"EXERCISES"

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1. Matroid basics

Let E be a finite set. A *matroid* on E is a nonempty collection of subsets of E, called *bases* of the matroid, that satisfies the exchange property:

For any bases B_1, B_2 and $e_1 \in B_1 \backslash B_2$, there is $e_2 \in B_2 \backslash B_1$ such that $\big(B_1 \backslash e_1\big) \cup e_2$ is a basis.

An *independent set* is a subset of a basis, a *dependent set* is a subset of E that is not independent, a *circuit* is a minimal dependent set, the *rank* of a subset of E is the cardinality of any one of its maximal independent subsets, and a *flat* is a subset of E that is maximal for its rank. The *rank* of a matroid is the cardinality of any one of its bases. For any unexplained matroid terms and facts, see Oxley's book [Oxl11].

Exercise 1.

(1) Show that the set of circuits $\mathcal{C}$ of a matroid satisfies the following properties:

  (a) For any distinct $C_1, C_2 \in \mathcal{C}$, we have $C_1 \nsubseteq C_2$.

  (b) For any distinct $C_1, C_2 \in \mathcal{C}$ and $e \in C_1 \cap C_2$, there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \backslash e$.

(2) Show that, if $\mathcal{C}$ is a family of subsets of E satisfying (a) and (b), then the collection of maximal sets not containing any member of $\mathcal{C}$ is the set of bases of a matroid on E.

2. Correlation constants

Let M be a matroid on E, and fix a set of positive weights $w = (w_e)$ on the elements e of E. Randomly pick a basis B of the matroid so that the probability of selecting an individual basis b is proportional to the product of the weights of its elements:

$$\mathbb{P}(\mathrm{B} = b) \ \propto \ \prod_{e \in b} w_e.$$

For any matroid M, we define a nonnegative real number $\alpha(\mathrm{M})$ by

$$\alpha(\mathrm{M}) = \sup \Big\{ \mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \, / \, \mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}) \Big\},$$

where the supremum is over all distinct non-loop non-coloop elements i and j in M and all sets of positive weights w on the elements of M. When every element of M is either a loop or a coloop, we set $\alpha(\mathrm{M}) = 0$.

Exercise 2. Show that the following statements are equivalent:

(1) $\mathbb{P}(i \in \mathrm{B} \mid j \in \mathrm{B}) \leqslant \mathbb{P}(i \in \mathrm{B})$.

(2) $\mathbb{P}(i \notin \mathrm{B} \mid j \notin \mathrm{B}) \leqslant \mathbb{P}(i \notin \mathrm{B})$.

(3) $\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B}) \leqslant \mathbb{P}(i \in \mathrm{B})\, \mathbb{P}(j \in \mathrm{B})$.

(4) $\mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \leqslant \mathbb{P}(i \notin \mathrm{B})\, \mathbb{P}(j \notin \mathrm{B})$.

(5) $\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \leqslant \mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B})$.

In this case, we say that the events $i \in \mathrm{B}$ and $j \in \mathrm{B}$ are *negatively correlated*.

Exercise 3. Let G_1, G_2 be subgroups of a finite group G, and let g be an element of G chosen uniformly at random. Show that the events $\mathrm{g} \in G_1$ and $\mathrm{g} \in G_2$ are positively correlated.

Exercise 4. Let $\mathrm{M}^\perp$ be the dual matroid of M. Show that

$$\alpha(\mathrm{M}) = \alpha(\mathrm{M}^\perp).$$

Exercise 5. Let M be a minor of another matroid N. Show that

$$\alpha(\mathrm{M}) \leqslant \alpha(\mathrm{N}).$$

Exercise 6. Suppose M_1 and M_2 have an element that is neither a loop nor a coloop. Show that

$$\alpha(\mathrm{M}_1 \oplus \mathrm{M}_2) = \max\big\{\alpha(\mathrm{M}_1), \alpha(\mathrm{M}_2), 1\big\}.$$

Exercise 7. Show that either $\alpha(\mathrm{M}) = 0$ or $\alpha(\mathrm{M}) \geqslant \frac{1}{4}$.

