Nordfjordeid Summer School 2018: Combinatorics and Hodge Theory
The algebraic geometry of Kazhdan-Lusztig-Stanley polynomials

Lecture 1: The combinatorics of Kazhdan-Lusztig-Stanley polynomials

Exercises

Let P be a ranked poset. Recall that we defined the **characteristic function** $\chi := \zeta^{-1}\bar{\zeta} \in I(P)$, where $\zeta_{xy}(t) := 1$ and thus $\bar{\zeta}_{xy}(t) = t^{r_{xy}}$ for all $x \leq y$. Let $f \in I(P)$ the the **right KLS-function** associated with χ, which is characterized by the following properties:

- $f_{xx}(t) = 1$ for all $x \in P$
- $\deg f_{xy}(t) < r_{xy}/2$ for all $x < y \in P$
- $\bar{f} = \chi f$, i.e. $t^{r_{xz}} f_{xz}(t^{-1}) = \sum_{x \leq y \leq z} \chi_{xy}(t) f_{yz}(t)$ for all $x \leq z \in P$.

1. Suppose that P has a unique minimal element 0 (of rank 0) and a unique maximal element 1 (of rank d). Let W_i be the number of elements of P of rank i.

a) Show that $\chi_{01}(t) = t^d - W_1 t^{d-1} +$ lower order terms.
Hint: First show that $\zeta^{-1}_{0x}(t) = -1$ for every rank 1 element $x \in P$.

b) Show that $f_{01}(t) = 1 + (W_{d-1} - W_1)t +$ higher order terms.

c) Try to find formulas for the coefficient of t^{d-2} in $\chi_{01}(t)$ and the coefficient of t^2 in $f_{01}(t)$. Your answers should involve the numbers

$$W_{ij} := |\{(x, y) \in P^2 \mid x \leq y, \operatorname{rk} x = i, \operatorname{rk} y = j\}|.$$

(A solution will be given in Lecture 5.)

2. Let P_n be the poset whose elements are subsets of $[n]$ of cardinality not equal to $n-1$, with $\operatorname{rk} S = |S|$ for all proper subsets and $\operatorname{rk}[n] = n - 1$.

a) Show that, for every proper subset $S \subsetneq [n]$, $\chi_{\emptyset S}(t) = (t-1)^{|S|}$.

b) Show that $\chi_{\emptyset[n]}(t) = \dfrac{(t-1)^n + (-1)^n(t-1)}{t}$.

3. Let Π_n be the poset of partitions of n, ordered by refinement, where the rank of a partition is equal to n minus the number of parts. Thus the minimal element 0 is the partition into n singletons, which has rank 0, and the maximal element 1 is the partition into a single part, which has rank $n-1$.

a) Compute $f_{01}(t)$ for small n. Hint: For $n \leq 5$, you can get the answer using Problem 1(b). For $n \leq 7$, you can get the answer using Problem 1(c).

b) Show that $\chi_{01}(t) = (t-1)(t-2)\cdots(t-(n-1))$.

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Lecture 2: Introduction to intersection cohomology

Exercise

Let X be a variety over $\mathbb{F}_q$. Let $P_X(t) := \sum \dim IH^{2i}(X)t^i$, and for all $p \in X$, let $P_{X,p}(t) := \sum \dim IH^{2i}(\mathrm{IC}_{X,p})t^i$. Recall that, assuming that all cohomolgy groups are **chaste** (meaning that they vanish in odd degree and the Frobenius automorphism acts on as scalar multiplication by q^i on the degree $2i$ part), the Grothendieck-Lefschetz trace formula and Poincaré duality combine to tell us that

$$q^{\dim X} P_X(q^{-1}) = \sum_{p \in X} P_{X,p}(q).$$

Recall also that, if X is an affine cone with cone point $0 \in X$, then $P_{X,0}(t) = P_X(t)$, and this polynomial has degree strictly less than the dimension of X (unless X is a point).

1. Fix a prime power q, and let X_n be the variety of $n \times n$ nilpotent matrices over $\mathbb{F}_q$ with rank at most 1. This is a singular variety with a stratification into two pieces, namely the nilpotent matrices of rank exactly 1 (which admits a transitive action of $\mathrm{GL}_n(\mathbb{F}_q)$ and is therefore smooth) and the zero matrix.

a) Compute $|X_n \smallsetminus \{0\}|$. Hint: The image and the kernel determine the matrix up to scale.

b) Use part (a) to compute the Poincaré polynomial for the intersection cohomology of X_n. (You may assume that everything is chaste.)

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Lecture 3: The main theorem

Exercises

1. Let $C = \operatorname{Spec} R$ be an affine variety.
a) Show that an action of $\mathbb{G}_m$ on C is equivalent to a $\mathbb{Z}$-grading on R.
b) Show that this action contracts C to a single point if and only if R is generated in positive degrees.

