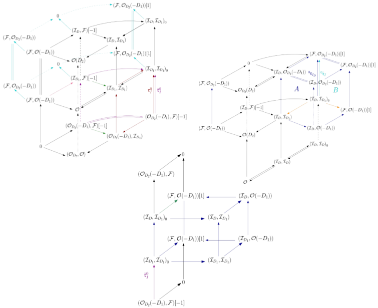
Wall-crossing for holomorphic Donaldson invariants and applications

Arkadij Bojko

Academia Sinica, Institute of Mathematics

December 12, 2023

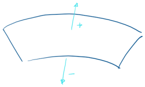


Structure of the talk.

1. The flavour and spice of Calabi–Yau fourfolds
2. Construction of families of vertex algebras in geometry
3. wall-crossing with insertions
   3.1 The main statement.
   3.2 Are the invariants well defined?
   3.3 Which stability conditions do I use?
4. Application to DT/PT wall-crossing for 3-folds
   4.1 Wall-crossing between PT^0 and PT^1 as an example.
   4.2 Simplifying assumptions.
   4.3 Elliptic fibration over a 3-fold.
5. Further applications coming eventually.

The flavour and spice of Calabi–Yau fourfolds - orientations

1. Existence of orientations is necessary to define invariants and changing



orientations introduces a sign.

[1] More generally compactly supported perfect complexes.

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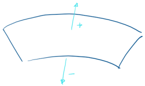

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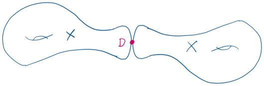

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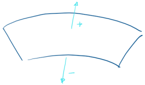

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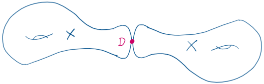
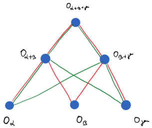


3. Wall-crossing is expressed in terms of taking direct sums of sheaves. So, comparing orientations under direct sums is needed.



Here α, β, γ some topological data and comparison gives signs $\epsilon_{\alpha,\beta}$.

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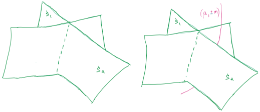
The flavour and spice of Calabi–Yau fourfolds - surfaces and obstruction theories

1. Increasing the dimension leads to a larger freedom of the dimension of support of sheaves. This makes the interplay between virtual dimension and insertions richer and offers more playground with stability conditions.



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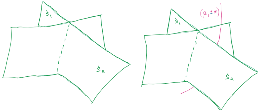
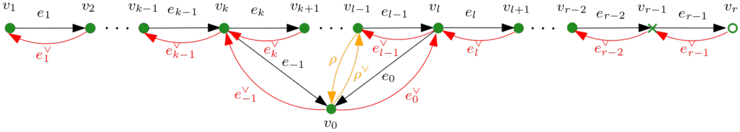


2. Since the obstruction theories are not of Behrend–Fantechi type, I need to find a new way of obtaining self-dual obstruction theories on enhanced master spaces.



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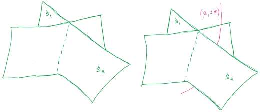

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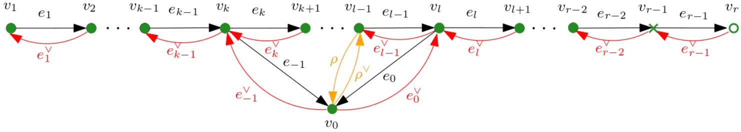
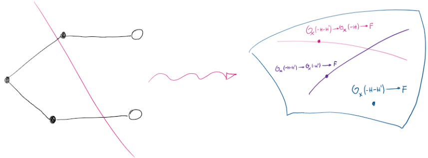


3. The proof that the invariants counting semistable sheaves are well-defined needs to be direct.



Definition of families of vertex algebras.