Exercise 8. Classify all matroids with $\alpha(\mathrm{M}) = \frac{1}{4}$.

Two elements x, y of a matroid are *parallel* if $\{x\}$, $\{y\}$, and $\{x, y\}$ have rank 1. A *parallel extension* of M is a matroid obtained by adding a new element parallel to a nonloop element of M.

Exercise 9. Let M' be a parallel extension of M. Show that

$$\alpha(\mathrm{M}) \geqslant 1 \Longrightarrow \alpha(\mathrm{M}') = \alpha(\mathrm{M}).$$

Find a matroid M and its parallel extension M' satisfying $\alpha(\mathrm{M}') > \alpha(\mathrm{M})$.

By definition, the *correlation constant* $\alpha_{\mathbb{F}}$ of a field $\mathbb{F}$ is the real number

$$\alpha_{\mathbb{F}} = \sup\Big\{\alpha(\mathrm{M})\Big\},$$

where the supremum is over all matroids M realizable over $\mathbb{F}$. The *correlation constant of matroids*, denoted α_{Mat}, is defined in the same way by taking the supremum over all matroids.

Exercise 10.

(1) Show that $\alpha_{\mathbb{F}}$ is at least 1 for any field $\mathbb{F}$.

(2) How would you show that $\alpha_{\mathbb{F}}$ is finite?

(3) How would you show that α_{Mat} is finite?

Exercise 11. Show that the value of α_{Mat} remains unchanged if we take the supremum only over matroids with constant weights.

Here are some wild guesses on the correlation constant $\alpha_{\mathbb{F}}$. I think these statements are true, but I wasn't brave enough to put (2) and (3) as conjectures in [HSW].

Exercise 12. Prove or disprove:

(1) The correlation constant $\alpha_{\mathbb{F}_2}$ is $\frac{8}{7}$.

(2) The correlation constant $\alpha_{\mathbb{F}}$ of any field $\mathbb{F}$ is $\frac{8}{7}$.

(3) The correlation constant α_{Mat} of matroids is $\frac{8}{7}$.

To prove (1), it is enough to show that, for all finite projective geometry over $\mathbb{F}_2$,

$$\alpha(\mathbb{P}^{d-1}_{\mathbb{F}_2}) \leqslant \frac{8}{7}.$$

Similarly, to prove (2), it is enough to show that, for all d and all q,

$$\alpha(\mathbb{P}^{d-1}_{\mathbb{F}_q}) \leqslant \frac{8}{7}.$$

Here is a more precise guess on finite projective geometries.

Exercise 13. Prove or disprove that

$$\alpha(\mathbb{P}^{d-1}_{\mathbb{F}_q}) = \begin{cases} 9/8 & \text{if } d = 4, \\ 8/7 & \text{if } d \geqslant 5. \end{cases}$$

Since the automorphism group of $\mathbb{P}^{d-1}_{\mathbb{F}_q}$ is doubly transitive, Exercise 13 amounts to finding the global maximum of a single explicit homogeneous rational function on the positive orthant.

3. Correlation in simplicial complexes

Let n be a positive integer, and let S_n be the $(n-1)$-dimensional skeleton of the $(2n-1)$-dimensional simplex. Thus S_2 is the complete graph K_4 and S_3 is a union of two copies of the minimal triangulation of the real projective plane. A *spanning tree* of S_n is a maximal subset of $(n-1)$-dimensional simplices in S_n that does not contain any $(n-1)$-dimensional cycle over $\mathbb{F}_2$.

Exercise 14.

(1) Show that the set of spanning trees of S_n is the set of bases of a matroid.

(2) Find a spanning tree of S_3 that is not contractible.

Let i_n and j_n be any two disjoint maximal simplices in S_n, and let T_n be a uniform random spanning tree of S_n.

Exercise 15.

