TUTTE POLYNOMIALS OF GENERALIZED PARALLEL CONNECTIONS

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ABSTRACT. We use weighted characteristic polynomials to compute Tutte polynomials of generalized parallel connections in the case in which the simplification of the maximal common restriction of the two constituent matroids is a modular flat of the simplifications of both matroids. In particular, this includes cycle matroids of graphs that are identified along complete subgraphs. We also develop formulas for Tutte polynomials of the k-sums that are obtained from such generalized parallel connections.

1. INTRODUCTION

The Tutte polynomial $t(M;x,y)$ of a matroid M encodes a great deal of information about the matroid, including the rank and corank, whether the matroid is connected, the numbers of independent sets of each cardinality, and the numbers of flats of each rank having sufficiently large cardinality [6, 7]. The Tutte polynomial is also intimately related to many important and difficult problems in a wide variety of areas, such as graph theory, linear coding theory, arrangements of hyperplanes, and knot theory [7, 13]. Computing the Tutte polynomial of a matroid is known to be $\#P$-hard, so there is great interest in formulas that reduce this computation to simpler computations. One of the most basic instances of such a formula is that for the Tutte polynomial of direct sums of matroids: $t(M_1 \oplus M_2;x,y) = t(M_1;x,y)\,t(M_2;x,y)$. Knowing this formula, a natural next step is to develop formulas for Tutte polynomials of matroids that are connected but that can be written in terms of other matroids via operations such as parallel connections, series connections, 2-sums, and their generalizations. The previously known formulas of this type are for parallel connections, series connections, and 2-sums [3, 6], and 3-sums and generalized parallel connections along three-point lines (see [1], including the results of J. Oxley stated at the end of that paper). In this paper, we treat all generalized parallel connections in which the simplification of the maximal common restriction of the two constituent matroids is a modular flat of the simplifications of both matroids. In this case, we develop a formula for the Tutte polynomial of the generalized parallel connection in terms of the Tutte polynomials of contractions of the constituent matroids by flats of the maximal common restriction. We also derive such a formula for the Tutte polynomials of the k-sums that are obtained from such generalized parallel connections. Our main tool is the weighted characteristic polynomial, which is equivalent to the Tutte polynomial. (For other recent applications of the weighted characteristic polynomial, see [2, 9].)

The key points needed about Tutte polynomials and weighted characteristic polynomials are reviewed in Section 2; further background can be found in [6, 7, 8]. The necessary background on generalized parallel connections and k-sums is given in Section 3; more information can be found in [4, 10]. The formula for the Tutte polynomial of a generalized parallel connection is developed in Section 4 and that for the Tutte polynomial of a k-sum is treated in Section 5. Our terminology and notation largely follow [10] with a few mild variations that are made explicit in the paper.

2. Tutte Polynomials and Weighted Characteristic Polynomials

The *Tutte polynomial* $t(M;x,y)$ of a matroid M on the set S is given by

$$t(M;x,y) = \sum_{A\subseteq S}(x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}. \tag{1}$$

The *characteristic polynomial* $\chi(M;\lambda)$ of M is, up to sign, a special evaluation of the Tutte polynomial of M:

$$\chi(M;\lambda) = (-1)^{r(M)}t(M;1-\lambda,0).$$

The characteristic polynomial can be formulated in a variety of ways. Equation (1) yields the following Boolean expansion of the characteristic polynomial.

$$\chi(M;\lambda) = \sum_{A\subseteq S}(-1)^{|A|}\lambda^{r(M)-r(A)}$$

The characteristic polynomial can also be expressed in the following way:

$$\chi(M;\lambda) = \sum_{\substack{\text{flats } F\\ \text{of } M}} \mu(\emptyset,F)\lambda^{r(M)-r(F)},$$

where μ is the Möbius function of M (see [11, 14]). It follows easily from any of these formulations that $\chi(M;\lambda)=0$ if M has loops. Thus, for a contraction M/Z of M, the characteristic polynomial $\chi(M/Z;\lambda)$ is nonzero only if Z is a flat of M.

Brylawski [5] defined the *weighted characteristic polynomial* $\overline{\chi}(M;x,y)$ of M to be

$$\overline{\chi}(M;x,y) = \sum_{\substack{\text{flats } F\\ \text{of } M}} x^{|F|}\chi(M/F;y).$$

(By the remark above about characteristic polynomials of contractions, we see that this sum could be taken over all subsets F of the ground set of M. The weighted characteristic polynomial is, upon switching the variables, the coboundary polynomial of [8]. In Section 6.3.F of [7], the notation $\overline{\chi}(M;x,y)$ is used for the coboundary polynomial.)

