0. Introduction

Our aim is to pursue the following speculative analogy:

Genus 0 GW invariants of a variety V $\rightsquigarrow$ Frobenius structure on $M \subset H^*(V, \mathbb{C})$

DT invariants of a CY_3 Δ-category $\mathcal{D}$ $\overset{??}{\rightsquigarrow}$ Joyce structure on $M = \mathrm{Stab}(\mathcal{D})$

It seems to have something highly non-trivial to say about non-perturbative aspects of topological string theory. For example for the CY_3 category associated to the A_2 quiver we end up considering the Painlevé I τ-function! But so far we can only deal with a few simple examples.

Remarks 0.1. (a) A Joyce structure is something like a non-linear analogue of a Frobenius structure, obtained by replacing the structure group $\mathrm{GL}_n(\mathbb{C})$ by the group of symplectic automorphisms of an algebraic torus $(\mathbb{C}^*)^n$.

(b) The relation with enumerative invariants in the two cases is *completely different*. Both structures involve pencils of flat connections on the tangent bundle. In the GW case the connnection 1-form is given by the triple partial derivatives of the $g = 0$ GW generating function. In the DT cases the connections are given implicitly by their Stokes data.

(c) The top arrow is not so well understood if one wants genuine rather than formal Frobenius structures. The bottom arrow has a more global flavour from the start, and is even less well understood. We will at least need to assume a sub-exponential growth condition on the DT invariants

$$\sum_{\gamma \in \mathbb{Z}^{\oplus n}} |\Omega(\gamma)| e^{-R\|\gamma\|} < \infty.$$

This is expected to hold for the derived category of coherent sheaves on a local Calabi-Yau threefold but not for compact ones. There are lots of examples of $\mathcal{D}$ defined using quivers with potential where it holds.

1. Lecture 1: Stokes data

We explain the definition of Stokes data in the simplest possible case. Then we explain why DT invariants can be viewed as non-linear Stokes data.

1.1. **Stokes data.** Since passing from Frobenius to Joyce structures involves changing the structure group it is worth being a bit pennickety about this now. Fix a finite dimensional complex vector space T and set $\mathfrak{g} = \mathfrak{gl}(T)$ and $G = \mathrm{GL}(T)$. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We the get a root system $\Phi \subset \mathfrak{h}^*$ and a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Define

$$\mathfrak{h}^{\mathrm{reg}} = \{U \in \mathfrak{h} : U(\alpha) \neq 0 \text{ for all } \alpha \in \Phi\} \subset \mathfrak{g}, \qquad \mathfrak{g}^{\mathrm{od}} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \subset \mathfrak{g}.$$

Example 1.1. We can take $T = \mathbb{C}^n$ so that $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ is the space of $n \times n$ matrices and take $\mathfrak{h} \subset \mathfrak{g}$ to be the subspace of diagonal matrices. Then $\Phi = \{\alpha_{ij} = e_) i^* - e_j^* : 1 \leq i, j \leq n\}$ and $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C} \cdot E_{ij}$. The subset $\mathfrak{h}^{\mathrm{reg}}$ consists of diagonal matrices with distinct eigenvalues and $\mathfrak{g}^{\mathrm{od}}$ is matrices with zeroes on the diagonal.

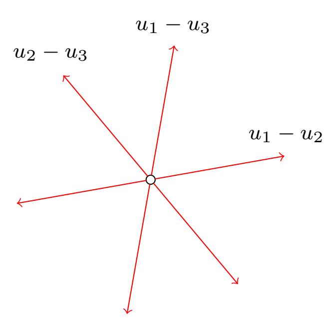
Define a meromorphic connection on the trivial G-bundle over $\mathbb{P}^1$

$$\nabla = d - \left(\frac{U}{\epsilon^2} + \frac{V}{\epsilon}\right) d\epsilon.$$

Remark 1.2. We can equivalently think of ∇ as a connection on the associated trivial vector bundle $\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} W$. The above G-bundle is the frame bundle of this. Flat sections of the G-bundle connection are given by bases of flat sections of the vector bundle connection.

The connection ∇ has a regular singularity (simple pole) at $\epsilon = \infty$ but an irregular singularity at $\epsilon = 0$. We should consider the generalised monodromy data at this point.