(1) Show that the events $i_n \in \mathrm{T}_n$ and $j_n \in \mathrm{T}_n$ are negatively correlated for $n = 2$.

(2) Show that the events $i_n \in \mathrm{T}_n$ and $j_n \in \mathrm{T}_n$ are positively correlated for $n = 3$.

Prove or disprove the following statements.

(3) The events $i_n \in \mathrm{T}_n$ and $j_n \in \mathrm{T}_n$ are positively correlated for $n \geqslant 4$.

(4) The events $i_n \in \mathrm{T}_n$ and $j_n \in \mathrm{T}_n$ are eventually independent:

$$\limsup_{n\to\infty} \mathbb{P}(i_n \in \mathrm{T}_n, j_n \in \mathrm{T}_n)\, \mathbb{P}(i_n \notin \mathrm{T}_n, j_n \notin \mathrm{T}_n) \,/\, \mathbb{P}(i_n \in \mathrm{T}_n, j_n \notin \mathrm{T}_n)\, \mathbb{P}(i_n \notin \mathrm{T}_n, j_n \in \mathrm{T}_n) = 1.$$

It feels to me that everything becomes eventually negatively correlated. Can you disprove the following statement? For any sequence of connected matroids M_n of rank and corank at least n, any distinct elements i_n and j_n of M_n, and uniform random basis B_n of M_n, we have

$$\lim_{n\to\infty} \mathbb{P}(i_n \in \mathrm{B}_n, j_n \in \mathrm{B}_n)\, \mathbb{P}(i_n \notin \mathrm{B}_n, j_n \notin \mathrm{B}_n) \,/\, \mathbb{P}(i_n \in \mathrm{B}_n, j_n \notin \mathrm{B}_n)\, \mathbb{P}(i_n \notin \mathrm{B}_n, j_n \in \mathrm{B}_n) \leqslant 1.$$

4. Correlation in spikes

According to Jim Geelen, "it all goes wrong for spikes" [Gee08]:

> *There is something utopian about matroids representable over finite fields. One does not need to go far outside the class before matroid theory reveals its true nature. In fact, all of the horrors are inherent to spikes. From a structural point of view, it is hard to imagine a more benign looking class.*

Exercise 16. A rank d *spike* with tip t is a matroid on the ground set

$$E = \{t, x_1, y_1, \ldots, x_d, y_d\}.$$

A spike has four different types of circuits:

(1) The first collection of circuits $\mathcal{C}_1$ consists of all sets of the form $\{t, x_i, y_i\}$, $1 \leqslant i \leqslant d$.

(2) The second collection of circuits $\mathcal{C}_2$ consists of all sets of the form $\{x_i, y_i, x_j, y_j\}$, $1 \leqslant i < j \leqslant d$.

(3) The third collection of circuits $\mathcal{C}_3$, possibly empty, consists of sets of the form

$$\{z_1, \ldots, z_d \mid z_i \text{ is either } x_i \text{ or } y_i, \text{ for all } 1 \leqslant i \leqslant d\}.$$

It is required that no two members of $\mathcal{C}_3$ share $d-1$ elements.

(4) The fourth collection of circuits $\mathcal{C}_4$ consists of all $(d+1)$-element subsets of E that contain no member of $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$.

Show that $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ is the set of circuits of a rank d matroid on E.

Definition.

(1) A matroid M is *negatively correlated* if, for a basis B of M chosen uniformly at random and for any distinct elements i and j of M,

$$\mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}) - \mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \geqslant 0.$$

A matroid M is *balanced* if M and all its minors are negatively correlated.

(2) A matroid M is *Rayleigh* if, for any set of positive weights $w = (w_e)$, a basis B of M chosen randomly according to the distribution

$$\mathbb{P}(\mathrm{B} = b) \propto \prod_{e \in b} w_e,$$

and any distinct elements i and j of M,

$$\mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}) - \mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \geqslant 0.$$

Exercise 17.

(1) Prove that a spike is negatively correlated.

(2) Prove that a spike need not be balanced.