The following well-known formulas make precise the statement that the Tutte polynomial $t(M;x,y)$ and the weighted characteristic polynomial $\overline{\chi}(M;x,y)$ are equivalent.

$$t(M;x,y) = \frac{\overline{\chi}\big(M;y,(x-1)(y-1)\big)}{(y-1)^{r(M)}} \tag{2}$$

$$\overline{\chi}(M;x,y) = (x-1)^{r(M)}\ t\left(M;\frac{y}{x-1}+1,x\right) \tag{3}$$

3. Generalized Parallel Connections and k-Sums

Assume that the following conditions hold for the matroids M_1 and M_2 on the ground sets S_1 and S_2, respectively:

(G1) $M_1|T = M_2|T$ where $T = S_1\cap S_2$,
(G2) $\mathrm{cl}_{M_1}(T)$ is a modular flat of M_1, and
(G3) each element of $\mathrm{cl}_{M_1}(T)-T$ is either a loop or parallel to an element of T.

Let N denote the common restriction $M_1|T = M_2|T$. The *generalized parallel connection* of M_1 and M_2 at T is the matroid $P_N(M_1,M_2)$ whose flats are precisely the subsets A of $S_1\cup S_2$ such that $A\cap S_1$ is a flat of M_1 and $A\cap S_2$ is a flat of M_2. Equivalently, the flats of $P_N(M_1,M_2)$ are the subsets of $S_1\cup S_2$ of the form $A_1\cup A_2$ where A_1 and A_2 are flats of M_1 and M_2, respectively, and $A_1\cap T = A_2\cap T$. Observe that if T is the empty set, then $P_N(M_1, M_2)$ is the direct sum $M_1 \oplus M_2$. While it is not required by the general definition, the main results of this paper assume the following two additional conditions:

(G4) $\mathrm{cl}_{M_2}(T)$ is a modular flat of M_2, and
(G5) each element of $\mathrm{cl}_{M_2}(T) - T$ is either a loop or parallel to an element of T.

In the case of graphs whose maximal common restriction is a clique, the generalized parallel connection of the corresponding cycle matroids yields the cycle matroid of their clique-sum [12]. Another special case that has received considerable attention is that of the *parallel connection* $P(M_1, M_2)$ of two matroids M_1 and M_2; in this case, the common restriction T is a singleton $\{p\}$; since we must have $M_1|T = M_2|T$, either p is a loop of both M_1 and M_2 or p is a loop of neither M_1 nor M_2. (See [3, 10].) The *series connection* $S(M_1, M_2)$ of two matroids M_1 and M_2 is given by the dual operation: $S(M_1, M_2) = \big(P(M_1^*, M_2^*)\big)^*$.

An operation that is closely related to parallel connections is the construction of 2-sums. This operation plays an important role in matroid theory; for instance, a result proved independently by Bixby, Cunningham, and Seymour says that a 2-connected matroid is not 3-connected if and only if it can be written as a 2-sum of two of its proper minors (see [10, Section 8.3]). The operation of 3-sum has been defined for binary matroids (see [10]). Consistent with these definitions, we define the k-sum of two matroids as follows. Assume that M_1 and M_2 are matroids on the ground sets S_1 and S_2, that conditions (G1)–(G3) hold, and that $r(T) = k - 1$. The *k-sum* of M_1 and M_2 is the deletion $P_N(M_1, M_2)\backslash T$ of the generalized parallel connection of M_1 and M_2.

Our results on Tutte polynomials of k-sums will follow relatively directly from our work on Tutte polynomials of generalized parallel connections and one technical lemma that we develop in Section 5, hence the rest of this section focuses on properties of generalized parallel connections.

We first turn to the ranks of flats in $P_N(M_1, M_2)$. The rank of a flat A of a matroid is the number of flats other than $\mathrm{cl}(\emptyset)$ in a saturated chain of flats from $\mathrm{cl}(\emptyset)$ to A. Using this perspective on rank together with the definition of the flats of $P_N(M_1, M_2)$, it follows that the rank of a flat $A_1 \cup A_2$ of $P_N(M_1, M_2)$, where A_1 and A_2 are flats of M_1 and M_2, respectively, with $W = A_1 \cap T = A_2 \cap T$, is

$$r_N(W) + \big(r_{M_1}(A_1) - r_N(W)\big) + \big(r_{M_2}(A_2) - r_N(W)\big),$$

that is,

$$r(A_1 \cup A_2) = r_{M_1}(A_1) + r_{M_2}(A_2) - r_N(W). \tag{4}$$

In particular, the rank of the matroid $P_N(M_1, M_2)$ is

$$r\big(P_N(M_1, M_2)\big) = r(M_1) + r(M_2) - r(N). \tag{5}$$

The following result contains the parts of [10, Proposition 12.4.14] that we need in this paper.

