq 2$ for any $z \in X[2]$. Given such Σ, we may again construct F_Σ as above, but we need to show that this is a layer of the type D_n arrangement. As in type B, we need only worry about F_{Σ_z} being a layer. But since $|\Sigma_z|$ is never 2, we will have F_{Σ_z} as a connected component of the intersection $H_{\Sigma_z} = \bigcap_{i,j\in\Sigma_z}(H_{ij} \cap H'_{ij})$.

On the other hand, suppose that we have a layer F of the type D arrangement and construct the corresponding partition $\Sigma \in \mathscr{P}_n(X)$. If $\Sigma_z = \{i, \bar{i}\}$, then there is no j such that H_{ij} or H'_{ij} contains F, contradicting the fact that F is a layer. □

Remark 2 The analogous statement for type A_{n-1} is clear, because the poset $\mathscr{C}_n$ in this case is equivalent to the partition lattice of the set $[n]$.

Remark 3 In the linear case, our description is equivalent to that given by Barcelo and Ihrig [2, Theorems 3.1 & 4.1]. They showed that the poset in question is also isomorphic to the lattice of parabolic subgroups of the Weyl group. It is also worth noting that in the type B/C linear case, this is the Dowling lattice. But in other cases, this labelling helps us take into account the more complicated structure of having multiple connected components of intersections.

4 Representation stability

Our goal is to show representation stability for the cohomology of our arrangements, but we first briefly review representation stability and its main tool of $\mathrm{FI}_{\mathscr{W}}$-modules. Throughout this section, we let $\mathscr{W}_n$ denote either the symmetric group S_n (type A) or the hyperoctahedral group W_n (type B/C). For more details on the theory, we refer the reader to [8,9] for the case of S_n (and much more) and [16,17] for the case of W_n (as well as the type D Weyl group).

Note that we are working over characteristic zero throughout this paper. A representation of a group G will always mean a group homomorphism $G \to \mathrm{GL}(V)$ where V is a finite-dimensional vector space over $\mathbb{Q}$. Unless otherwise stated, cohomology will always be with rational coefficients, and we will write $H^*(X)$ to mean $H^*(X; \mathbb{Q})$ for a space X.

4.1 $\mathscr{W}_n$-representation stability

To discuss representation stability for a sequence of groups, one needs a consistent way of describing the irreducible representations. There are many cases in which this can be done, including the classical families of Weyl groups.

For the symmetric group S_n, irreducible representations are indexed by partitions of n. If we consider a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of k with $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ and $n \geq \lambda_1 + k$, we may write $V(\lambda)_n$ to denote the irreducible representation of S_n indexed by the partition $\lambda[n] := (n-k, \lambda_1, \ldots, \lambda_\ell)$. For example, in this notation, $V(0)_n$ is always the trivial representation and $V(1)_n$ is always the standard representation.

For the hyperoctahedral group W_n, irreducible representations are indexed by pairs of partitions $\lambda = (\lambda^+, \lambda^-)$ where $|\lambda^+| + |\lambda^-| = n$. Given a pair of partitions $\lambda = (\lambda^+, \lambda^-)$, where λ^- is a partition of k, and n large enough, we may write $V(\lambda)_n$ to be the irreducible representation of W_n corresponding to $(\lambda^+[n-k], \lambda^-)$. For example, $V(0,0)_n$ is always the trivial representation.

We start with a *consistent* sequence $\{V_n\}$ of $\mathscr{W}_n$-representations, that is, each V_n is a $\mathscr{W}_n$-representation along with $\mathscr{W}_n$-equivariant maps $\varphi_n : V_n \to V_{n+1}$. Such a sequence is said to be *uniformly representation stable* with stable range $n \geq N$ if for $n \geq N$:

(1.) The map φ_n is injective,
(2.) The image $\varphi_n(V_n)$ generates V_{n+1} as a $\mathbb{Q}[\mathscr{W}_{n+1}]$-module, and
(3.) $V_n = \bigoplus_{\lambda} c_\lambda V(\lambda)_n$, where the multiplicities c_λ do not depend on n.

4.2 FI$_{\mathscr{W}}$-modules

Consider the category FI$_{\mathscr{W}}$ (where $\mathscr{W}$ denotes either type A or type B/C) consisting of objects $\mathbf{n}$ (with $\mathbf{0} = \emptyset$) and morphisms $f : \mathbf{m} \to \mathbf{n}$ which are injections such that $f(\bar{k}) = \overline{f(k)}$ for all $k \in \mathbf{m}$, also requiring that $f([n]) \subseteq [n]$ if $\mathscr{W}$ is type A. An FI$_{\mathscr{W}}$*-module* is a functor V from the category FI$_{\mathscr{W}}$ to the category of $\mathbb{Q}$-modules. We denote by V_n the image of $\mathbf{n}$. Since $\mathrm{End}(\mathbf{n}) = \mathscr{W}_n$ in the category FI$_{\mathscr{W}}$, the $\mathbb{Q}$-module V_n comes equipped with an action of $\mathscr{W}_n$. Moreover, the natural inclusions $\mathbf{n} \hookrightarrow \mathbf{n+1}$ induce $\mathscr{W}_n$-equivariant maps $V_n \to V_{n+1}$, making the sequence $\{V_n\}$ a consistent sequence of $\mathscr{W}_n$-representations.

A map of FI$_{\mathscr{W}}$-modules is a natural transformation. We say U is a *sub*-FI$_{\mathscr{W}}$*-module* of V if there is a map $U \to V$ such that U_n is a subrepresentation of V_n for all n. An FI$_{\mathscr{W}}$-module V is *finitely generated* if there is a finite set of elements of $\sqcup V_n$ that are not contained in any proper sub-FI$_{\mathscr{W}}$-module. An FI$_{\mathscr{W}}$-module V has *stability degree* $\leq s$ if $(V_{n+a})_{\mathscr{W}_n} \cong (V_{n+1+a})_{\mathscr{W}_{n+1}}$ for every $a \geq 0$ and $n \geq s$, where the subscript denotes the coinvariants. We say that V has *weight* $\leq d$ if for all n, every irreducible representation $V(\lambda)_n$ appearing with nonzero multiplicity in V_n satisfies $|\lambda| \leq d$ (if λ is a partition) or $|\lambda^+| + |\lambda^-| \leq d$ (if $\lambda = (\lambda^+, \lambda^-)$ is a pair of partitions). Again, we refer the reader to [16,17] for more details on these concepts; we state here the main properties and example on which our results rely.