Exercise 18. Let e be an element of a spike S, and let B be a basis of $\mathrm{S}\backslash e$ chosen uniformly at random.

(1) Show that, for any distinct elements i and j of $\mathrm{S}\backslash e$,

$$\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \,/\, \mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}) \leqslant \frac{8}{7}.$$

(2) Show that the equality is achieved by exactly one matroid of the form $\mathrm{S}\backslash e$, where S is a spike.

(3) Show that the spike S from (2) is binary.

Welsh conjectured that the class of balanced matroids have infinitely many excluded minors [Oxl11, Conjecture 15.8.1].

Exercise 19. Show that there are only finitely many excluded minors for the class of balanced matroids that are of the form $\mathrm{S}\backslash e$, where S is a spike. Can you estimate how many there are?

5. Correlation in paving matroids

Exercise 20. Let $\mathcal{C}$ be a collection of d-element subsets of E satisfying the following condition:

> If the symmetric difference of $C_1, C_2 \in \mathcal{C}$ has cardinality 2, then every d-element subset of $C_1 \cup C_2$ is in $\mathcal{C}$.

Let $\mathcal{C}'$ be the collection of $(d+1)$-element subsets of E that contain no member of $\mathcal{C}$. Show that $\mathcal{C} \cup \mathcal{C}'$ is the set of circuits of a rank d matroid on E.

Matroids of the above kind are called *paving*. Mayhew, Newman, Welsh, and Whittle conjectured that, asymptotically, almost every matroid is paving [Oxl11, Conjecture 15.5.8].

Definition. A *Steiner system* $S(r,k,n)$ is a collection of k-element subsets of an n-element set E, called *blocks* of the Steiner system, with the following property:

Every r-element subset of E is contained in exactly one block.

We fix a Steiner system $S(r,k,n)$, and let $d = r+1$.

Exercise 21. Let $\mathcal{C}$ be the collection of d-element subsets of E contained in some block of $S(r,k,n)$, and let $\mathcal{C}'$ be the collection of $(d+1)$-element subsets of E that contain no member of $\mathcal{C}$. Show that $\mathcal{C} \cup \mathcal{C}'$ is the set of circuits of a rank d paving matroid on E.

We abuse notation and write the above paving matroid by $S(r,k,n)$.

Exercise 22. Show that the paving matroid $S(r,k,n)$ is negatively correlated.

Let i and j be distinct elements of E, and write the set of blocks $\mathbb{V}$ as the disjoint union

$$\mathbb{V} = \mathbb{V}_{ij} \cup \mathbb{V}^{ij} \cup \mathbb{V}^i_j \cup \mathbb{V}^j_i.$$

Here $\mathbb{V}_{ij}$ is the set of blocks containing i and containing j, $\mathbb{V}^{ij}$ is the set of blocks not containing i and not containing j, $\mathbb{V}^i_j$ is the set of blocks not containing i and containing j, and $\mathbb{V}^j_i$ is the set of blocks containing i and not containing j.

Exercise 23. Let $\mathcal{C}$ be the collection of d-element subsets of E that is contained in a block in $\mathbb{V}^i_j \cup \mathbb{V}^j_i$, and let $\mathcal{C}'$ be the collection of $(d+1)$-element subsets of E that contain no member of $\mathcal{C}$. Show that $\mathcal{C} \cup \mathcal{C}'$ is the set of circuits of a rank d paving matroid on E.

We abuse notation and write the above paving matroid by $S(r,k,n,i,j)$.

Exercise 24.