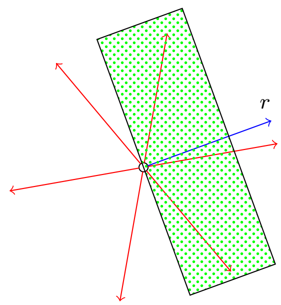
Definition 1.3. A ray $\ell = \mathbb{R}_{>0} \cdot \zeta \subset \mathbb{C}^*$ is called Stokes if it is of the form $\mathbb{R}_{>0} \cdot U(\alpha)$ for $\alpha \in \Phi$.



In general there will be $\frac{1}{2}n(n-1)$ Stokes rays but for non-generic U they could have collided.

Theorem 1.4 (Balser, Jurkat, Lutz, 1970s)**.** *For each* **non**-*Stokes ray $r \subset \mathbb{C}^*$ there exists a unique flat section $Y_r \colon \mathbb{H}_r \to G$ of ∇ defined on the half-plane $\mathbb{H}_r$ centered on r such that $Y_r(\epsilon) \cdot e^{U/\epsilon)} \to 1$ as $\epsilon \to 0$.*

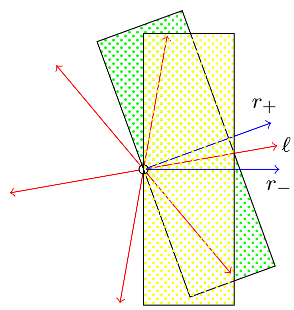
Note that if $V = 0$ the flat sections of ∇ are given by $Y(\epsilon) = C \cdot e^{-U/\epsilon}$ for $C \in G$.



Definition 1.5. The Stokes factor $S_\ell \in G$ associated to a Stokes ray $\ell \subset \mathbb{C}^*$ is defined by

$$Y_{r_-}(\epsilon) = Y_{r_+}(\epsilon) \cdot S_\ell,$$

where $r_\pm = \exp(\pm i\pi\delta)$ are small non-Stokes perturbations of ℓ.



Exercise 1.6. Show that

$$S_\ell \in \exp\Big(\bigoplus_{U(\alpha)\in\ell} \mathfrak{g}_\alpha\Big) \subset G.$$

If we fix U we get a holomorphic (but not algebraic) Stokes map $\mathcal{S}_U\colon \mathfrak{g}^{\mathrm{od}} \to \mathfrak{g}^{\mathrm{od}}$ sending an element V to the sum of the elements $\log(S_\ell)$ over the set of Stokes rays ℓ.

Theorem 1.7. *For each $U \in \mathfrak{h}^{\mathrm{reg}}$ the Stokes map $\mathcal{S}_U$ is a local biholomorphism.*

That is, after having fixed U, we can uniquely reconstruct the connection ∇ from its Stokes data, at least locally. We will see how to do this in practice later using Riemann-Hilbert problems. To get a complete reconstruction one must add a discrete amount of extra monodromy data (a choice of the log of the monodromy, and the central connection matrix).

1.2. **DT invariants as Stokes data.** Consider a CY_3 triangulated category $\mathcal{D}$. Assume $\Gamma = K_0(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$. Recall the skew-symmetric Euler form

$$\langle [E], [F]\rangle = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(E, F[i])\colon \Gamma\times\Gamma \to \mathbb{Z}.$$

Introduce the algebraic torus

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n, \qquad \mathbb{C}[\mathbb{T}] = \bigoplus_{\gamma\in\Gamma} \mathbb{C}\cdot x_\gamma.$$

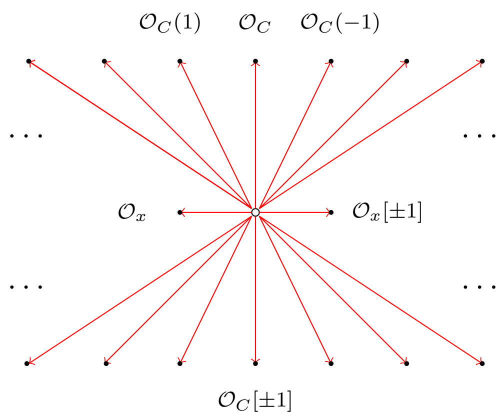
with the invariant Poisson structure

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta\rangle \cdot x_{\alpha+\beta}.$$

Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{D}$. Given extra assumptions we get DT invariants $\mathrm{DT}_\sigma(\gamma) \in \mathbb{Q}$ and $\Omega_\sigma(\gamma) \in \mathbb{Q}$ related by the multi-cover formula

$$\mathrm{DT}_\sigma(\gamma) = \sum_{n\in\mathbb{N}:\gamma=n\alpha} \frac{\Omega_\sigma(\alpha)}{n^2}.$$

A ray $\ell \subset \mathbb{C}^*$ is called Stokes if $\ell = \mathbb{R}_{>0}\cdot Z(\gamma)$ for some class $\gamma\in\Gamma$ with $\mathrm{DT}_\sigma(\gamma)\neq 0$. There are countably many Stokes rays in general. The following picture is for a stability condition on $\mathcal{D}^b\,\mathrm{Coh}(X))$ with X the resolved conifold.



Try to define, for each Stokes ray $\ell \subset \mathbb{C}^*$, a Poisson automorphism $\mathbb{S}_\ell\colon \mathbb{T}\to\mathbb{T}$:

$$\mathbb{S}_\ell^*(x_\beta) = \exp\left\{ \sum_{Z(\gamma)\in\ell} \mathrm{DT}_\sigma(\gamma)\cdot x_\gamma, -\right\}(x_\beta) = x_\beta \cdot \prod_{Z(\gamma)\in\ell} (1-x_\gamma)^{\Omega(\gamma)\cdot\langle\beta,\gamma\rangle}$$

We are ignoring quadratic refinements here: really we should replace $\mathbb{T}$ by a torsor over it which introduces some signs. In any case work is required to make rigorous sense of $\mathbb{S}_\ell$. Three possible approaches are:

- Assume there are only finitely many nonzero $\Omega(\gamma)$: then the $\mathbb{S}_\ell$ are well-defined birational automorphisms of $\mathbb{T}$,
- Assume a sub-exponential growth condition on the DT invariants

$$\sum_{\gamma\in\mathbb{Z}^{\oplus n}} |\Omega(\gamma)| e^{-R\|\gamma\|} < \infty.$$

Then the $\mathbb{S}_\ell$ are well-defined on analytic open subsets of $\mathbb{T}$.

- Work with formal series: replace $\operatorname{Aut}\mathbb{C}[x_i^{\pm 1}]$ with $\operatorname{Aut}\mathbb{C}[[x_i]]$. This allows a rigorous statement of the wall-crossing formula but is not good for more global statements.

Consider the Lie algebra

$$\mathfrak{g} = \operatorname{vect}_{\{-,-\}}(\mathbb{T}) = \mathfrak{h} \oplus \mathfrak{g}^{\mathrm{od}}$$

of algebraic vector fields on $\mathbb{T}$ whose flows preserve $\{-,-\}$. Temporarily assume that $\langle -,-\rangle$ is non-degenerate. The Cartan subalgebra consists of translation-invariant vector fields:

$$\mathfrak{h} = T_e(\mathbb{T}) = \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}).$$

The subalgebra $\mathfrak{g}^{\mathrm{od}}$ consists of Hamiltonian vector fields:

$$\mathfrak{g}^{\mathrm{od}} = \bigoplus_{\gamma\in\Gamma^\times} \mathfrak{g}_\alpha = \bigoplus_{\gamma\in\Gamma^\times} \mathbb{C}\cdot x_\gamma.$$

The root system is $\Gamma^\times = \Gamma \setminus \{0\}$, and the root decomposition of a function is its Fourier decomposition. Note the lack of a well-defined exponential map (time 1 flow)

$$\exp\colon \mathfrak{g} = \operatorname{vect}_{\{-,-\}}(\mathbb{T}) \dashrightarrow G = \operatorname{Aut}_{\{-,-\}}(\mathbb{T})$$

The $\mathbb{S}_\ell$ defined above look like the Stokes factors for a meromorphic connection on the trivial G-bundle over $\mathbb{P}^1$

$$\nabla_\sigma = d - \left(\frac{Z}{\epsilon^2} + \frac{\mathrm{Ham}_F}{\epsilon}\right) d\epsilon,$$

where $Z \in \mathfrak{h}$ is the central charge $Z\colon \Gamma \to \mathbb{C}$, and $F = \sum_{\gamma\in\Gamma^\times} F_\gamma \cdot x_\gamma \in \mathfrak{g}^{\mathrm{od}}$ is a function on $\mathbb{T}$. Further evidence for this comes from the wall-crossing formula.