We start by stating a proposition on finitely generated FI$_{\mathscr{W}}$-modules, bounding the weight and stability degree for kernels, cokernels, and extensions. The statements on stability degree were made in [8, Lemma 6.3.2.] for type A. Wilson extended this to FI$_{\mathscr{W}}$ to establish part (1) in [16, Prop. 4.18], and the same argument gives part (2). The statement on weight in part (1) follows from Definition 4.1 in [16], and part (2) for weights follows from semisimplicity of the representations (since we are in characteristic zero).

Proposition 1 *1. Assume that $f : U \to V$ is a map of FI$_{\mathscr{W}}$-modules which are finitely generated with weight $\leq d$ and stability degree $\leq s$. Then,* $\ker(f)$ *and* $\mathrm{coker}(f)$ *are both finitely generated with weight $\leq d$ and stability degree $\leq s$.*
2. Assume that $0 \to U \to V \to Q \to 0$ is a short exact sequence of FI$_{\mathscr{W}}$-modules, where U and Q are both finitely generated with weight $\leq d$ and stability degree $\leq s$. Then so is V.

Now, since finitely generated FI$_{\mathscr{W}}$-modules form an abelian category, a spectral sequence of finitely generated FI$_{\mathscr{W}}$-modules converges to a finitely generated FI$_{\mathscr{W}}$-module. See, for example, [13, Cor. 2.5] or [12, Thm. 3.3] in the type A case. However, it takes a little work to get a bound on the stability degree of the abutment. See, for example, [8, Thm. 6.3.1] for an argument on the bounds for weight and stability degree in the case of configuration spaces. One could make the same kind of argument for

the spaces which we work with, but we state a formulation which could be applied more generally. We state it for sequences which have $E_3 = E_\infty$, but it could (with a bit more book-keeping) be stated for sequences with $E_r = E_\infty$. The idea is that Proposition 1 tells us how to use the weight and stability degree on one page to bound the weight and stability degree on the next, continuing until the sequence collapses. We benefit from the fact that all of our sequences collapse early.

Proposition 2 *Suppose that E_*^{pq} is a first quadrant spectral sequence of $FI_{\mathscr{W}}$–modules which converges to the $FI_{\mathscr{W}}$-module H^{p+q} and assume that $E_3^{pq} = E_\infty^{pq}$. If E_2^{pq} is finitely generated with weight $\leq d_{p+2q}$ and stability degree $\leq s_{p+2q}$, then H^i is finitely generated with weight $\leq \max_{0\leq p\leq i}\{d_{2i-p}\}$ and stability degree $\leq \max_{0\leq p\leq i}\{s_{2i-p}\}$.*

Proof Since each of the $\mathrm{FI}_{\mathscr{W}}$-modules in a sequence $E_2^{p-2,q+1} \to E_2^{p,q} \to E_2^{p+2,q-1}$ has weight $\leq d_{p+2q}$ and stability degree $\leq s_{p+2q}$, part (1) of Proposition 1 implies that $E_\infty^{p,q} = E_3^{p,q}$ has weight $\leq d_{p+2q}$ and stability degree $\leq s_{p+2q}$.

Fix i, and consider the filtration $F_0 \subseteq \cdots \subseteq F_i = H^i$, where $F_j/F_{j-1} = E_\infty^{j,i-j}$. Since F_0 and F_1/F_0 are both finitely generated, so is F_1 by part (2) of Proposition 1, with weight $\leq \max\{d_{2i}, d_{2i-1}\}$ and stability degree $\leq \max\{s_{2i}, s_{2i-1}\}$. Repeating this for each F_{j-1} and F_j/F_{j-1} $(j = 1, 2, \ldots, i)$ gives that H^i is finitely generated with the desired bounds on weight and stability degree. □

Now having the bounds on weight and stability degree is what allows us to get a stable range for our sequences, as the following theorem says, due to Church–Ellenberg–Farb in type A and Wilson in type B/C.

Theorem 2 *[16, Thm. 4.26],[8, Thm. 2.58] If V is an $FI_{\mathscr{W}}$-module, with $\mathscr{W}$ of type A or B/C, which is finitely generated with weight $\leq d$ and stability degree $\leq s$, then the sequence $\{V_n\}$ of $\mathscr{W}_n$-representations with maps $V_n \to V_{n+1}$ induced by the natural inclusions $\mathbf{n} \to \mathbf{n+1}$ is uniformly representation stable with stable range $n \geq d + s$.*

In the following example, we describe particularly nice $\mathrm{FI}_{\mathscr{W}}$-modules, which will be useful for us in the next section.

Example 7 ([17, Ex. 1.5.5]) Let U be a $\mathscr{W}_k$-representation which is finite dimensional. Consider the $\mathrm{FI}_{\mathscr{W}}$-module $\mathrm{Ind}^{FI_{\mathscr{W}}}(U)$ which takes $\mathbf{n}$ to 0 if $n < k$ and otherwise to the $\mathscr{W}_n$-representation $\mathrm{Ind}^{\mathscr{W}_n}_{\mathscr{W}_k\times\mathscr{W}_{n-k}} U \boxtimes \mathbb{Q}$, where $U \boxtimes \mathbb{Q}$ is the external tensor product of U with the trivial $\mathscr{W}_{n-k}$-representation $\mathbb{Q}$. $\mathrm{Ind}^{FI_{\mathscr{W}}}(U)$ is a finitely generated $\mathrm{FI}_{\mathscr{W}}$-module with weight $\leq k$ and stability degree $\leq k$. Thus, the sequence of induced representations

$$\left\{\mathrm{Ind}^{\mathscr{W}_n}_{\mathscr{W}_k\times\mathscr{W}_{n-k}} U \boxtimes \mathbb{Q}\right\}$$

is representation stable with stable range $n \geq 2k$.