(1) Show that the numbers of bases of $S(r,k,n,i,j)$ containing i and containing j is

$$|\mathbb{B}_{ij}| = \binom{n-2}{r-1}.$$

(2) Show that the numbers of bases of $S(r,k,n,i,j)$ not containing i and not containing j is

$$|\mathbb{B}^{ij}| = \binom{n-2}{r+1} - 2\binom{k-1}{r+1}\left[\frac{\binom{n-1}{r-1}}{\binom{k-1}{r-1}} - \frac{\binom{n-2}{r-2}}{\binom{k-2}{r-2}}\right].$$

(3) Show that the numbers of bases of $S(r,k,n,i,j)$ not containing i and containing j is

$$|\mathbb{B}^j_i| = \binom{n-2}{r} - \binom{k-1}{r}\left[\frac{\binom{n-1}{r-1}}{\binom{k-1}{r-1}} - \frac{\binom{n-2}{r-2}}{\binom{k-2}{r-2}}\right].$$

By Peter Keevash's big theorem, we know that $S(r,k,n)$ exists if and only if

$$\binom{k-m}{r-m} \text{ divides } \binom{n-m}{r-m} \text{ for all nonnegative } m \leqslant r.$$

Call parameters (r,k,n) *valid* if it satisfies the above divisibility condition.

Exercise 25. Find valid parameters (r, k, n) such that $S(r, k, n, i, j)$ is not negatively correlated. How many did you find? How large is the ratio $|\mathbb{B}_{ij}||\mathbb{B}^{ij}|/|\mathbb{B}_j^i||\mathbb{B}_i^j|$?

Here is an easy construction of paving matroids related to the above example. For $r < n$, let H be any hypergraph on the vertex set $[n]$ such that all pairwise intersections of hyperedges have cardinality less than r. Let $d = r + 1$.

Exercise 26. Show that the collection of d-element subsets of $[n]$ that is not contained in any hyperedge is the set of bases of a matroid.

This construction may be useful in proving Welsh's conjecture on excluded minors for balanced matroids.

6. Correlation in sparse paving matroids

Exercise 27. Let $\mathcal{C}$ be a collection of d-element subsets of E satisfying the following condition:

> The symmetric difference of $C_1, C_2 \in \mathcal{C}$ does not have cardinality 2 for any $C_1, C_2 \in \mathcal{C}$.

Let $\mathcal{C}'$ be the collection of $(d + 1)$-element subsets of E that contain no member of $\mathcal{C}$. Show that $\mathcal{C} \cup \mathcal{C}'$ is the set of circuits of a rank d matroid on E.

Matroids of the above kind are called *sparse paving*. It has been conjectured that, asymptotically, almost every matroid is sparse paving. This is not stronger than the previously introduced conjecture on paving matroids, by the following result.

Exercise 28. A paving matroid is sparse paving if and only if its dual is paving.

Mark Jerrum proved that every sparse paving matroid is balanced [Jer06]. Is every sparse paving matroid Rayleigh? A proof is given in [Eri08, Theorem 4.2.1]. The proof actually claims something stronger:

> For a sparse paving matroid, the coefficients of the polynomial $|\mathbb{B}_j^i(w)||\mathbb{B}_i^j(w)| - |\mathbb{B}_{ij}(w)||\mathbb{B}^{ij}(w)|$ are nonnegative.

Benjamin Schröter noticed that something's strange.

Exercise 29. Check that the graphic matroid of the complete graph K_4 is sparse paving. Is the stronger claim in [Eri08] true for K_4?

Exercise 30. Prove or disprove that all sparse paving matroids are Rayleigh.

7. ENTROPY OF SIMPLICIAL COMPLEXES

Recall that the *Shannon entropy* $H(\mathrm{X})$ of a discrete random variable X is, by definition,

$$H(\mathrm{X}) = -\sum_k \mathbb{P}(\mathrm{X}=k)\log \mathbb{P}(\mathrm{X}=k),$$

where the logarithm is in base 2 and the sum is over all values of X with nonzero probability. For a simplicial complex Δ, let I_Δ be the dimension of a face drawn uniformly at random from the collection of all faces of Δ. Kruskal-Katona theorem gives a complete numerical characterization of f-vectors of simplicial complexes.[1]

Exercise 31. State the Kruskal-Katona theorem.

The following statements feel true to me (do you agree?), but I could not prove them.