Theorem 1. *Assume that M_1 and M_2 are matroids on the ground sets S_1 and S_2, and that conditions (G1)–(G3) hold. Then the following properties hold.*

(i) $P_N(M_1, M_2)|S_1 = M_1$ *and* $P_N(M_1, M_2)|S_2 = M_2$.
(ii) *If* $x \in S_1 - T$, *then* $P_N(M_1, M_2)\backslash x = P_N(M_1\backslash x, M_2)$.
(iii) *If* $x \in S_1 - \mathrm{cl}_{M_1}(T)$, *then* $P_N(M_1, M_2)/x = P_N(M_1/x, M_2)$.
(iv) *If* $x \in S_2 - T$, *then* $P_N(M_1, M_2)\backslash x = P_N(M_1, M_2\backslash x)$.
(v) *If* $x \in S_2 - \mathrm{cl}_{M_2}(T)$, *then* $P_N(M_1, M_2)/x = P_N(M_1, M_2/x)$.
(vi) *If* $x \in T$, *then* $P_N(M_1, M_2)/x = P_{N/x}(M_1/x, M_2/x)$.
(vii) $P_N(M_1, M_2)/T = (M_1/T) \oplus (M_2/T)$.

Using condition (G3), one can adapt these formulas to identify contractions by elements in $\mathrm{cl}_{M_1}(T)-T$; however, there are no corresponding results for contractions by elements in $\mathrm{cl}_{M_2}(T)-T$ in the absence of condition (G5). Note that parts (iii) and (vi) of Theorem 1 rely on the following lemma [4, Corollary 3.9], which we also use below.

Lemma 2. *Assume that F is a modular flat of a matroid M and that A is a subset of the ground set of M. Then $\mathrm{cl}_{M/A}(F-A)$ is a modular flat of M/A.*

To apply the definition of the weighted characteristic polynomial to a generalized parallel connection $P_N(M_1,M_2)$ under the added assumptions (G4) and (G5), we need to know the characteristic polynomials of contractions of $P_N(M_1,M_2)$ by flats. The next lemma, which requires conditions (G4) and (G5), addresses these contractions.

Lemma 3. *Assume that M_1 and M_2 are matroids on the ground sets S_1 and S_2, and that conditions (G1)–(G5) hold. For a flat F of $P_N(M_1,M_2)$ with $F\cap S_1=A_1$, $F\cap S_2=A_2$, and $F\cap T=W$, we have*

$$P_N(M_1,M_2)/F = P_{N/W}(M_1/A_1, M_2/A_2). \tag{6}$$

Proof. By viewing the contraction $P_N(M_1,M_2)/F$ as a sequence of contractions by points (i.e., flats of rank-1), it suffices to show that equation (6) is true if F is a point, and (so that the argument may be applied recursively) that the corresponding contractions M_1/A_1 and M_2/A_2 also satisfy conditions (G1)–(G5). By Lemma 2, the closures of $T-W$ in M_1/A_1 and M_2/A_2 are modular flats so conditions (G2) and (G4) are valid for these contractions. Throughout this proof, cl denotes the closure operator of $P_N(M_1,M_2)$. By conditions (G3), (G5), and the description of flats of $P_N(M_1,M_2)$, the closure $\mathrm{cl}(T)$ is clearly $\mathrm{cl}_{M_1}(T)\cup\mathrm{cl}_{M_2}(T)$.

Assume that F is a point and that $r\big(F\cap\mathrm{cl}_{M_1}(T)\big)=1$; by conditions (G3) and (G5), this is equivalent to the case in which $r\big(F\cap\mathrm{cl}_{M_2}(T)\big)=1$. By condition (G3), the point F contains at least one element, say x, in T that is not a loop. By part (vi) of Theorem 1, we have $P_N(M_1,M_2)/x = P_{N/x}(M_1/x,M_2/x)$. Each element z of $F-x$ is a loop of $P_N(M_1,M_2)/x$ and of one or both of M_1/x and M_2/x, according to whether z is in S_1-T, S_2-T, or T. Since contracting a loop is the same as deleting the loop, one of parts (ii), (iv), or (vi) of Theorem 1 applies to any loop. Thus equation (6) follows. Since $A_1\subseteq\mathrm{cl}_{M_1}(T)$ and $A_2\subseteq\mathrm{cl}_{M_2}(T)$, we get $\mathrm{cl}_{M_1/A_1}(T-W)\subseteq\mathrm{cl}_{M_1}(T)$ and $\mathrm{cl}_{M_2/A_2}(T-W)\subseteq\mathrm{cl}_{M_2}(T)$, from which it follows that conditions (G3) and (G5) are valid for M_1/A_1 and M_2/A_2. Since x is in $F\cap T$, condition (G1) is also valid for these contractions.