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for $\deg(u) = 0, \deg(z) = -2$ extending to a (u)-adic continuous $\mathbb{Q}[\![u]\!]$-linear map

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It must additionally induce

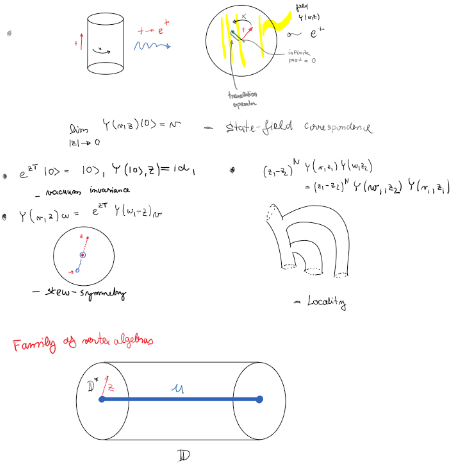
$$Y_u(v, z) = \sum_{n\in\mathbb{Z}} v_{u,n} z^{-n-1} : V_\bullet \to V_\bullet(\!(z)\!)[\![u]\!]\,,$$

for each $v \in V_\bullet \subset V_\bullet[\![u]\!]$ and

$$v_{u,n} : V_\bullet \to V_\bullet[\![u]\!]$$

linear for each $n \in \mathbb{Z}$.

Families of Vertex algebras in a picture



2

[2] Should the pole along $z = 0$ be constant for all u or should it vary?

Axioms of families of vertex algebras.

They are required to satisfy the following set of conditions:

1. *(vacuum)* $T\,|0\rangle = 0\,,\ Y_u(|0\rangle, z) = \mathrm{id}\,,\ Y_u(v, z)|0\rangle \in v + zV_\bullet[\![u, z]\!]\,,$

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3. *(locality)* for any $v, w \in V_\bullet$ and $k \geq 0$, there is an $N \gg 0$ such that the *k'th order deformations of the fields*

$$Y_{\leq k}(v,z) := \sum_{n=0}^{k} u^n\,[t^n]\{\,Y_t(v,z)\,\}\,. \tag{1}$$

satisfy

$$(z_1 - z_2)^N [Y_{\leq k}(v, z_1), Y_{\leq k}(w, z_2)] = 0\,,$$

where the supercommutator is defined on $\mathrm{End}(V_\bullet)[\![u]\!]$ by

$$[A, B] = A \circ B - (-1)^{|A||B|} B \circ A\,.$$

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The reason for introducing the finite order deformations and $Y_u(v,z)$ mapping to $V_\bullet(\!(z)\!)[\![u]\!]$ rather than $V_\bullet[\![u]\!](\!(z)\!)$ is motivated by the geometric construction.

[3] Infinitesimal version of skew-symmetry (assuming locality).

Geometric construction I

The underlying vector space for the geometric construction of the vertex algebras is

$$V_\bullet = H_{\bullet + \mathrm{vdim}_{\mathbb{C}}}(\mathcal{M}_X)$$

where $\mathcal{M}_X$ is the stack of sheaves[4].

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2. There is a K-theory class Θ on $\mathcal{M}_X \times \mathcal{M}_X$ given by the dual of

$$\mathrm{Ext}^\bullet(E, F) \quad \text{at} \quad (E, F) \in \mathcal{M}_X \times \mathcal{M}_X\,.$$

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   It is clearly *additive* with respect to taking direct sums and *multiplicative* with respect to ρ.
3. (New) Consider the trivial $\mathbb{C}^*$-action on $\mathcal{M}_X \times \mathcal{M}_X$ and e^u the weight one line bundle. Take an equivariant K-theory class Ω_u on $\mathcal{M}_X \times \mathcal{M}_X$ satisfying the same additivity and scaling properties $\Theta^\vee$ did. I then introduce

   $$\Theta_u = \Theta + \Omega_u^\vee + \sigma^*\Omega_u\,.$$

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Geometric construction II

1. Example: Let M_α be a (projective) moduli space of stable complexes of class α and define
$$L^{[\alpha]} = \pi_{M_\alpha,*}({\pi_X}^*(L) \cdot \mathcal{F})$$
using the projections to the factors of $X \times M_\alpha$, L a line bundle on X and $\mathcal{F}$ the universal complex. Extending $e^u L^{[\alpha]}$ additively to $\mathcal{M}_X \times \mathcal{M}_X$ one obtains a prime example of Ω_u.