4.3 Arrangements associated to root systems

Let Φ_n be a root system of type B, C, or D, and let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve. Then recall our notation of $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$ for the corresponding arrangement, with complement $M(\mathscr{A}_n)$ in X^n and poset of layers $\mathscr{C}_n = \mathscr{C}(X, \Phi_n)$. Again, by Remark 1, we will work with the action of the hyperoctahedral group W_n until the corollaries at the end of this section. Throughout this section, cohomology is assumed to have rational coefficients unless otherwise stated.

Consider the Leray spectral sequence of the inclusion $f : M(\mathscr{A}_n) \hookrightarrow X^n$, which is given by

$$E_2^{pq}(n) = H^p(X^n; R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}) \implies H^{p+q}(M(\mathscr{A}_n)),$$

where $R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}$ is the sheaf associated to the presheaf which assigns to an open set U the group $H^q(U \cap M(\mathscr{A}_n))$.

We start by analysing the E_2-term as a representation, breaking this up into two lemmas. The first describes the action on the E_2-term and gives a decomposition akin to that of Totaro [15] using our description of layers from Theorem 1, and the second refines and reorganizes this decomposition in order to write it as a sum of induced representations. Finally, in Theorem 3, we show that this decomposition gives us a decomposition into finitely generated $\mathrm{FI}_{\mathscr{W}}$-modules and use techniques from the previous section to establish stability of the cohomology.

Lemma 2 *Let Φ_n be a root system of type B, C, or D, and let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve. Consider the corresponding arrangement $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$ and Leray spectral sequence. For each $p, q \geq 0$, there is a decomposition*

$$E_2^{pq}(n) \cong \bigoplus_{\Sigma} H^p(F_\Sigma) \otimes \mathrm{OS}^q(\Sigma)$$

where the sum is over $\Sigma \in \mathscr{C}_n = \mathscr{C}(X, \Phi_n)$ such that $\mathrm{rk}(\Sigma) = q$, and $\mathrm{OS}^q(\Sigma)$ is the degree-q part of the Orlik–Solomon algebra of the poset $\{\Sigma' \in \mathscr{C}_n \mid \Sigma' \leq \Sigma\}$.

Moreover, the natural action of $w \in W_n$ on the left induces an action on the right-hand side which permutes the summands according to the action on $\mathscr{C}_n$ and uses the isomorphisms $H^p(F_\Sigma) \to H^p(F_{w\Sigma})$ and $\mathrm{OS}^q(\Sigma) \to \mathrm{OS}^q(w\Sigma)$.

Proof For the decomposition, we follow the method of [4, Lemma 3.1] specialized to our arrangements. A key to the proof is an examination of the stalks of $R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}$, where for $x \in X^n$ we have

$$(R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)})_x \cong \bigoplus_{\substack{\mathrm{rk}(\Sigma)=q \\ F_\Sigma \ni x}} \mathrm{OS}^q(\Sigma).$$

To see how this arises, we must examine $H^q(U \cap M(\mathscr{A}_n))$ for small neighbourhoods U of x. For a small enough neighbourhood, recall from Sect. 2.1 that the cohomology of $U \cap M(\mathscr{A}_n)$ is isomorphic to the Orlik–Solomon algebra of the subposet of layers

which contain x, and this isomorphism respects inclusions of smaller neighbourhoods $U' \hookrightarrow U$. Let Σ be the maximum element of $\mathscr{C}_n$ for which F_Σ contains x, so that the subposet of layers which contain x is the poset $\{\Sigma' \in \mathscr{C}_n \mid \Sigma' \leq \Sigma\}$. Then using Brieskorn's decomposition of the Orlik–Solomon algebra [5, p. 27] we obtain

$$H^q(U \cap M(\mathscr{A}_n)) \cong \mathrm{OS}^q(\Sigma) \cong \bigoplus_{\substack{\mathrm{rk}(\Sigma')=q \\ \Sigma' \leq \Sigma}} \mathrm{OS}^q(\Sigma').$$

Next, we will show that $R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}$ is isomorphic to $\bigoplus_{\mathrm{rk}(\Sigma)=q} (i_\Sigma)_*(\mathrm{OS}^q(\Sigma))$ where $i_\Sigma : F_\Sigma \hookrightarrow X^n$ is the inclusion and $\mathrm{OS}^q(\Sigma)$ is considered as a constant sheaf on F_Σ. Let $\Sigma \in \mathscr{C}_n$ so that F_Σ is a rank-q layer of $\mathscr{A}_n$, and let $\mathscr{A}_\Sigma = \{H \in \mathscr{A}_n \mid H \supseteq F_\Sigma\}$, an arrangement in X^n. Then with $g : M(\mathscr{A}_\Sigma) \hookrightarrow X^n$, the inclusion $M(\mathscr{A}_n) \hookrightarrow M(\mathscr{A}_\Sigma)$ induces a morphism

$$R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)} \to R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}. \tag{1}$$

If F is a rank-q layer of $\mathscr{A}_\Sigma$ with inclusion $i_F : F \hookrightarrow X^n$, then we have a natural map $R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)} \to (i_F)_*(i_F)^* R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)}$. Since the sheaf $R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)}$ is supported on the rank-q layers of $\mathscr{A}_\Sigma$ (similar to our examination of stalks above), and these layers are disjoint closed subvarieties, these maps form an isomorphism

$$\bigoplus_{\mathrm{rk}(F)=q} (i_F)_*(i_F)^* R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)} \cong R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)}. \tag{2}$$

Since F_Σ is one of these rank-q layers, there is a morphism

$$(i_\Sigma)_*(i_\Sigma)^* R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)} \to \bigoplus_{\mathrm{rk}(F)=q} (i_F)_*(i_F)^* R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)}. \tag{3}$$