Exercise 32. Prove or disprove the following statements:

(1) For any $\epsilon > 0$, there is a simplicial complex Δ of dimension d such that

$$(1-\epsilon)\log d \leqslant H(\mathrm{I}_\Delta) \leqslant \log d.$$

(2) For any $\epsilon > 0$, there is a pure simplicial complex Δ of dimension d such that

$$(1-\epsilon)\log d \leqslant H(\mathrm{I}_\Delta) \leqslant \log d.$$

8. SUBMODULAR FUNCTIONS

A real-valued function c on $2^{[n]}$ is said to be *strictly submodular* if, for any pair of incomparable subsets $I_1, I_2 \subseteq [n]$, we have

$$c_{I_1} + c_{I_2} > c_{I_1 \cap I_2} + c_{I_1 \cup I_2}.$$

The function c is said to be *submodular* if the equality is allowed in the displayed inequality. The main theorem of Hodge theory for matroids is a statement about strictly submodular functions.

Exercise 33.

(1) For every n, construct an explicit strictly submodular function on $2^{[n]}$.

(2) How many independent S_n-invariant strictly submodular functions on $2^{[n]}$ can you find?

A submodular function is *symmetric* if $c_I = c_{[n]\setminus I}$ for all I.

Exercise 34. For every n, construct an explicit symmetric strictly submodular function on $2^{[n]}$.

A submodular function is *strictly increasing* if $c_I < c_J$ for all $I \subsetneq J$.

Exercise 35. For every n, construct an explicit strictly increasing strictly submodular function on $2^{[n]}$.

[1] According to Ziegler, a similar characterization of f-vectors of pure simplicial complexes is "probably impossible" [Zie95, Exercise 8.16].

A submodular function is *strictly decreasing* if $c_I > c_J$ for all $I \subsetneq J$.

Exercise 36. For every n, construct an explicit strictly decreasing strictly submodular function on $2^{[n]}$.

Exercise 37. Let G be a graph with the vertex set $[n]$, and define the cut function of G by

$$c_I = (\text{the number of edges joining a vertex in } I \text{ and not in } I).$$

(1) Show that the cut function is a submodular function on $2^{[n]}$.

(2) When is the cut function strictly submodular?

(3) Define a hypergraph analog of the cut function.

Exercise 38. Let $G_1, G_2, \ldots, G_n$ be subgroups of a finite group G, and let g be an element of G chosen uniformly at random. Show that the function

$$c_I = -\log \mathbb{P}(\mathrm{g} \in G_i \text{ for all } i \in I)$$

is a submodular function on $2^{[n]}$.

Show that the above statement is a special case of the next result, due to Claude Shannon.

Exercise 39. Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ be any sequence of discrete random variables on a probability space Ω. For a finite subset $I \subseteq [n]$, write X_I for the random variable $(\mathrm{X}_i)_{i \in I}$. Show that the entropy

$$c_I = H(\mathrm{X}_I)$$

is a submodular function on $2^{[n]}$.

I think the joint entropy function tends to be strictly submodular, if there are dependencies between the random variables. Can you make this precise?

9. The Chow ring of a matroid and its variation

9.1. Let M be a loopless matroid on E of rank $d+1$, and let $\mathscr{L}$ be the lattice of flats of M. Introduce variables x_F, one for each nonempty proper flat F of M, and consider the polynomial ring

$$S(\mathrm{M}) = \mathbb{R}[x_F]_{F \neq \varnothing, F \neq E, F \in \mathscr{L}}.$$

The *Chow ring* $A(\mathrm{M})$ is the quotient of $S(\mathrm{M})$ by the ideal generated by the linear forms

$$\sum_{e_1 \in F} x_F - \sum_{e_2 \in F} x_F,$$

one for each pair of distinct elements e_1 and e_2 of E, and the quadratic monomials

$$x_{F_1} x_{F_2},$$

one for each pair of incomparable nonempty proper flats F_1 and F_2 of M. We denote the degree q component of $A(\mathrm{M})$ by $A^q(\mathrm{M})$.