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2. Letting $\mathcal{M}_\alpha$ denote the union of connected components associated to an $\alpha \in K^0(X)$, I also define the modified pairing
$$\chi_\Omega(\alpha,\beta) = \chi(\alpha,\beta) + \kappa(\alpha,\beta) + \kappa(\beta,\alpha)\,, \text{ where } \kappa(\alpha,\beta) = \mathrm{rk}(\Omega_u|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta})\,.$$

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Definition

Construct the formal family of vertex algebras on $V_\bullet$ by setting
$$\begin{aligned} Y_u(v,z)v' =& (-1)^{\kappa(\alpha,\beta)+a\chi_\Omega(\beta,\beta)}\epsilon_{\alpha,\beta} z^{\chi_\Omega(\alpha,\beta)} \\ & \Sigma_*\Big[(e^{zT} \otimes \mathrm{id})\big(v \boxtimes v' \cap c_{z^{-1}}(\Theta_u)\big)\Big]\,, \end{aligned}$$
where
$$\deg(u) = a$$
and Σ is the direct sum map $\mathcal{M}_X \times \mathcal{M}_X \to \mathcal{M}_X$.

The axioms hold

1. (Important) Consider the prototypical example

$$\Omega_u|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta} = e^u\, \mathcal{O}^{\oplus \chi(\alpha,\beta)}\,,$$

then

$$c_{z^{-1}}\left(\Omega_u|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta}\right) = \left(1 + z^{-1}u\right)^{\chi(\alpha,\beta)}\,.$$

[5] Because for $\chi(\alpha, \beta) < 0$, we have an infinite power-series in z^{-1}.

[6] While the construction is heavily inspired by the original work, the proof is different.

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2. It becomes clear that the families of vertex algebras will not in general have constant orders of poles[5], and one needs to put restrictions on powers of u for any kind of vertex algebra axioms to be satisfied.

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Theorem (B.(??) generalizing Joyce(17'))

[6] *The data* $(V_\bullet, Y_u, T, |0\rangle)$ *introduced above satisfies the axioms of a formal u-family of vertex algebras.*

[5] Because for $\chi(\alpha, \beta) < 0$, we have an infinite power-series in z^{-1}.

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Families of Lie algebras

To understand how formal families of vertex algebras are applied to wall-crossing, first, construct a formal family of Lie algebras by

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Starting from a formal u-family of vertex algebras $(V_\bullet, Y_u, T, |0\rangle)$, define a formal u-family of Lie algebras $(Q_\bullet, [-,-]_u)$ for

$$Q_\bullet = V_{\bullet+2}/TV_\bullet$$

by

$$[\overline{v}, \overline{w}]_u = \overline{v_{u,0}w}\,, \quad \forall v, w \in V_\bullet[\![u]\!]\,.$$

Here $\overline{(-)}$ denotes the associated class in the quotient $Q_\bullet[\![u]\!]$. [7]

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Outside of the 0 component, we have $Q_\bullet = H_{\bullet+\mathrm{vdim}_\mathbb{C}}(\mathcal{M}_X^{\mathrm{rig}})$, where $\mathcal{M}_X^{\mathrm{rig}}$ is the quotient by the action of $[*/\mathbb{G}_m]$ and we use a non-standard symmetric obstruction theory on it.

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What is the main statement?

Assumption

Fix two stability conditions σ_0, σ_1 and assume that a list of assumptions holds. One of them is that the enhanced master spaces are projective for a choice of a family of stability conditions interpolating between σ_0 and σ_1.

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Then there should exist classes $\langle \mathcal{M}_\alpha^{\sigma_i} \rangle_u \in Q_\bullet[u]$ independent of choices counting σ_i-semistables in class α such that

$$\langle \mathcal{M}_\alpha^{\sigma_i} \rangle_u = \left[M_\alpha^{\sigma_i} \right]_{\mathrm{vir}} \cap c_{\mathrm{rk}}(\Delta^* \Omega_u)$$

when there are no strictly semistables. Here, the class $[M_\alpha^{\sigma_i}]_{\mathrm{vir}}$ is the pushforward along the open embedding $M_\alpha^{\sigma_i} \hookrightarrow \mathcal{M}_X^{\mathrm{rig}}$ of $[M_\alpha^{\sigma_i}]^{\mathrm{vir}}$.