Assume that F is a point that intersects $\mathrm{cl}_{M_1}(T)$ and $\mathrm{cl}_{M_2}(T)$ only in loops. Let x be an element in F that is not a loop; we may assume x is in $S_1-\mathrm{cl}_{M_1}(T)$. Thus, x is not in $\mathrm{cl}(T)$. We have $P_N(M_1,M_2)/x = P_N(M_1/x,M_2)$ by part (iii) of Theorem 1. It follows that each element z of $F-x$ is a loop of $P_N(M_1,M_2)/x$ and of one or both of M_1/x and M_2, so equation (6) follows as above. Note that

$$(M_1/A_1)|(T-W) = M_1|(T-W) = M_2|(T-W) = (M_2/A_2)|(T-W)$$

since F is not in $\mathrm{cl}(T)$, so condition (G1) holds for M_1/A_1 and M_2/A_2. From equation (4) we have $r_{M_2}(A_2)=0$; therefore $\mathrm{cl}_{M_2/A_2}(T-W)\subseteq\mathrm{cl}_{M_2}(T)$, so M_2/A_2 satisfies condition (G5). Finally, to address condition (G3) assume u is in $\mathrm{cl}_{M_1/A_1}(T-W)-\mathrm{cl}_{M_1}(T)$. Thus, u is in $\mathrm{cl}_{M_1}(T\cup\{x\})-\mathrm{cl}_{M_1}(T)$. Now $\mathrm{cl}_{M_1}(T)$ is a modular hyperplane of the restriction $M_1|\big(\mathrm{cl}_{M_1}(T\cup\{x\})\big)$, so the line $\mathrm{cl}_{M_1}(\{x,u\})$, which is not contained in $\mathrm{cl}_{M_1}(T)$, must intersect T in a set of rank 1. It follows that in M_1/A_1, the element u is parallel to some element of $T-W$; from this we see that M_1/A_1 satisfies condition (G3). □

Turning to the characteristic polynomial, part (vii) of Theorem 1 and basic properties of the characteristic polynomial give the following formula:

$$\chi\big(P_N(M_1,M_2)/T;\lambda\big) = \chi(M_1/T;\lambda)\chi(M_2/T;\lambda). \tag{7}$$

The characteristic polynomial of $P_N(M_1,M_2)$ is given by the following result due to Brylawski [4, Theorem 7.8].

Theorem 4. *Assume that M_1 and M_2 are matroids on the ground sets S_1 and S_2, and that conditions (G1)–(G3) hold. If the maximal common restriction N of M_1 and M_2 has no loops, then*

$$\chi\big(P_N(M_1,M_2);\lambda\big) = \frac{\chi(M_1;\lambda)\chi(M_2;\lambda)}{\chi(N;\lambda)}.$$

Combining Brylawski's theorem with Lemma 3 gives the following result.

Theorem 5. *Assume M_1 and M_2 are matroids on the ground sets S_1 and S_2, and that conditions (G1)–(G5) hold. For a flat F of $P_N(M_1,M_2)$ with $F\cap S_1 = A_1$, $F\cap S_2 = A_2$, and $F\cap T = W$, we have*

$$\chi\big(P_N(M_1,M_2)/F;\lambda\big) = \frac{\chi(M_1/A_1;\lambda)\chi(M_2/A_2;\lambda)}{\chi(N/W;\lambda)}.$$

Theorem 5 applies even in the case $W = T$ if we take the characteristic polynomial of the empty matroid to be 1. With this convention, equation (7) is a special case of Theorem 5.

4. Tutte Polynomials of Generalized Parallel Connections

In this section, we prove the following theorem on Tutte polynomials of generalized parallel connections and we specialize this to Tutte polynomials of parallel connections and series connections [3, 6], and to Tutte polynomials of generalized parallel connections along three-point lines [1].

Theorem 6. *Assume M_1 and M_2 are matroids on the ground sets S_1 and S_2, and that conditions (G1)–(G5) hold. The Tutte polynomial $t\big(P_N(M_1,M_2);x,y\big)$ of the generalized parallel connection $P_N(M_1,M_2)$ is given by the following formula:*

$$(y-1)^{r(T)} \sum_{\substack{\text{flats } W\\ \text{of } N}} \frac{1}{y^{|W|}\,\chi\big(N/W;(x-1)(y-1)\big)}$$
$$\times\left(\sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\frac{y^{|Z|}}{(y-1)^{r(Z)}}t(M_1/Z;x,y)\right)$$
$$\times\left(\sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\frac{y^{|Z|}}{(y-1)^{r(Z)}}t(M_2/Z;x,y)\right),$$

where μ_N is the Möbius function of N.