We may now state the hard Lefschetz theorem and the Hodge-Riemann relations for matroids. The function "deg" in Theorem 9.1 is the unique isomorphism $A^d(\mathrm{M}) \simeq \mathbb{R}$ with the property

$$\deg(x_{F_1}x_{F_2}\cdots x_{F_d}) = 1 \text{ for any chain of nonempty proper flats } F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d \text{ in } \mathrm{M}.$$

Exercise 40.

(1) Show that the squarefree monomials of the form $x_{F_1}x_{F_2}\cdots x_{F_d}$ span $A^d(\mathrm{M})$.

(2) Show that any two squarefree monomials of the form $x_{F_1}x_{F_2}\cdots x_{F_d}$ are equal in $A^d(\mathrm{M})$.

(3) Show that any squarefree monomial of the form $x_{F_1}x_{F_2}\cdots x_{F_d}$ is nonzero in $A^d(\mathrm{M})$.

Theorem 9.1. Let c be a strictly submodular function on 2^E satisfying $c_\varnothing = c_E = 0$, and let L be the element

$$\mathrm{L} = \sum_F c_F x_F \in A^1(\mathrm{M}),$$

where the sum is over all nonempty proper flats of M.

(1) (Poincaré duality) For every nonnegative integer $q \leqslant \frac{d}{2}$, we have a nondegenerate pairing

$$A^q(\mathrm{M}) \times A^{d-q}(\mathrm{M}) \longrightarrow \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto \deg(\eta_1\, \eta_2).$$

(2) (Hard Lefschetz theorem) For every nonnegative integer $q \leqslant \frac{d}{2}$, the multiplication by L defines an isomorphism

$$A^q(\mathrm{M}) \longrightarrow A^{d-q}(\mathrm{M}), \qquad \eta \longmapsto \mathrm{L}^{d-2q}\, \eta.$$

(3) (Hodge-Riemann relations) For every nonnegative integer $q \leqslant \frac{d}{2}$, the multiplication by L defines a symmetric bilinear form

$$A^q(\mathrm{M}) \times A^q(\mathrm{M}) \longrightarrow \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^q \deg(\eta_1\eta_2\mathrm{L}^{d-2q})$$

that is positive definite on the kernel of L^{d-2q+1}.

Exercise 41.

(1) Prove the above theorem when $d = 2$.

(2) Prove the above theorem when $d = 3$.

9.2. Let N be a loopless matroid on E of rank d, and let $\mathscr{L}$ be the lattice of flats of N. Introduce variables x_F, one for each proper flat F of N, and consider the polynomial ring

$$\tilde{S}(\mathrm{N}) = \mathbb{R}[x_F]_{F \neq E, F \in \mathscr{L}}.$$

The *Chow ring* $\tilde{A}(\mathrm{N})$ is the quotient of $\tilde{S}(\mathrm{N})$ by the ideal generated by the quadratic monomials

$$x_{F_1}x_{F_2},$$

one for each pair of incomparable proper flats F_1 and F_2 of N, and the quadratic forms

$$y_i x_F, \quad y_i = (\text{the sum of all } x_G \text{ for proper flats } G \text{ not containing } i),$$

for every element i and every proper flat F not containing i. The function "deg" in Theorem 9.2 is the unique isomorphism $\tilde{A}^d(\mathrm{N}) \simeq \mathbb{R}$ with the property

$$\deg(x_\varnothing x_{F_1} \cdots x_{F_d}) = 1 \quad \text{for any chain of nonempty proper flats } \varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_d \text{ in N.}$$

Exercise 42.

(1) Show that the squarefree monomials of the form $x_\varnothing x_{F_1} \cdots x_{F_d}$ span $\tilde{A}^d(\mathrm{N})$.

(2) Show that any two squarefree monomials of the form $x_\varnothing x_{F_1} \cdots x_{F_d}$ are equal in $\tilde{A}^d(\mathrm{M})$.

(3) Show that any squarefree monomial of the form $x_\varnothing x_{F_1} \cdots x_{F_d}$ is nonzero in $\tilde{A}^d(\mathrm{M})$.