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Claim (Writing of the proof is in progress)

Let σ_i be two stability conditions for $i = 0, 1$, then for some set $\mathcal{E} \subset K^0(X)$ of emergent classes, $\langle M_\alpha^{\sigma_i} \rangle_u$ satisfy

$$\langle M_\alpha^{\sigma_1} \rangle_u = \sum_{\vec{\alpha} \vdash \alpha} (\mathit{coeff.}) \Big[\cdots \Big[\langle M_{\alpha_1}^{\sigma_0} \rangle_u, \langle M_{\alpha_2}^{\sigma_0} \rangle_u \Big]_u, \cdots, \langle M_{\alpha_k}^{\sigma_0} \rangle_u \Big]_u$$

whenever the Assumptions hold.

Definition of $\langle \mathcal{M}_\alpha^\sigma \rangle_u$ in the case of torsion-free sheaves.

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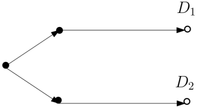
4. A major deviation from Joyce(22') and Mochizuki(09') is how I prove that these classes are independent of the choice of D.

While they use some adapted version of a master space,



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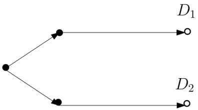

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where $D_1 + D_2$ can be assumed to be sufficiently positive again. Then I compare the obstruction theories of the two moduli spaces to obtain Park's virtual pullback diagram.

Two different ways of proving virtual pullback

1. One way to obtain Park's compatibility diagram of obstruction theories is by direct diagram chasing giving

$$\begin{tikzcd}
\tilde{\mathbb{F}}^\vee[2] \arrow[r, "\bar{\mu}"] \arrow[d, "\kappa"] & \mathbb{F}_{D_1} \arrow[r, "\nu"] \arrow[d, "\mu"] & \mathbb{V}_{D_2}^\vee[1] \arrow[r] \arrow[d, equal] & \tilde{\mathbb{F}}^\vee[3] \arrow[d] \\
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3. Unlike the classical case, this picture can not be lifted to the derived setting directly.

Speculation about a more intuitive approach

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4. Derived Lagrangian correspondence $\iff$ Park's compatibility diagram

Proof of wall-crossing for coherent sheaves

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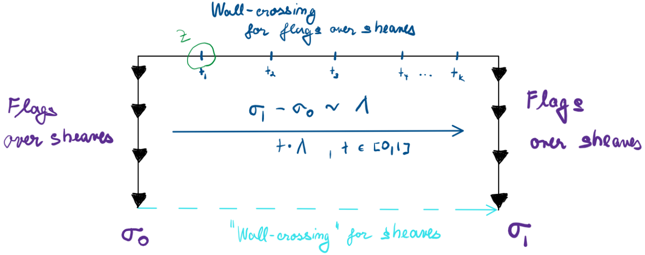

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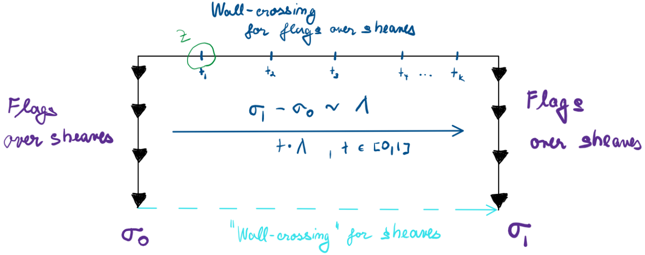