2. Lecture 2: Frobenius and Joyce structures

A (tame) Frobenius structure on a complex manifold M is essentially an isomonodromic family of meromorphic connections of the type considered last time. Replacing linear connections by non-linear connections leads to the notion of a Joyce structure.

2.1. **Frobenius manifolds and iso-Stokes.**

A Frobenius structure on a complex manifold M consists of a holomorphic metric g, a commutative and associative multiplication $*\colon T_M \times T_M \to T_M$, and identity and Euler vector fields e and E subject to a system of axioms.

The most well-known consequence of this structure is a pencil (= 1-dimensional family) of flat, torsion-free connections on the tangent bundle

$$\nabla^{(\epsilon)}_X(Y) = \nabla^{LC}_X(Y) + \epsilon^{-1} X * Y.$$

Here $\epsilon \in \mathbb{C}^* \subset \mathbb{P}^1$ is the parameter in the pencil, ∇^{LC} is the Levi-Civita connection for the metric g, and X and Y are vector fields on M.

In fact this pencil is part of a bigger structure. Consider the projection $p\colon M \times \mathbb{P}^1 \to M$. Then there is a flat meromorphic connection on the bundle $p^*(T_M)$ by the formula

$$\nabla = p^*(\nabla^{LC}) + \epsilon^{-1}\Theta - \left(\frac{U}{\epsilon^2} + \frac{V}{\epsilon}\right) d\epsilon.$$

Here $\Theta \in T_M^* \otimes \mathrm{End}(T_M)$ is given by $\Theta_X(Y) = X * Y$ and $U, V \in \mathrm{End}(T_M)$ are given by $U(X) = E * X$ and $V(X) = \nabla_X^{LC}(E) + \frac{1}{2}(d-2)\cdot X$. The constant $d \in \mathbb{C}$ is the charge or conformal dimension of the Frobenius structure.

For each $m \in M$ we can restrict to $p^{-1}(m) = \mathbb{P}^1$ to get a connection

$$\nabla_m = d - \left(\frac{U_m}{\epsilon^2} + \frac{V_m}{\epsilon}\right) d\epsilon.$$

of the form considered above, where $W = T_{M,m}$. The multiplication gives a map $W \to \mathfrak{g} = \mathfrak{g}l(W)$ and when the multiplication is semi-simple this is injective. Since the multiplication is commutative the image $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, and we have $V \in \mathfrak{g}^{\mathrm{od}}$. The Frobenius manifold is called tame if $U \in \mathfrak{h}$ has distinct eigenvalues: then we are in the setting considered above.

The fact that the connections ∇_m for different $m \in M$ are related by the sideways connection implies the Stokes factors of ∇_m are constant. More precisely, since the Stokes rays may collide and separate as U_m varies, the correct way to state the iso-Stokes condition is the following.

Theorem 2.1. *For any convex sector $\Delta \subset \mathbb{C}^*$ the clockwise product*

$$S_\Delta(m) := \prod_{\ell \in \Delta}^{\curvearrowright} S_\ell(m) \in G$$

is constant as $m \in M$ varies in any domain where the boundary rays of Δ are never Stokes.

The wall-crossing formula in DT theory is the exact same statement for the DT automorphisms $\mathbb{S}_\ell$ consisdered above. We see that over $\mathrm{Stab}(\mathcal{D})$ there should be an analogue of a Frobenius structure where instead of linear automorphisms of $\mathbb{C}^n$ we deal with Poisson automorphisms of $(\mathbb{C}^*)^n$. For simplicity we assume the Poisson structure / Euler form is non-degenerate. Thus we replace pencils of flat linear connections with pencils of non-linear symplectic connections. This gives the notion of a Joyce structure.

2.2. **Non-linear connections.** Let $\pi\colon X \to M$ be a map of complex manifolds. Consider the tangent sequence

$$0 \longrightarrow V_{X/M} \xrightarrow{\ i\ } T_X \xrightarrow{\ \pi_*\ } \pi^*(T_M)$$

where π_* is the derivative of π and $V_{X/M} \subset T_X$ is the sub-bundle of vertical tangent vectors. A non-linear connection on π is a splitting of this sequence:

$$h\colon \pi^*(T_M) \to T_X, \qquad \pi_* \circ h = \mathrm{id}\,.$$

Then π is necessarily a submersion, and the tangent bundle

$$T_X = V_{X/M} \oplus \mathrm{im}(h)$$

decomposes into vertical and horizontal directions.