Proof. By the definition of the weighted characteristic polynomial and the second formulation of the flats of the generalized parallel connection, it follows that the weighted characteristic polynomial $\overline{\chi}\big(P_N(M_1,M_2);x,y\big)$ of $P_N(M_1,M_2)$ is given by the following formula:

$$\sum_{\substack{\text{flats } W\\ \text{of } N}} \sum_{\substack{\text{flats } A_1 \text{ of } M_1,\\ A_2 \text{ of } M_2 \text{ with}\\ A_1\cap T=W=A_2\cap T}} x^{|A_1|+|A_2|-|W|}\chi\big(P_N(M_1,M_2)/(A_1\cup A_2);y\big).$$

This summation simplifies as follows. First use Theorem 5 to get

$$\sum_{\substack{\text{flats } W\\ \text{of } N}} \sum_{\substack{\text{flats } A_1 \text{ of } M_1,\\ A_2 \text{ of } M_2 \text{ with}\\ A_1\cap T=W=A_2\cap T}} x^{|A_1|+|A_2|-|W|}\chi\big(P_N(M_1,M_2)/(A_1\cup A_2);y\big)$$

$$= \sum_{\substack{\text{flats } W\\ \text{of } N}} \frac{1}{x^{|W|}} \sum_{\substack{\text{flats } A_1 \text{ of } M_1,\\ A_2 \text{ of } M_2 \text{ with}\\ A_1\cap T=W=A_2\cap T}} x^{|A_1|}x^{|A_2|}\frac{\chi(M_1/A_1;y)\chi(M_2/A_2;y)}{\chi(N/W;y)}$$

$$= \sum_{\substack{\text{flats } W\\ \text{of } N}} \frac{1}{x^{|W|}\,\chi(N/W;y)} \sum_{\substack{\text{flats } A_1 \text{ of } M_1\\ \text{with } A_1\cap T=W}} x^{|A_1|}\chi(M_1/A_1;y) \tag{8}$$

$$\times \sum_{\substack{\text{flats } A_2 \text{ of } M_2\\ \text{with } A_2\cap T=W}} x^{|A_2|}\chi(M_2/A_2;y). \tag{9}$$

We next show that the sums in lines (8) and (9) can be written as sums of weighted characteristic polynomials as follows:

$$\sum_{\substack{\text{flats } A_1 \text{ of } M_1\\ \text{with } A_1\cap T=W}} x^{|A_1|}\chi(M_1/A_1;y) = \sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\,x^{|Z|}\overline{\chi}(M_1/Z;x,y), \tag{10}$$

$$\sum_{\substack{\text{flats } A_2 \text{ of } M_2\\ \text{with } A_2\cap T=W}} x^{|A_2|}\chi(M_2/A_2;y) = \sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\,x^{|Z|}\overline{\chi}(M_2/Z;x,y). \tag{11}$$

To see equation (10), note that the right side of this equation is

$$\sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\,x^{|Z|} \sum_{\substack{\text{flats } A_1 \text{ of } M_1\\ \text{with } Z\subseteq A_1}} x^{|A_1|-|Z|}\chi(M_1/A_1;y).$$

Thus, the term $x^{|A_1|}\chi(M_1/A_1;y)$ occurs in this sum with coefficient

$$\sum_{\substack{\text{flats } Z \text{ of } N:\\ W\subseteq Z\subseteq A_1\cap T}} \mu_N(W,Z).$$

Thus, this coefficient is 1 if $A_1\cap T = W$, and 0 otherwise, as needed to prove equation (10). By using equations (10) and (11) to rewrite the sums in lines (8) and (9), we get

$$\overline{\chi}\big(P_N(M_1,M_2);x,y\big) = \sum_{\substack{\text{flats } W\\ \text{of } N}} \frac{1}{x^{|W|}\chi(N/W;y)} \times \left(\sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\,x^{|Z|}\overline{\chi}(M_1/Z;x,y)\right) \times \left(\sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\,x^{|Z|}\overline{\chi}(M_2/Z;x,y)\right).$$

The formula in the theorem now follows by applying equations (2) and (5) to the last equation. □

We now turn to several corollaries of Theorem 6.

It follows from the definition of the parallel connection that if the ground sets of two matroids M_1 and M_2 intersect in a single element p that is a loop of both M_1 and M_2, then their parallel connection $P(M_1, M_2)$ is the direct sum $M_1 \oplus (M_2 \backslash p)$. In this case, we therefore have $t\big(P(M_1, M_2); x, y\big) = (1/y)\, t(M_1; x, y)\, t(M_2; x, y)$. Dually, if the ground sets of two matroids M_1 and M_2 intersect in a single element that is an isthmus of both M_1 and M_2, then $t\big(S(M_1, M_2); x, y\big) = (1/x)\, t(M_1; x, y)\, t(M_2; x, y)$. Corollary 7 considers a slightly more general setting than [3, Theorem 6.15] in that it considers matroids whose grounds sets intersect in a point, that is, a flat of rank 1, thereby allowing this flat to contain loops and parallel elements. (See also Section 4 of [6].)