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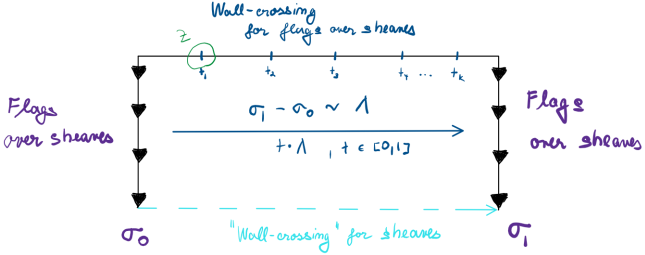

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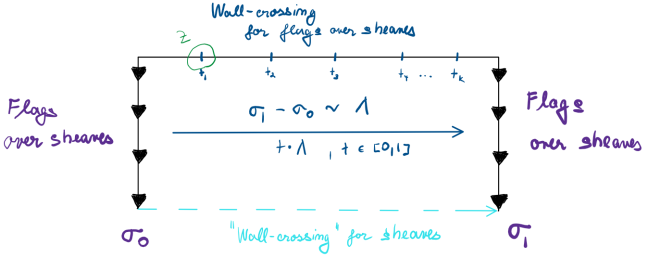

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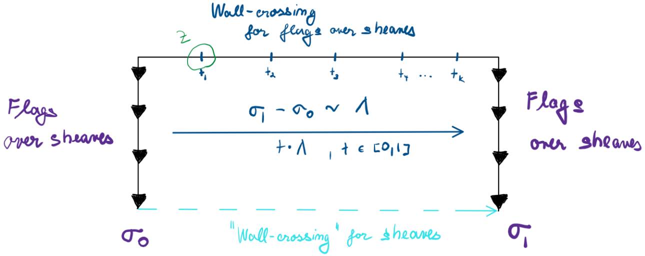

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5. To get the bottom dashed arrow, I need self-dual obstruction theories on the flag-bundles compatible with the obstruction theories for Joyce-Song pairs which were used to define the sheaf invariants.

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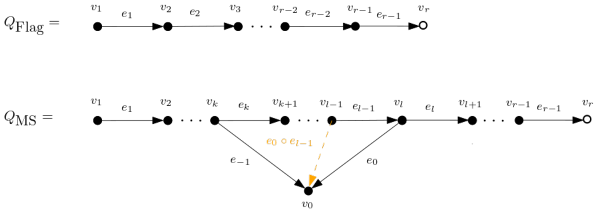

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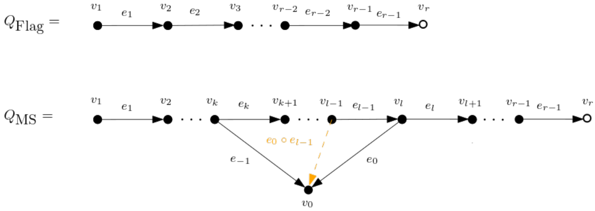

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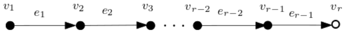
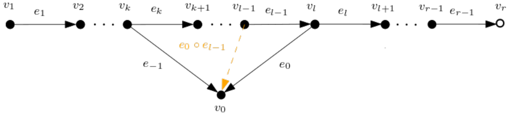
$Q_{\mathrm{Flag}} =$ 

$Q_{\mathrm{MS}} =$ 

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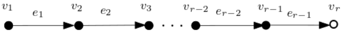
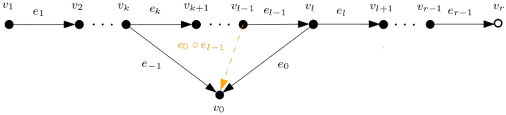
$Q_{\mathrm{Flag}} =$ 

$Q_{\mathrm{MS}} =$ 

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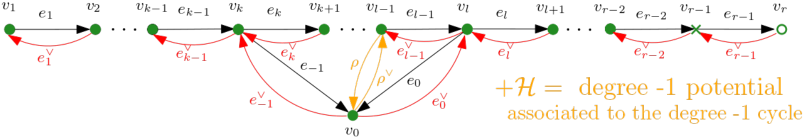