Now, since $\mathscr{A}_\Sigma$ is locally a linear arrangement, for every point $x \in F_\Sigma$, we have small neighbourhoods U for which $(i_\Sigma)^* R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)}(U) \cong \mathrm{OS}^q(\Sigma)$. This means that the sheaf $(i_\Sigma)^* R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)}$ is locally constant on F_Σ. Moreover, we have a canonical identification of the stalks via translation, and so this sheaf is in fact constant on F_Σ. Thus, we have an isomorphism

$$(i_\Sigma)_* \mathrm{OS}^q(\Sigma) \cong (i_\Sigma)_*(i_\Sigma)^* R^q g_* \mathbb{Q}_{M(\mathscr{A}_\Sigma)}. \tag{4}$$

By composing (1), (2), (3), and (4), we obtain a morphism

$$(i_\Sigma)_* \mathrm{OS}^q(\Sigma) \to R^q f_* \mathbb{Q}_{M(\mathscr{A})},$$

and by summing over all Σ of rank q, we obtain the desired morphism

$$\bigoplus_{\mathrm{rk}(\Sigma)=q} (i_\Sigma)_* \mathrm{OS}^q(\Sigma) \to R^q f_* \mathbb{Q}_{M(\mathscr{A})}. \tag{5}$$

It is an isomorphism because it is an isomorphism on stalks.

Finally, from this isomorphism of sheaves we obtain the decomposition

$$\begin{aligned} H^p(X; R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}) &\cong H^p(X; \bigoplus_{\mathrm{rk}(\Sigma)=q} (i_\Sigma)_*(\mathrm{OS}^q(\Sigma))) \\ &\cong \bigoplus_{\mathrm{rk}(\Sigma)=q} H^p(X; (i_\Sigma)_*(\mathrm{OS}^q(\Sigma))) \\ &\cong \bigoplus_{\mathrm{rk}(\Sigma)=q} H^p(F_\Sigma) \otimes \mathrm{OS}^q(\Sigma). \end{aligned} \tag{6}$$

It remains to show that the action is as described, and so we start by reviewing the action on the left-hand side. For an element $w \in W_n$, there is an isomorphism

$$(w^{-1})^* R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)} \to R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)} \tag{7}$$

which can be seen in the isomorphisms $H^q(w^{-1}(U) \cap M(\mathscr{A}_n)) \to H^q(U \cap M(\mathscr{A}_n))$ from the homeomorphisms $w^{-1} : U \cap M(\mathscr{A}_n) \to w^{-1}(U \cap M(\mathscr{A}_n)) = w^{-1}(U) \cap M(\mathscr{A}_n)$ for small open sets U.

The isomorphism (7) induces an isomorphism in cohomology

$$H^p(X^n; (w^{-1})^* R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}) \to H^p(X^n; R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}).$$

Precomposing with the natural isomorphism on cohomology arising from the homeomorphism $w^{-1} : X^n \to X^n$,

$$H^p(X^n; R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}) \to H^p(X^n; (w^{-1})^* R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)}),$$

we obtain the action of w on $H^p(X^n; R^q f_* \mathbb{Q}_{M(\mathscr{A}_n)})$.

Now we must see that this action gives the claimed action on the decomposition (6). The isomorphisms $H^q(w^{-1}(U) \cap M(\mathscr{A}_n)) \to H^q(U \cap M(\mathscr{A}_n))$ along with Brieskorn's decomposition show that (5) and (7) induce an isomorphism

$$(w^{-1})^* \bigoplus_{\mathrm{rk}(\Sigma)=q} (i_\Sigma)_* \mathrm{OS}^q(\Sigma) \to \bigoplus_{\mathrm{rk}(\Sigma)=q} (i_\Sigma)_* \mathrm{OS}^q(\Sigma)$$

which maps the summand corresponding to Σ to that of $w\Sigma$ using the isomorphism $\mathrm{OS}^q(\Sigma) \to \mathrm{OS}^q(w\Sigma)$. Hence, also the action on the decomposition (6) takes the summand corresponding to Σ to that of $w\Sigma$ using this isomorphism along with the isomorphism $H^p(F_\Sigma) \to H^p(F_{w\Sigma})$ which corresponds to the homeomorphism $w^{-1} : w(F_\Sigma) = F_{w\Sigma} \to F_\Sigma$. □

Lemma 3 *Let Φ_n be a root system of type B, C, or D, and let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve. Consider the corresponding arrangement $\mathscr{A}(X, \Phi_n^+)$ and Leray spectral sequence. Assume that $p, q \geq 0$ and $n \geq p + 2q$.*

There are W_k–representations $V(\lambda, r, \alpha)$ indexed by some finite set $I = \{(\lambda, r, \alpha)\}$, where $k \leq p + 2q$ depends on (λ, r, α), such that

$$E_2^{pq}(n) \cong \bigoplus_I \mathrm{Ind}_{W_k \times W_{n-k}}^{W_n} V(\lambda, r, \alpha) \boxtimes \mathbb{Q}.$$

Proof Fix the notation of Φ_n, X, $\mathscr{A}_n = \mathscr{A}(X, \Phi_n)$, $\mathscr{C}_n = \mathscr{C}(X, \Phi_n)$, and $\mathscr{Q}_n = \mathscr{Q}_n(X)$. Also assume that $p, q \geq 0$ and $n \geq p + 2q$.

First we recall the W_n-orbits in the indexing set $\mathscr{C}_n$. In Lemma 1, we saw that for every $\lambda \in \mathscr{Q}_n$, the set $\{\Sigma \in \mathscr{C}_n \mid \widehat{\Sigma} = \lambda\}$ is either empty or a W_n-orbit. We will introduce a more convenient way to index these orbits using labelled partitions of q, $\mathscr{Q}_q$, so that it is independent of n.