Corollary 7. *Assume that M_1 and M_2 are matroids on the ground sets S_1 and S_2, that $N := M_1|T = M_2|T$, where $T = S_1 \cap S_2$, and that $r(N) = 1$. Assume that N contains h elements, j of which are loops. The Tutte polynomial of $P_N(M_1, M_2)$ is given by the following formula.*

$$\begin{aligned} t\big(P_N(M_1, M_2); x, y\big) &= \frac{y-1}{(xy-x-y)y^j}\, t(M_1; x, y)\, t(M_2; x, y) \\ &\quad - \frac{y^{h-j}}{xy-x-y}\, t(M_1; x, y)\, t(M_2/T; x, y) \\ &\quad - \frac{y^{h-j}}{xy-x-y}\, t(M_1/T; x, y)\, t(M_2; x, y) \\ &\quad + \frac{y^{2h-j} + y^h(xy-x-y)}{(xy-x-y)(y-1)}\, t(M_1/T; x, y)\, t(M_2/T; x, y) \end{aligned}$$

In particular, if $T = \{p\}$ and $r(T) = 1$, then the Tutte polynomial of the parallel connection $P(M_1, M_2)$ is given by the following formula.

$$\begin{aligned} t\big(P(M_1, M_2); x, y\big) &= \frac{y-1}{xy-x-y}\, t(M_1; x, y)\, t(M_2; x, y) \\ &\quad - \frac{y}{xy-x-y}\, t(M_1; x, y)\, t(M_2/p; x, y) \\ &\quad - \frac{y}{xy-x-y}\, t(M_1/p; x, y)\, t(M_2; x, y) \\ &\quad + \frac{xy}{xy-x-y}\, t(M_1/p; x, y)\, t(M_2/p; x, y) \end{aligned}$$

Assume that M_1 and M_2 are matroids on the sets S_1 and S_2, that $S_1 \cap S_2 = \{p\}$, and that p is not an isthmus of either M_1 or M_2. The Tutte polynomial of the series connection $S(M_1, M_2)$ is given by the following formula.

$$\begin{aligned} t\big(S(M_1, M_2); x, y\big) &= \frac{x-1}{xy-x-y}\, t(M_1; x, y)\, t(M_2; x, y) \\ &\quad - \frac{x}{xy-x-y}\, t(M_1; x, y)\, t(M_2 \backslash p; x, y) \\ &\quad - \frac{x}{xy-x-y}\, t(M_1 \backslash p; x, y)\, t(M_2; x, y) \\ &\quad + \frac{xy}{xy-x-y}\, t(M_1 \backslash p; x, y)\, t(M_2 \backslash p; x, y) \end{aligned}$$

In the case of a three-point line, the formula in the next corollary of Theorem 6 reduces to formulas in [1].

Corollary 8. *Assume that M_1 and M_2 are matroids on the ground sets S_1 and S_2, that $T := S_1 \cap S_2 = \{p_1, p_2, \ldots, p_n\}$, that the restrictions $M_1|T$ and $M_2|T$ are isomorphic to the uniform matroid $U_{2,n}$, and that conditions (G2)–(G5) hold. Let $a(x, y) = xy - x - y$ and $b(x, y) = xy - x - y - n + 2$. The Tutte polynomial of the generalized parallel* *connection $P_N(M_1,M_2)$ is given by the following formula.*

$$\begin{aligned}
&\frac{(y-1)^2}{a(x,y)b(x,y)}t(M_1;x,y)t(M_2;x,y)\\
&-\frac{y(y-1)}{a(x,y)b(x,y)}\sum_{i=1}^{n}\big(t(M_1/p_i;x,y)t(M_2;x,y)+t(M_1;x,y)t(M_2/p_i;x,y)\big)\\
&+\frac{(n-1)y^n}{a(x,y)b(x,y)}\big(t(M_1/T;x,y)t(M_2;x,y)+t(M_1;x,y)t(M_2/T;x,y)\big)\\
&+\frac{y^2}{a(x,y)b(x,y)}\sum_{\substack{1\le i,j\le n\\ i\neq j}}t(M_1/p_i;x,y)t(M_2/p_j;x,y)\\
&+\frac{y\big(b(x,y)+y\big)}{a(x,y)b(x,y)}\sum_{i=1}^{n}t(M_1/p_i;x,y)t(M_2/p_i;x,y)\\
&-\frac{y^n(x+n-2)}{a(x,y)b(x,y)}\sum_{i=1}^{n}\big(t(M_1/p_i;x,y)t(M_2/T;x,y)+t(M_1/T;x,y)t(M_2/p_i;x,y)\big)\\
&+\frac{y^n\left((n-1)^2y^n+ny^{n-1}b(x,y)+a(x,y)b(x,y)\right)}{a(x,y)b(x,y)(y-1)^2}t(M_1/T;x,y)t(M_2/T;x,y)
\end{aligned}$$