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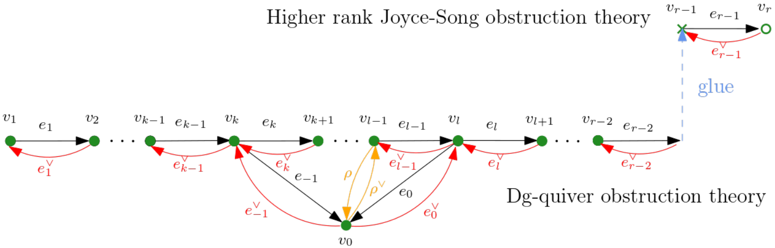

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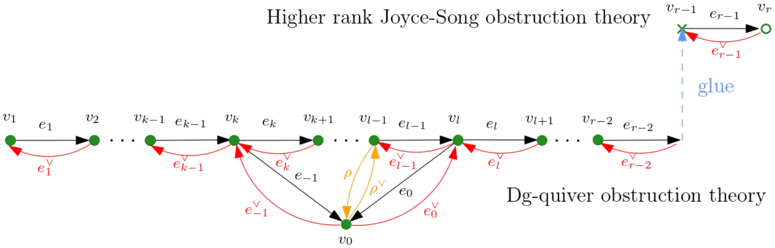

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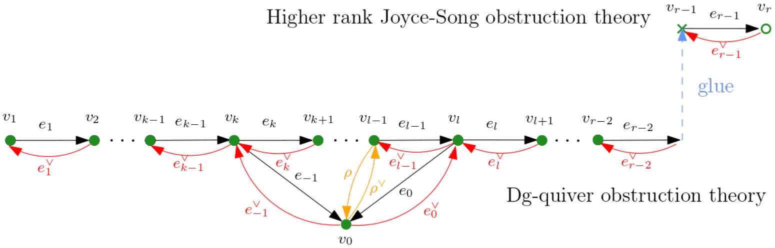

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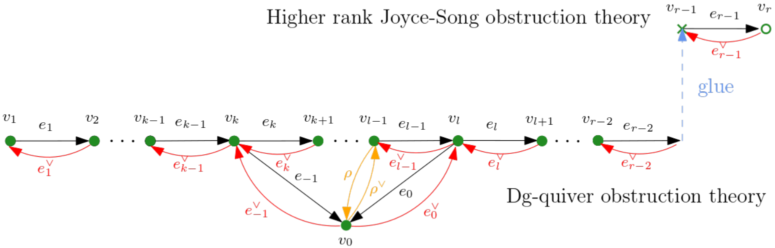

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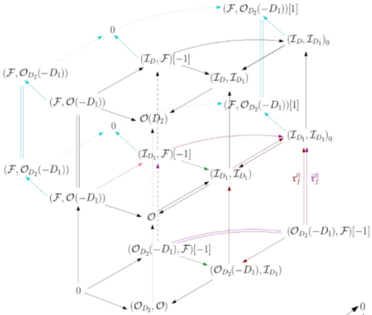
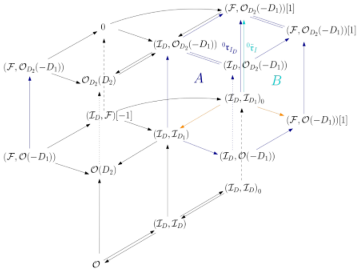
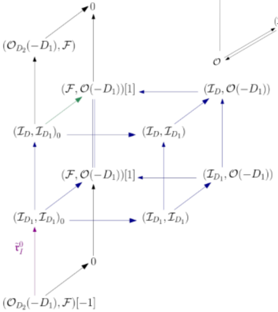
   vanish almost for trivial reasons.
3. The diagram chasing takes places in stable ∞-categories to make everything independent of choices without making it too wild.