For $\lambda \in \mathscr{Q}_q$, define $\lambda\langle n\rangle \in \mathscr{Q}_n$ as follows: say $(\lambda\langle n\rangle)_z = \lambda_z$ and if $(\lambda_1, \ldots, \lambda_\ell)$ are the unlabelled parts of λ with $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$, let $(\lambda_1 + 1, \ldots, \lambda_\ell + 1, 1, \ldots, 1)$ be the unlabelled parts of $\lambda\langle n\rangle$. In order for this to be a partition of n, note that we must add $n - q - \ell$ ones to the end of the partition and it will have $n - q$ unlabelled parts. For example, if λ is the labelled partition $(1_e, 2, 1)$ of 4, then $\lambda\langle 8\rangle$ is the partition $(1_e, 3, 2, 1, 1)$. Note that for every $\Sigma \in \mathscr{C}_n$ with rank q, there is some $\lambda \in \mathscr{Q}_q$ such that $\lambda\langle n\rangle = \widehat{\Sigma}$, so that indexing by $\mathscr{Q}_q$ will cover all of our W_n-orbits.

With this in mind, we may rewrite the decomposition of Lemma 2 as

$$E_2^{pq}(n) \cong \bigoplus_{\lambda \in \mathscr{Q}_q} \bigoplus_{\widehat{\Sigma} = \lambda\langle n\rangle} H^p(F_\Sigma) \otimes \mathrm{OS}^q(\Sigma),$$

which is a decomposition into W_n-representations indexed by $\lambda \in \mathscr{Q}_q$. But these representations can and should be decomposed further.

Let $\Sigma \in \mathscr{C}_n$. Suppose that $\{i_1\}, \{\bar{i_1}\}, \ldots, \{i_s\}, \{\bar{i_s}\}$ are all of the singleton parts in Σ. Referring to our coordinate-wise description of F_Σ in Sect. 3.3, we see that it factors as

$$F_\Sigma = F'_\Sigma \times X_{i_1} \times \cdots \times X_{i_s}$$

where each subscript i_j denotes the coordinate in which the factor X appears. This means we can use the Künneth formula to write

$$H^p(F_\Sigma) \cong \bigoplus_{r + \sum a_i = p} H^r(F'_\Sigma) \otimes H^{a_1}(X_{i_1}) \otimes \cdots \otimes H^{a_s}(X_{i_s}).$$

We denote $a = (a_1, \ldots, a_s)$, and let $\widehat{a} \vdash (p - r)$ be the partition which lists the nonzero elements of a in decreasing order. For example, if $a = (0, 2, 0, 1, 2)$, then $\widehat{a} = (2, 2, 1)$. Note that we may consider a as an n-tuple where $a_1, \ldots, a_s$ are the coordinates corresponding to $i_1, \ldots, i_s$ and we extend by 0. So we have an action of W_n via the action of S_n by permuting coordinates. The orbits of this action on a's are indexed by $\alpha \vdash (p - r)$.

Given $\lambda \in \mathscr{Q}_q$, $r \in \{0, \ldots, p\}$, and $\alpha \vdash (p-r)$, we will define the following:

$$E(\lambda, r, \alpha)_n = \bigoplus_{\widehat{\Sigma}=\lambda\langle n\rangle} \bigoplus_{\widehat{a}=\alpha} H^r(F'_\Sigma) \otimes H^{a_1}(X_{i_1}) \otimes \cdots \otimes H^{a_s}(X_{i_s}) \otimes \mathrm{OS}^q(\Sigma).$$

Our claim is that each $E(\lambda, r, \alpha)_n$ is a W_n-representation, giving us the following decomposition and the indexing set in the statement of the theorem:

$$E_2^{pq}(n) \cong \bigoplus_{\lambda \in \mathscr{Q}_q} \bigoplus_{r=0}^{p} \bigoplus_{\alpha \vdash (p-r)} E(\lambda, r, \alpha)_n.$$

The action of $w \in W_n$ sends the summand indexed by (Σ, r, a) to that indexed by $(w\Sigma, r, wa)$. The orbit of the index (Σ, r, a) is then indexed by (λ, r, α), which gives the desired decomposition of W_n-representations, but moreover means that W_n acts transitively on the (nontrivial) summands of each $E(\lambda, r, \alpha)_n$.

Now it remains to find the value k (dependent on (λ, r, α) and independent of n) and W_k-representation $V(\lambda, r, \alpha)$ for which $E(\lambda, r, \alpha)_n$ is the desired induced representation. Since W_n acts transitively on the summands, for an arbitrary summand $V(\Sigma, r, a)$ with stabilizer denoted by G, we have $E(\lambda, r, \alpha)_n = \mathrm{Ind}_G^{W_n} V(\Sigma, r, a)$. We can pick a particularly nice choice of Σ and a, by "left-justifying" in the same way Church does. Take $a = (\alpha_1, \ldots, \alpha_t, 0, \ldots, 0)$ and Σ to have singletons $\{n-s+1\}$, $\{\overline{n-s+1}\}$, $\ldots$, $\{n\}$, $\{\bar{n}\}$ along with some fixed partition of **n-s** (independent of n since $n - s = q + \ell$).

Let ℓ be the number of unlabelled parts of λ and $\ell(\alpha) = t$, and define $k = q + \ell + t$. This is independent of n, but note that $k = n - s + t$. Consider W_k as the subgroup of W_n which acts on $\mathbf{k}$, and consider W_{n-k} as acting on $\mathbf{n}\backslash\mathbf{k}$. The stabilizer G of our summand $V(\Sigma, r, \alpha)$ satisfies $W_{n-k} \subseteq G \subseteq W_k \times W_{n-k}$, and moreover, W_{n-k} acts trivially on $V(\Sigma, r, a)$. Thus, we can write $G = H \times W_{n-k}$ for some $H \subseteq W_k$ and view $V(\Sigma, r, a)$ as a representation over H. We define $V(\lambda, r, \alpha) = \mathrm{Ind}_H^{W_k} V(\Sigma, r, a)$. Note that by our choice of Σ and a, the W_k–representation $V(\lambda, r, \alpha)$ does not depend on n.