Proof. By Theorem 6, $t\big(P_N(M_1,M_2);x,y\big)$ is given by the following expression.

$$\begin{aligned}
&\frac{(y-1)^2}{a(x,y)b(x,y)}\times\\
&\quad\left(t(M_1;x,y)-\left(\sum_{i=1}^{n}\frac{y}{y-1}t(M_1/p_i;x,y)\right)+(n-1)\frac{y^n}{(y-1)^2}t(M_1/T;x,y)\right)\times\\
&\quad\left(t(M_2;x,y)-\left(\sum_{i=1}^{n}\frac{y}{y-1}t(M_2/p_i;x,y)\right)+(n-1)\frac{y^n}{(y-1)^2}t(M_2/T;x,y)\right)\\
&+\frac{(y-1)^2}{y\cdot a(x,y)}\sum_{i=1}^{n}\left(\left(\frac{y}{y-1}t(M_1/p_i;x,y)-\frac{y^n}{(y-1)^2}t(M_1/T;x,y)\right)\times\right.\\
&\qquad\left.\left(\frac{y}{y-1}t(M_2/p_i;x,y)-\frac{y^n}{(y-1)^2}t(M_2/T;x,y)\right)\right)\\
&+\frac{(y-1)^2}{y^n}\frac{y^n}{(y-1)^2}t(M_1/T;x,y)\frac{y^n}{(y-1)^2}t(M_2/T;x,y)
\end{aligned}$$

The corollary follows from algebraic manipulation of this expression. □

5. Tutte Polynomials of k-Sums

By the deletion-contraction formula, the Tutte polynomial of each single-element deletion by a non-loop, non-isthmus can be written in terms of the Tutte polynomials of the matroid and the corresponding single-element contraction. Specifically, we have that if e is neither a loop nor an isthmus of M, then

$$t(M\backslash e;x,y)=t(M;x,y)-t(M/e;x,y). \tag{12}$$

If e is a loop of M, we can write the Tutte polynomial of the single-element deletion $M\backslash e$ in terms of the Tutte polynomial of M via the following equation:

$$t(M\backslash e;x,y)=y^{-1}\,t(M;x,y). \tag{13}$$

The formula for the Tutte polynomial of a k-sum that we give in Theorem 10 follows by combining Theorem 6 with the extension of equations (12) and (13) that we develop in the next several paragraphs and summarize in Lemma 9.

Let D be a subset of the ground set of M for which $r(M\backslash D) = r(M)$. From this it follows that no element e of D is an isthmus of $M\backslash E$ for any subset E of $D - e$. Since contractions do not introduce isthmuses, we also have that no element e of D is an isthmus of $M/F\backslash E$ for any disjoint subsets E and F of $D - e$.

Assume D is $\{e_1, e_2, \ldots, e_k\}$. For h with $1 \leq h \leq k$, let D_h be $\{e_1, e_2, \ldots, e_h\}$. Computing $t(M\backslash D; x, y)$ using equations (12) and (13) gives

$$t(M\backslash D; x, y) = t(M\backslash D_{k-1}; x, y) - t(M/e_k\backslash D_{k-1}; x, y) \tag{14}$$

if e_k is not a loop of $M\backslash D_{k-1}$ (or equivalently, of M), and

$$t(M\backslash D; x, y) = y^{-1}\, t(M\backslash D_{k-1}; x, y) \tag{15}$$

if e_k is a loop of M. All the Tutte polynomials that arise upon iteration have the form $t(M/E\backslash D_h; x, y)$ where $E \subseteq D - D_h$; note however that due to equation (13), not every Tutte polynomial of this form necessarily appears in this expansion. Applying equations (12) and (13) to $t(M/E\backslash D_h; x, y)$ gives

$$t(M/E\backslash D_h; x, y) = t(M/E\backslash D_{h-1}; x, y) - t(M/E \cup e_h\backslash D_{h-1}; x, y)$$

if e_h is not a loop of M/E and

$$t(M/E\backslash D_h; x, y) = y^{-1}\, t(M/E\backslash D_{h-1}; x, y)$$

if e_h is a loop of M/E.