What happens if you work only with triangulated categories







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3. I compare obstruction theories along the arrow $\pi_{\overline{d},\alpha}$ in

$$\begin{tikzcd}
 & P^D_\alpha \arrow[dr] & \\
\mathrm{Flag}^D_{\overline{d},\alpha} \arrow[ur, "\pi_{\overline{d},\alpha}"] \arrow[dr] & & \mathcal{M}_\alpha \\
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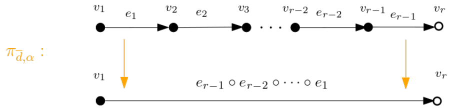

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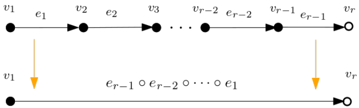

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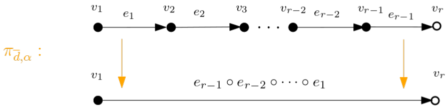

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4. Q.E.D.

Example: relating PT^0 and PT^1 invariants (unfinished).

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$$\mathcal{A}^p = \langle \mathrm{Coh}_{\geq 2}(X), \mathrm{Coh}_{\leq 1}(X)[-1] \rangle \, .$$
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$$\begin{aligned}
&\rho_0 : H^0(X) \oplus H^2(X) \to \mathbb{H}\,, \qquad (\beta, n) \mapsto -n + i(\beta \cdot H)\,,\\
&-\rho_1 : H^4(X) \to \mathbb{H}\,, \qquad \gamma \mapsto -\gamma \cdot H^2\,,\\
&-\rho_3 : H^8(X) \to \mathbb{H}\,, \qquad r \mapsto r(-t+i)\,.
\end{aligned}$$

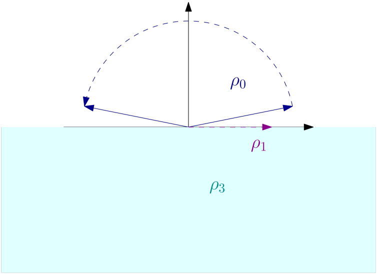

Figure: The cyan region represents $\leq$ 1-dimensional sheaves which are distributed across the lower half-plane. Wall-crossing happens whenever ρ_0 crosses a ray of the phase $\arctan(-\beta \cdot H/n)$ for some $(\beta, n) \in N_{\leq 1}(X)$.

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The PT^0/PT^1 wall-crossing formula

1. After checking assumptions, the last example of stability conditions will give the wall-crossing formula

$$[\mathrm{PT}^{(0)}_{(\gamma,\delta)}]_{\mathrm{vir}} = \sum_{\substack{\underline{\delta}\vdash\delta \\ l(\underline{\delta})=k+1 \\ \frac{n_i}{\beta_i\cdot H}\leq\frac{n_{i+1}}{\beta_{i+1}\cdot H} \text{ if } 0<i}} \frac{(-1)^k}{k!}\Big[\cdots\Big[[\mathrm{PT}^{(1)}_{(\gamma,\delta_0)}]_{\mathrm{vir}}, \big[M_{\delta_1}\big]^{\mathrm{in}}\Big], \cdots, \big[M_{\delta_k}\big]^{\mathrm{in}}\Big],$$

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$$\begin{aligned}\Omega_u|_{\{\mathcal{O}\to F_1\},\{\mathcal{O}\to F_2\}} &= e^u\mathrm{Ext}^\bullet(\mathcal{O}, F_2\otimes L) = \Omega_u|_{\{\mathcal{O}\to F_1\},\{F_2\}}\,,\\ \Omega_u|_{\{F_1\},\{F_2\}} &= 0\,.\end{aligned}$$

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3. Notice that the expression depends only on γ if the orthogonality assumption $\delta c_1(L) = 0$ holds. This motivates the following

Assumption

In the $\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ wall-crossing formula, assume that

$$(\delta - \delta_0) \cdot c_1(L)$$

always holds.

Conjecture of Bae–Kool–Park

1. Under this orthogonality assumption, taking $\left[u^{\frac{\gamma}{2}(c_1(L)^2+\gamma)}\right]\left\{ - \right\}$ in the wall-crossing formula with insertions leads to

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$$\begin{tikzcd} X \arrow[r, "\pi"] & B \arrow[l, bend left, "i"] \end{tikzcd}$$

with base B and section i. Set $L = \pi^* L_B$ which will satisfy the orthogonality assumption for any line bundle L_B because $\beta - \beta_0$ will be the multiple of a fiber class.