2. Recall that we defined $V_n := \{p \in \mathbb{A}^n \mid p_1 + \cdots + p_n = 0\}$, and we defined Y_n to be the closure of V_n inside of $(\mathbb{P}^1)^n$. We then computed the local intersection cohomology Poincaré polynomial of Y_n at its most singular point $(\infty, \ldots, \infty)$. Compute the **global** intersection cohomology Poincaré polynomial of Y_n. (We will come back to this in Lecture 5.)

3. **For people familiar with toric geometry.** Let T be a split algebraic torus over $\mathbb{F}_q$ with cocharacter lattice N and let Σ be a rational fan in $N_\mathbb{R}$ containing the zero cone. We order Σ by reverse inclusion and equip it with the rank function given by codimension. Thus the maximal element is 0, and its rank is equal to the dimension of T.

Let Y be the T-toric variety associated with Σ. The cones of Σ are in bijection with T-orbits in Y and with T-invariant affine open subsets of Y. Given $\sigma \in \Sigma$, let V_σ denote the corresponding orbit, let W_σ denote the corresponding affine open subset, and let $T_\sigma \subset T$ be the stabilizer of V_σ. We then have $\dim V_\sigma = \operatorname{codim} \sigma$, and

$$\sigma \leq \tau \iff \bar{V}_\sigma \subset V_\tau \iff W_\sigma \supset W_\tau \iff W_\sigma \supset V_\tau.$$

For each $\sigma \in \Sigma$, we have a canonical identification $V_\sigma \cong T/T_\sigma$, and we define $e_\sigma \in V_\sigma$ to be the identity element of T/T_σ. In particular, we have $T_\sigma \subset T \cong V_0 \subset Y$ for all σ, and we define

$$C_\sigma := W_\sigma \cap \bar{T}_\sigma.$$

The character lattice of T_σ is equal to $N_\sigma := N \cap \mathbb{R}\sigma$, C_σ is isomorphic to the T_σ-toric variety associated with the cone $\sigma \subset N_{\sigma,\mathbb{R}}$, and $e_\sigma \in C_\sigma$ is the unique fixed point. If $\sigma \leq \tau$, then $U_{\sigma\tau} := C_\sigma \cap V_\tau$ is equal to the T_σ-orbit in C_σ corresponding to the face τ of σ.

a) For each $\sigma \in \Sigma$, find a homomorphism $\rho_\sigma : \mathbb{G}_m \to T \subset \operatorname{Aut}(Y)$ that contracts C_σ to e_σ.
b) For each $\sigma \in \Sigma$, find a subgroup $G_\sigma \subset T$ such that the action map $G_\sigma \times C_\sigma \to Y$ is an open immersion.
c) For all $\sigma \leq \tau \in \Sigma$, compute $|C_\sigma \cap V_\tau(\mathbb{F}_{q^s})|$, and show that it is equal to $\kappa_{\sigma\tau}(q)$ for a certain (very simple!) polynomial $\kappa_{\sigma\tau}(t)$. Our main theorem says that $\kappa \in I(\Sigma)$ is a Σ-kernel. Prove directly that κ is a Σ-kernel without all of the geometry.

If Σ is equal to the cone over a polytope Δ along with all of its faces, then the largest right KLS-function associated with κ (the intersection cohomology Poincaré polynomial of Y) is known as the ***g*-polynomial** of Δ. Note that this makes sense even if Σ is not rational, because we can still define κ and show that it is a Σ-kernel.

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Lecture 4: Hyperplane arrangements

Exercises

1. Fix a prime power q and a positive integer n, and consider the vector space $V = \mathbb{F}_q^n$. Let $\mathcal{P}_{q,n}$ be the hyperplane arrangement consisting of **all** hyperplanes in V. Prove that $\chi_{\mathcal{P}_{q,n}}(t) = (t-q)(t-q^2)\cdots(t-q^n)$.

Hint: First prove that q is a root. With some more work, you can prove that q^m is a root for all $m \leq n$. Then use Problem 1(a) from the first set of exercises.

2. Let $\mathcal{B}_n$ be the arrangement in k^n/k consisting of the $\binom{n}{2}$ hyperplanes $\{x_i = x_j\}$. Recall that flats of $\mathcal{B}_n$ are in bijection with partitions of the set $[n]$. Let $P = (P_1, \ldots, P_m)$ be such a partition into m parts, and let F be the associated flat.

a) Show that the contraction $(\mathcal{B}_n)_F$ is isomorphic to $\mathcal{B}_m$.
b) Show that the localization $\mathcal{B}_n^F$ is isomorphic to $\mathcal{B}_{|P_1|} \times \cdots \times \mathcal{B}_{|P_m|}$.