To state the effect of applying equations (12) and (13) repeatedly until no deletions remain, we introduce some notation. For a subset E of D and an element e_i in D, let E_i^+ be $\{e_j \mid e_j \in E \text{ and } j > i\}$. Let the functions α_D and β_D on the set of subsets of D be defined as follows. For $E \subseteq D$, let $\beta_D(E)$ be the number of elements e_i in $D - E$ such that e_i is a loop of M/E_i^+. For $E \subseteq D$, let

$$\alpha_D(E) = \begin{cases} 0, & \text{if some } e_j \text{ in } E \text{ is a loop of } M/E_j^+; \\ (-1)^{|E|} y^{-\beta_D(E)}, & \text{otherwise.} \end{cases}$$

Clearly α_D and β_D depend upon M and the linear ordering that is implicit in our listing of the elements $e_1, e_2, \ldots, e_k$ of D, but we suppress this dependence in the notation.

The considerations above lead to Lemma 9, which can be proved by induction using equations (14) and (15) together with the following observations. Let $\alpha'_{D_{k-1}}$ be defined in the same manner as α_D but considering D_{k-1} to be in M/e_k instead of M. It follows that if e_k is not a loop of M, then $\alpha'_{D_{k-1}}(E) = -\alpha_D(E \cup e_k)$ for $E \subseteq D_{k-1}$. Similarly, if e_k is not a loop of M, then $\alpha_{D_{k-1}}(E) = \alpha_D(E)$; if e_k is a loop of M, then $\alpha_{D_{k-1}}(E) = y\, \alpha_D(E)$.

Lemma 9. *Assume that for some subset D of the ground set of a matroid M, we have $r(M\backslash D) = r(M)$. Then*

$$t(M\backslash D; x, y) = \sum_{E \subseteq D} \alpha_D(E)\, t(M/E; x, y).$$

Note that if N is a restriction of M that contains the set D, then the definition of β_D does not depend upon whether D is considered to be in N or in M; the same is true of α_D.

With Lemma 9, we can prove the main result of this section.

Theorem 10. *Assume that M_1 and M_2 are matroids on the ground sets S_1 and S_2, and that conditions (G1)–(G5) hold. Assume that N has rank $k - 1$. Let B be a basis of the* *contraction* $P_N(M_1,M_2)/\big((S_1\cup S_2)-T\big)$. *The Tutte polynomial* $t\big(P_N(M_1,M_2)\backslash T;x,y\big)$ *of the* k*-sum* $P_N(M_1,M_2)\backslash T$ *is given by the following formula.*

$$\begin{aligned} x^{-|B|} &\sum_{E:E\subseteq T-B} \alpha_{T-B}(E)\,(y-1)^{r(T)+r(E)} \\ &\times \sum_{\substack{\text{flats } W \text{ of } N\\ \text{with } E\subseteq W}} \frac{1}{y^{|W|+|E|}\,\chi\big(N/W;(x-1)(y-1)\big)} \\ &\times \left(\sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\frac{y^{|Z|}}{(y-1)^{r(Z)}}t(M_1/Z;x,y)\right) \\ &\times \left(\sum_{\substack{\text{flats } Z \text{ of } N\\ \text{with } W\subseteq Z}} \mu_N(W,Z)\frac{y^{|Z|}}{(y-1)^{r(Z)}}t(M_2/Z;x,y)\right) \end{aligned}$$

Proof. Note that the elements of B are isthmuses of $P_N(M_1,M_2)\backslash(T-B)$, so

$$x^{|B|}\,t\big(P_N(M_1,M_2)\backslash T;x,y\big) = t\big(P_N(M_1,M_2)\backslash(T-B);x,y\big).$$

Also note that $P_N(M_1,M_2)\backslash(T-B)$ and $P_N(M_1,M_2)$ have the same rank, so Lemma 9 can be applied to compute $t\big(P_N(M_1,M_2)\backslash(T-B);x,y\big)$. Using this lemma, we get

$$t\big(P_N(M_1,M_2)\backslash(T-B);x,y\big) = \sum_{E\subseteq T-B}\alpha_{T-B}(E)\,t\big(P_N(M_1,M_2)/E;x,y\big).$$

By part (vi) of Theorem 1, the contraction $P_N(M_1,M_2)/E$ is the generalized parallel connection $P_{N/E}(M_1/E,M_2/E)$. Note that since $E\subseteq T$, the contractions M_1/E and M_2/E satisfy conditions (G1)–(G5). The formula in the theorem now follows by applying Theorem 6 to evaluate the Tutte polynomial of each of these generalized parallel connections $P_{N/E}(M_1/E,M_2/E)$. □

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