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3. For the class $(\gamma,\delta) = \pi^*(\beta, n)$ for $(\beta, n) \in H^{\geq 4}(B)$ Bae–Kool–Park define

$$\langle\!\langle \mathrm{PT}^{(0)}_{\gamma,\delta}\rangle\!\rangle^L = \sum_{d\geq 0} \langle \mathrm{PT}^{(0)}_{\gamma,\delta+dE}\rangle^L q^d$$

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   Here E is the Poincare dual of a fiber class and PT stands for the usual PT stable pairs.

BKP conjecture

1. Up to a simple structural assumption on $\left[M_{dE,n}\right]^{\text{in}}$ that holds whenever B is a Fano of Picard rank 1 and $(d, n) = 1$ (with a sketch of how it works for any Fano 3-fold), and I expect to prove later, I can show that

Conjecture (Bae–Kool–Park)

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3. Note that if $\gamma=0$, then this additional assumption is not required in any geometry, so this expresses PT invariants in terms of just integrals of the form

$$\int_{[M_{\beta,0}]^{\mathrm{in}}} c_1(\mathcal{O}_X^{[\beta,0]})\,.$$

Application to 3-fold DT/PT

1. By the work in progress of Bae–Kool–Park, there is an identification of the moduli spaces
$$DT_{\beta,n} = \mathrm{PT}^{(0)}_{\gamma,\delta}\,, \qquad \mathrm{PT}_{\beta,n} = \mathrm{PT}^{(1)}_{\gamma,\delta}$$
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2. As a consequence of proving the BKP conjecture, one obtains:

Corollary

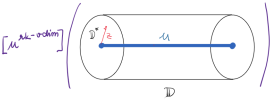
As long as the assumptions of BKP hold, we have the following DT/PT correspondence on the base B:

$$\langle\!\langle DT_\beta \rangle\!\rangle^{\mathcal{O}_B} = \langle\!\langle \mathrm{PT}_\beta \rangle\!\rangle^{\mathcal{O}_B} \langle\!\langle DT \rangle\!\rangle^{\mathcal{O}_B}\,.$$

where the generating series are defined exactly as they were for 4-folds but starting at $(\beta, 0)$.

Summary of everything new that my work introduces

1. Formal families of vertex algebras in relation to wall-crossing with insertions



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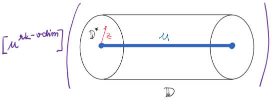

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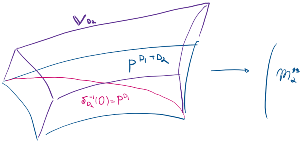

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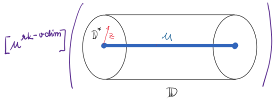

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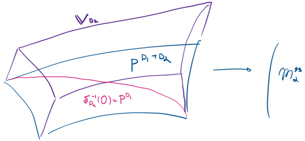

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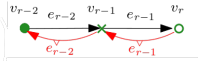

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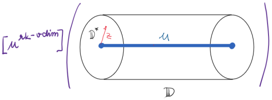

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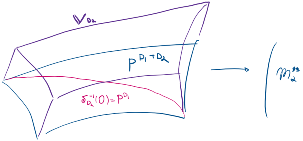

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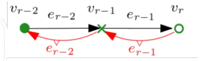
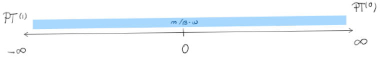


4. New family of stability conditions interpolating between the different surface counting theories.



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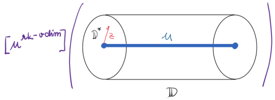

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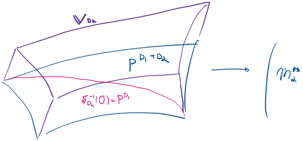

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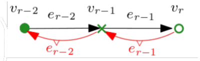

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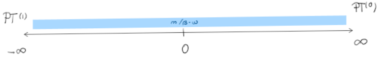

5. (Working on) a complete package for dealing with wall-crossing for stable pairs.