Finally,

$$\begin{aligned}
E(\lambda, r, \alpha)_n &= \mathrm{Ind}_G^{W_n} V(\Sigma, r, a) \\
&= \mathrm{Ind}_{H \times W_{n-k}}^{W_n} V(\Sigma, r, a) \boxtimes \mathbb{Q} \\
&= \mathrm{Ind}_{W_k \times W_{n-k}}^{W_n} \left(\mathrm{Ind}_{H \times W_{n-k}}^{W_k \times W_{n-k}} V(\Sigma, r, a) \boxtimes \mathbb{Q} \right) \\
&= \mathrm{Ind}_{W_k \times W_{n-k}}^{W_n} \left(\mathrm{Ind}_H^{W_k} V(\Sigma, r, a) \right) \boxtimes \mathbb{Q} \\
&= \mathrm{Ind}_{W_k \times W_{n-k}}^{W_n} V(\lambda, r, \alpha) \boxtimes \mathbb{Q}.
\end{aligned}$$

□

Remark 4 While the lemma required $n \geq p+2q$, one could still make sense of such a decomposition for any n. The point of making a restriction on n is so that the partitions actually determine representations of W_n. But one could just ignore the summands which do not make sense because n is too small.

Theorem 3 *Let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, and let W_n be the hyperoctahedral group. Let $\{\Phi_n\}$ be a sequence of root systems in type B, C, or D, and let $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$ be the corresponding arrangements in X^n. Then for each $i \geq 0$, the sequence $\{H^i(M(\mathscr{A}_n))\}$ of W_n-representations is uniformly representation stable with stable range $n \geq 4i$.*

Proof Note that for any inclusion $\iota : \mathbf{n} \hookrightarrow \mathbf{m}$ with $\iota(\bar{k}) = \overline{\iota(k)}$, we have induced maps $X^m \to X^n$ and $M(\mathscr{A}_m) \to M(\mathscr{A}_n)$. By functoriality of the Leray spectral sequence, we have maps from the Leray spectral sequence associated to $\mathscr{A}_n$ to that of $\mathscr{A}_m$. This makes the Leray spectral sequence a spectral sequence of FI_W-modules. We claim that our decomposition of $E_2^{pq}(n)$ in Lemma 3 actually gives us a decomposition of FI_W-modules

$$E_2^{pq} \cong \bigoplus_I \mathrm{Ind}^{FI_W}(V(\lambda, r, \alpha)).$$

Since each summand on the right-hand side is finitely generated with weight $\leq p+2q$ and stability degree $\leq p+2q$, the finite direct sum giving E_2^{pq} must be as well. In each of our cases, we have $E_3 = E_\infty$ (see Lemma 3.2 and Remark 3.5 in [4]), so the theorem then follows from Proposition 2 and Theorem 2.

To see our decomposition as FI_W-modules, the fact that $V(\lambda, r, \alpha)$ does not depend on n tells us that for each morphism (i.e., injection) $\iota : \mathbf{n} \to \mathbf{n+1}$ in the category FI_W, the following diagram commutes: □

$$\begin{array}{ccc} E_2^{pq}(n) & \longrightarrow & E_2^{pq}(n+1) \\ \uparrow & & \uparrow \\ \bigoplus \mathrm{Ind}_{W_k \times W_{n-k}}^{W_n} V(\lambda, r, \alpha) \boxtimes \mathbb{Q} & \longrightarrow & \bigoplus \mathrm{Ind}_{W_k \times W_{n+1-k}}^{W_{n+1}} V(\lambda, r, \alpha) \boxtimes \mathbb{Q} \end{array}$$

Remark 5 The argument in Lemma 3 is very similar to that given by Church [7] in the type A case, generalized to work with W_n and labelled partitions (rather than S_n and partitions). In a separate paper, Church, Ellenberg, and Farb [8, Theorem 6.2.1] provide an alternative proof of representation stability for the type A case, which uses the fact that the E_2-term is generated by the cohomology of the linear arrangement along with the cohomology of the ambient space. The fact that these generators are finitely generated FI-modules is enough to show that each piece of the E_2-term is. However, in the other cases, the lack of unimodularity makes it more complicated. We still have that the cohomology of the linear arrangement and the cohomology of the ambient space give finitely generated $\mathrm{FI}_{\mathscr{W}}$-modules [17]. However, these together are not enough to generate the E_2-term. Instead of dealing with these extra generators separately, we have decided to follow Church's original argument more closely.

There are a few easy consequences of the stability. First, if we consider $\mathscr{W}_n$ to be the type A or type D Weyl groups, both of which are a subgroup of the type B/C Weyl group W_n, then the restriction of an FI_W-module to an $\mathrm{FI}_{\mathscr{W}}$-module preserves finite generation [16, Prop. 3.22]. This gives us the following:

Corollary 1 *Let $\{\Phi_n\}$ be a sequence of root systems in type B, C, or D. Let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, and let $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$ be the corresponding arrangement in X^n. Let $\mathscr{W}_n$ be the type A, B/C, or D Weyl groups. Then, the sequence $\{H^i(M(\mathscr{A}_n))\}$ of $\mathscr{W}_n$-representations is uniformly representation stable.*

Because of this, we will state the other corollaries in this generality. We also obtain an analogue of [17, Cor. 5.10] on the polynomiality of characters.

Corollary 2 *Let $\{\Phi_n\}$ be a sequence of root systems in type A, B, C, or D, and let $\mathscr{W}_n$ be the corresponding Weyl groups. Let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, and let $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$ be the corresponding arrangement in X^n. Then, the sequence of characters of the $\mathscr{W}_n$-representations $H^i(M(\mathscr{A}_n))$ is given by a unique character polynomial of degree $\leq 2i$. In particular, we have that* $\dim H^i(M(\mathscr{A}_n))$ *is a polynomial in n of degree $\leq 2i$.*

Finally, since $H^i(M(\mathscr{A}_n)/\mathscr{W}_n) \cong H^i(M(\mathscr{A}_n))^{\mathscr{W}_n}$ and Theorem 3 implies stability of the multiplicity of the trivial representation, we can make a statement on homological stability. Arnol'd [1] established homological stability for the type A linear arrangement, and Church [7] gave a more general type A homological stability result. The other linear cases have been studied by Brieskorn [5], but as far as the author knows, it has not been stated for the toric and elliptic analogues in types B, C, and D.