3. Let $\mathcal{A}$ be a hyperplane arrangement in V such that the intersection of all the hyperplanes is $\{0\}$. Let $U_{\mathcal{A}} := V \smallsetminus \bigcup_{H \in \mathcal{A}} H$ be the complement of $\mathcal{A}$. Recall that the we have a natural injection

$$V \hookrightarrow \prod_{H \in \mathcal{A}} V/H \cong \prod_{H \in \mathcal{A}} \mathbb{A}^1,$$

and we define $Y_{\mathcal{A}}$ to be the closure of V inside of $\prod_{H \in \mathcal{A}} \mathbb{P}^1$. For any flat $F \subset V$, we made the following definitions:

$$(e_F)_H := \begin{cases} 0 & \text{if } F \subset H \\ \infty & \text{if } F \not\subset H \end{cases} \qquad V_F := \{p \in Y_{\mathcal{A}} \mid p_H = \infty \iff F \not\subset H\}$$
$$C_F := \{p \in Y_{\mathcal{A}} \mid p_H = 0 \iff F \subset H\}.$$

Here's what we still need to show in order to apply our main theorem and conclude that $f_{\mathcal{A}}(t) = P_{Y_{\mathcal{A}}, e_V}(t)$. (I think that parts (a) and (e) are the most satisfying to work through.)

a) $e_F \in Y_{\mathcal{A}}$. Hint: Choose a generic element of F and "let it run off to infinity".
b) $V_F = V + e_F$ (it is clear that the RHS is contained in the LHS).
c) $Y_{\mathcal{A}} = \coprod_F V_F$ (it is clear that the RHS is contained in the LHS).
d) For all pairs of flats $F \leq G$ (equivalently $G \subset F$), we have $C_F \cap V_G \cong U_{\mathcal{A}_F^G}$.
Hint: First show that $\overline{C_F} \cap \overline{V_G} \cong Y_{\mathcal{A}_F^G}$ and use this to reduce to the case $G = \{0\}$ and $F = V$.
e) By part (b), we have an isomorphism $V_F \cong V/\operatorname{Stab}(e_F) = V/F$. Fix a section $V_F \to V$ of the projection, and show that the map $V_F \times C_F \to Y_{\mathcal{A}}$ taking (v, p) to $v + p$ is an open immersion.
Hint: By dimension count, it's enough to prove injectivity. That is, if $v + p = v' + p'$, we need to show that $v = v'$. Since v and v' lie in a subspace complementary to $F \subset V$, it is enough to show that v and v' have the same image in V/F, which is equivalent to the statement that $v + e_F = v' + e_F$.

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Lecture 5: The Z-polynomial

Exercises

1. Let P be a ranked poset. Let $\kappa \in I(P)$ be a P-kernel, and let $f, g, Z \in I(P)$ be the associated right KLS-function, left KLS-function, and Z-function. Recall that this means the following:

- $f_{xx}(t) = 1 = g_{xx}(t)$ for all $x \in P$,
- $\deg f_{xy}(t)$ and $\deg g_{xy}(t)$ are both strictly less than $r_{xy}/2$ for all $x < y \in P$,
- $\bar{f} = \kappa f$ and $\bar{g} = g\kappa$,
- $Z = g\kappa f = g\bar{f} = \bar{g}f$.

Show that, if you know Z, you can compute f, g, and κ recursively.

2. Let $\mathcal{A}$ be a hyperplane arrangement in a vector space V of dimension d. Let $c_{\mathcal{A}}(i)$ be the coefficient of t^i in $f_{\mathcal{A}}(t)$. For any increasing sequence $k_1 \leq \cdots \leq k_i$, let

$$D_{k_1 \cdots k_i} := \left| \left\{ (F_1, \ldots, F_i) \;\middle|\; F_1 \subset \cdots \subset F_i \text{ and } \dim F_j = k_j \text{ for all } j \right\} \right|.$$

Recall that we proved that

$$c_{\mathcal{A}}(i) = \sum_{F} c_{\mathcal{A}_F}(\dim F - i) - \sum_{F \neq V} c_{\mathcal{A}^F}(i - \operatorname{codim} F).$$

We used this to show that $c_{\mathcal{A}}(1) = D_1 - D_{d-1}$ and $c_{\mathcal{A}}(2) = D_2 + D_{13} - D_{23} - D_{d-2} - D_{1(d-2)} + D_{2(d-2)}$. Find a similar formula for $c_{\mathcal{A}}(3)$.

Hint: Your formula should have 18 terms, half of which are positive and half of which are negative. Each term should be a D number with at most 3 indices.