Theorem 9.2. Let c be a strictly decreasing strictly submodular function on 2^E satisfying $c_E = 0$, and let L be the element

$$\mathrm{L} = \sum_F c_F x_F \in \tilde{A}^1(\mathrm{N}),$$

where the sum is over all proper flats of N.

(1) (Poincaré duality) For every nonnegative integer $q \leqslant \frac{d}{2}$, we have a nondegenerate pairing

$$\tilde{A}^q(\mathrm{N}) \times \tilde{A}^{d-q}(\mathrm{N}) \longrightarrow \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto \deg(\eta_1\, \eta_2).$$

(2) (Hard Lefschetz theorem) For every nonnegative integer $q \leqslant \frac{d}{2}$, the multiplication by L defines an isomorphism

$$\tilde{A}^q(\mathrm{N}) \longrightarrow \tilde{A}^{d-q}(\mathrm{N}), \qquad \eta \longmapsto \mathrm{L}^{d-2q}\, \eta.$$

(3) (Hodge-Riemann relations) For every nonnegative integer $q \leqslant \frac{d}{2}$, the multiplication by L defines a symmetric bilinear form

$$\tilde{A}^q(\mathrm{N}) \times \tilde{A}^q(\mathrm{N}) \longrightarrow \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^q \deg(\eta_1 \eta_2 \mathrm{L}^{d-2q})$$

that is positive definite on the kernel of L^{d-2q+1}.

Exercise 43.

(1) Prove the above theorem when $d = 2$.

(2) Prove the above theorem when $d = 3$.

One beautiful property of the Chow ring $\tilde{A}(\mathrm{N})$ is that it contains the *graded Möbius algebra* of the lattice of flats $\mathscr{L} = \mathscr{L}(\mathrm{N})$. Introduce symbols y_F, one for each flat F of N, and construct vector spaces

$$B^p(\mathrm{N}) = \bigoplus_{F \in \mathscr{L}^p} \mathbb{Q}\, y_F, \quad B^*(\mathrm{N}) = \bigoplus_{F \in \mathscr{L}} \mathbb{Q}\, y_F.$$

We equip $B^*(\mathrm{N})$ with the structure of a commutative graded algebra over $\mathbb{Q}$ by setting

$$y_{F_1} y_{F_2} = \begin{cases} y_{F_1 \vee F_2} & \text{if } \mathrm{rank}(F_1) + \mathrm{rank}(F_2) = \mathrm{rank}(F_1 \vee F_2), \\ 0 & \text{if } \mathrm{rank}(F_1) + \mathrm{rank}(F_2) > \mathrm{rank}(F_1 \vee F_2). \end{cases}$$

For simplicity, we write y_i instead of $y_{\{i\}}$.

Exercise 44. There is a unique graded algebra homomorphism

$$\varphi : B(\mathrm{N}) \longrightarrow \tilde{A}(\mathrm{N}), \qquad y_F \longmapsto \prod_{i \in B_F} y_i,$$

where B_F is any maximal independent set of N in F.

(1) Show that φ does not depend on the choice of bases B_F of F.

(2) Show that φ is a graded algebra homomorphism.

(3) Show that φ is injective.

(4) Show that $\deg \varphi(y_E) = 1$.

10. Volume polynomials

Let $b(w)$ be a homogeneous degree d polynomial in the variables $w = (w_1, \ldots, w_n)$ with positive coefficients. We write

$$b(w+1) = b_d(w) + \cdots + b_2(w) + b_1(w) + b_0(w),$$

where $b_k(w)$ is homogeneous of degree k.

Exercise 45. Are there implications between the following statements?

(1) The polynomial $b_2(w)$ is stable.

(2) The function $\log b(w)$ is concave on the positive orthant.

(3) The function $b(w)^{1/d}$ is concave on the positive orthant.

Exercise 46. Can we explicitly write down the volume polynomial for the Chow ring $\tilde{A}(\mathrm{N})$?

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