Corollary 3 *Let $\{\Phi_n\}$ be a sequence of root systems in type A, B, C, or D, and let $\mathscr{W}_n$ be the corresponding Weyl groups. Let X be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, and let $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$ be the corresponding arrangement in X^n. Then, the orbit spaces $M(\mathscr{A}_n)/\mathscr{W}_n$ enjoy rational homological stability. That is, for each i, $H^i(M(\mathscr{A}_n)/\mathscr{W}_n)$ does not depend on n for $n \geq 4i$.*

4.4 Examples and computations

Computations such as finding the stable multiplicities of irreducible representations and the character polynomials are difficult in general. In [6, Thm. 1(1)], Chen recently gave a generating function for the stable multiplicities of $H^i(M(\mathscr{A}(X, \Phi_n^+)))$ in the case that $X = \mathbb{C}$ and Φ_n is type A. But more general computations, even for the other linear or type A cases, are not known.

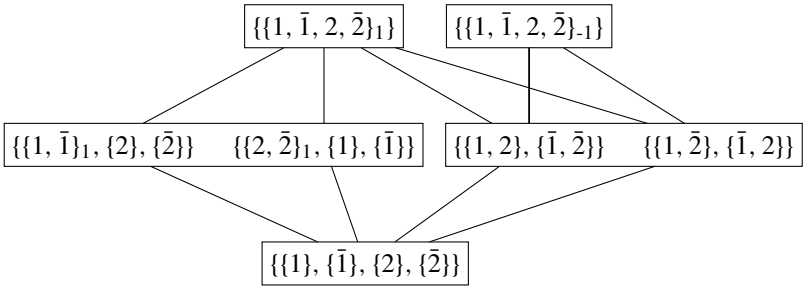
One aspect of the elliptic case that might make it harder is that not even the Betti numbers are known in general (there is a nice combinatorial description of the Betti numbers for linear and toric arrangements). But if one wanted to compute the stable multiplicities for the elliptic case, one might try to first compute them for the E_2-term. What you see, even in type A, is some tensor products of the linear case with an exterior algebra. Thus, even if the multiplicities of the linear case were known, one would have to deal with computation of the Kronecker coefficients from the tensor product.

We do show some work for the degree one cohomology. Even in degree two, though, it starts to get more complicated.

Example 8 Here are computations of the stable multiplicities of $H^1(M(\mathscr{A}_n))$ when Φ_n is type A and $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$.

1. If $X = \mathbb{C}$, then $H^1(M(\mathscr{A}_n)) = V(0) \oplus V(1) \oplus V(2)$ for $n \geq 4$. Church and Farb give this and a decomposition for degree two in [9].
2. If $X = \mathbb{C}^\times$, then $H^1(M(\mathscr{A}_n)) = V(0)^{\oplus 2} \oplus V(1)^{\oplus 2} \oplus V(2)$ for $n \geq 4$.
3. If X is an elliptic curve, then we have $E_\infty^{01}(n) = 0$ and hence for $n \geq 2$, $H^1(M(\mathscr{A}_n)) = E_2^{10}(n) = V(0)^{\oplus 2} \oplus V(1)^{\oplus 2}$.

Example 9 In this example, we demonstrate the decomposition of Lemma 3 for first-degree cohomology $H^1(M(\mathscr{A}(\mathbb{C}^\times, \Phi_2^+)))$ in the case that Φ_2 is type B. In the Leray spectral sequence for toric arrangements, we have that $E_2 = E_\infty$, and so the decompositions of E_2^{01} and E_2^{10} together give a decomposition of the cohomology. First, we consider our poset of labelled partitions, drawn so that orbits are grouped together.



Even though this example is about $n = 2$, there is one note that we can make here for general n. We have $E_2^{10} = H^1((\mathbb{C}^\times)^n)$, which is the first-degree part of an exterior algebra of $\mathbb{Q}^n$. The Weyl group acts in the standard way on $\mathbb{Q}^n$, giving us $E_2^{10}(n) = V((n-1), (1))$. This tells us in particular that $E_2^{10}(2) = V((1), (1))$, and stably we have $E_2^{10}(n) = V(\emptyset, (1))_n$.

For E_2^{01}, we must have $r = 0$ and $\alpha = 0$, and so our decomposition is indexed by the two orbits in the middle of the above picture. These correspond to the two labelled partitions of 1: (1_1) and (1). In the first orbit, let $\Sigma = \{\{1, \bar{1}\}_1, \{2\}, \{\bar{2}\}\}$, and in the second orbit, let $\Sigma' = \{\{1, 2\}, \{\bar{1}, \bar{2}\}\}$. We have:

$$\begin{aligned}
H^1(M(\mathscr{A}_2)) &= E_2^{10}(2) \oplus E_2^{01}(2) \\
&= V((1), (1)) \oplus E((1_1), 0, 0)_2 \oplus E((1), 0, 0)_2 \\
&= V((1), (1)) \oplus \mathrm{Ind}_{W_1 \times W_1}^{W_2} V(\Sigma, 0, 0) \oplus \mathrm{Ind}_{D_2}^{W_2} V(\Sigma', 0, 0) \\
&= V((1), (1)) \oplus \mathrm{Ind}_{W_1 \times W_1}^{W_2} \mathrm{OS}^1(\Sigma) \oplus \mathrm{Ind}_{D_2}^{W_2} \mathrm{OS}^1(\Sigma') \\
&= V((1), (1)) \oplus \mathrm{Ind}_{W_1 \times W_1}^{W_2} H^1(\mathbb{C}^2 \backslash H_1) \oplus \mathrm{Ind}_{D_2}^{W_2} H^1(\mathbb{C}^2 \backslash H_{12}) \\
&= V((1), (1)) \oplus \mathrm{Ind}_{W_1 \times W_1}^{W_2} V((1), \emptyset) \boxtimes \mathbb{Q} \oplus \mathrm{Ind}_{D_2}^{W_2} V((2), \emptyset)
\end{aligned}$$

Note that this does not give us the stable multiplicities, since we need $n \geq 4$. However, Wilson [17] gave a decomposition of E_2^{01}, which we can consider as the first-degree cohomology of the linear type B/C arrangement. This decomposition, as an FI_W-module is $\mathrm{Ind}^{FI_W}(1, \emptyset) \oplus \mathrm{Ind}^{FI_W}(2, \emptyset) \oplus \mathrm{Ind}^{FI_W}(\emptyset, 2)$. By decomposing this into irreducibles when $n = 4$ and using $E_2^{10}(n) = V(\emptyset, (1))_n$, one could compute the stable multiplicities.

If we had considered the type D_2 arrangement, we would have

$$H^1(M(\mathscr{A}_2)) = V((1), (1)) \oplus V((2), \emptyset) \oplus V(\emptyset, (2)).$$

The only difference from type B_2 is that we have only one orbit of rank one, indexed by the partition (1).

If we had considered the type C_2 arrangement, we would have three orbits: (1_1), (1_{-1}), and (1). The first and last would act as before; the new orbit would act just as (1_1) did. Thus, we would have the same decomposition as in the type B_2 case with an extra factor of $\mathrm{Ind}^{FI_W}(1, \emptyset)$.

Example 10 In this example, we demonstrate an aspect of the polynomiality of characters as in Corollary 2. For each of our arrangements $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$, we state the dimension of $H^1(M(\mathscr{A}_n))$. These formulas hold for all $n \geq 2$.

	$X = \mathbb{C}$	$X = \mathbb{C}^\times$	$X = E$
Type A_{n-1}	$\binom{n}{2}$	$\binom{n}{2} + n$	$2n$
Type B_n	$2\binom{n}{2} + n$	$2\binom{n}{2} + 2n$	$\binom{n}{2} + 2n$
Type C_n	$2\binom{n}{2} + n$	$2\binom{n}{2} + 3n$	$\binom{n}{2} + 5n$
Type D_n	$2\binom{n}{2}$	$2\binom{n}{2} + n$	$2n$

4.5 An improvement for type A

In our main results, we had ignored type A for ease of working only with W_n and because the result was already known, but the stable range for some type A arrangements can be improved. Recall that we know the sequence stabilizes once $n \geq 4i$, in each type (that is, for each X and family of root systems). Recently, Hersh and Reiner [11] improved the stable range for the type A linear case, showing that the i-th cohomology stabilizes for $n \geq 3i + 1$. We show an improvement for the elliptic case, and we wonder if it can be improved further, or if Hersh and Reiner's result can be used to improve the range of the toric case.

Proposition 3 *If $\{\Phi_n\}$ is a sequence of type A root systems and X is a complex elliptic curve, let $\mathscr{A}_n = \mathscr{A}(X, \Phi_n^+)$. Then for each $i \geq 1$, the stable range of the sequence $\{H^i(M(\mathscr{A}_n))\}$ of S_n-representations may be improved to $n \geq 4i - 2$.*

Proof Fix $i \geq 1$. We claim that the differential $d : E_2^{0,i}(n) \to E_2^{2,i-1}(n)$ is injective for all n, and hence $E_3^{0,i} = E_\infty^{0,i} = 0$ for all n. Thus, in our filtration $F_0 \subseteq \cdots \subseteq F_i = H^i$, the maximum weight (and similarly stability degree) among F_j and F_j/F_{j-1} is $2i-1$. This implies that $H^i(M(\mathscr{A}_n))$ is representation stable for $n \geq 2(2i-1) = 4i-2$.

First, we refer the reader to [15] for a description of a decomposition of and differential on the E_2 term, as well as [3] for a description of a basis, in the case of type A arrangements. Now to show injectivity of this differential, we pick our standard generators x_i, y_i $(1 \leq i \leq n)$ for $H^*(X^n)$ and g_{ij} $(1 \leq i < j \leq n)$ for

$$E_2^{0,1}(n) = \bigoplus_{1 \leq i < j \leq n} \mathrm{OS}^1(H_{ij})$$

where $\mathrm{OS}^1(H_{ij}) \cong H^1(\mathbb{C}^\times)$. The differential sends g_{ij} to the class of the diagonal H_{ij} in $H^2(X^n)$, which is given by $(x_i - x_j)(y_i - y_j)$. A basis for $E_2^{2,0} = H^2(X^n)$ is given by pairs $x_i x_j$, $y_i y_j$ $(i \neq j)$ along with pairs $x_i y_j$. So if we had a linear combination $\sum c_{ij} g_{ij}$ in the kernel of the differential, then its image $\sum c_{ij}(x_i - x_j)(y_i - y_j)$ would be zero. We can write this sum in terms of the basis as $\sum d_i x_i y_i - \sum c_{ij}(x_i y_j + x_j y_i)$ for some coefficients d_i. Since $x_i y_j$ appears once, with coefficient c_{ij}, we must have $c_{ij} = 0$.

To extend this to $E_2^{0,q} \to E_2^{2,q-1}$, we recall the basis given by Bezrukavnikov [3]. For $E_2^{0,q}$, we have monomials $g_{i_1 j_1} \ldots g_{i_q j_q}$ with $i_s > j_s$ for each s and with $i_1 > i_2 > \cdots > i_q$. Similarly, a basis for $E_2^{2,q-1}$ is given by $z_a z_b g_{i_1 j_1} \ldots g_{i_{q-1} j_{q-1}}$ where we have the same conditions on the indices on g's, z_a stands for x_a or y_a, z_b stands for x_b or y_b, and we have $a, b \notin \{i_1, \ldots, i_{q-1}\}$. Then suppose we have $d(\sum c_S g_S) = 0$ where S runs through the sets indexing our monomial basis and the c_S are some rational coefficients. Consider one such $S = \{(i_1, j_1), \ldots, (i_q, j_q)\}$. In the expansion of $d(\sum c_S g_S)$ in the monomial basis, the coefficient of $x_{i_q} y_{j_q} g_{i_1 j_1} \ldots g_{i_{q-1} j_{q-1}}$ is equal to $\pm c_S$. This is because we have $i_1 > i_2 > \cdots > i_{q-1} > i_q > j_q$; there is no other S' from which this monomial arises in the differential. Thus, we must have each $c_S = 0$. □

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