Given a path $\gamma\colon [0,1] \to M$ and a point $x \in \pi^{-1}(\gamma(0))$ we obtain a lifted path

$$\alpha\colon [0,\epsilon] \to X$$

for $0 \ll \epsilon < 1$ defined by the conditions $\alpha(0) = x$ and $\dot\alpha(t) = h(\dot\gamma(t))$. Varying x in a small neighbourhood $U_0 \subset \pi^{-1}(\gamma(0))$ we obtain parallel transport maps

$$\mathrm{PT}(\gamma)\colon U_0 \subset \pi^{-1}(\gamma(0)) \to U_t \subset \pi^{-1}(\gamma(t)),$$

well-defined for $0 \ll t < 1$ and $U_0 \subset \pi^{-1}(\gamma(0))$ small enough.

The connection h is called flat if it satisfies

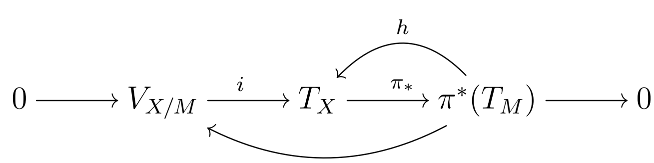
$$h([w_1, w_2]) = [h(w_1), h(w_2)]$$

for any vector fields w_1, w_2 on M. Then $\mathrm{PT}(\gamma)$ depends only on the homotopy class of γ.

2.3. **Pre-Joyce structures.** Let M be a complex manifold with tangent bundle $\pi\colon X = T_M \to M$. There is a canonical isomorphism $\nu\colon \pi^*(T_M) \to V_{X/M}$ coming from

$$V_{X/M,x} = T_{\pi^{-1}(\pi(x)),x} = T_{T_{M,\pi(x)},x} = T_{M,\pi(x)} = \pi^*(T_M)_x.$$

Take a non-linear connection $h\colon \pi^*(T_M) \to T_X$, and set $v = i \circ \nu$. We get a pencil of connections $h_\epsilon = h + \epsilon^{-1} v$ depending on $\epsilon^{-1} \in \mathbb{C}$.



Definition 2.2. A pre-Joyce structure on a complex manifold M consists of

(i) a holomorphic symplectic form ω on M,

(ii) a non-linear connection h on the tangent bundle $\pi\colon X = T_M \to M$,

such that each connection $h_\epsilon = h + \epsilon^{-1} v$ is flat and symplectic.

Here we say that a connection on π is symplectic if the partially-defined maps

$$\mathrm{PT}(\gamma)\colon T_{M,m} \to T_{M,m'},$$

are symplectic, i.e. take ω_m to $\omega_{m'}$.

2.4. **In co-ordinates.** Take co-ordinates z_i on M and write $\omega = \sum_{i<j} \omega_{ij}\, dz_i \wedge dz_j$. Assume ω_{ij} is a constant matrix (so we have more-or-less chosen Darboux co-ordinates) and let η_{ij} be the inverse matrix. Writing tangent vectors in the form $\sum_i \theta_i \cdot \frac{\partial}{\partial z_i}$ gives co-ordinates z_i, θ_i on $X = T_M$. The definition of v gives

$$v\Big(\frac{\partial}{\partial z_i}\Big) = \frac{\partial}{\partial \theta_i},$$

and the symplectic condition gives Hamiltonian functions $W_i\colon X \to \mathbb{C}$ such that

$$v\Big(\frac{\partial}{\partial z_i}\Big) = \frac{\partial}{\partial \theta_i}, \qquad h\Big(\frac{\partial}{\partial z_i}\Big) = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta_{pq} \cdot \frac{\partial W_i}{\partial \theta_p} \cdot \frac{\partial}{\partial \theta_q}.$$

Flatness implies that $W_i = \frac{\partial W}{\partial \theta_i}$, where $W\colon X \to \mathbb{C}$ must satisfy

$$\frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q}.$$

These are known as Plebanski's second heavenly equations.

2.5. **Complex hyperkähler structure on $X = T_M$.** A non-linear connection h on π gives

$$T_X = \mathrm{im}(v) \oplus \mathrm{im}(h) \cong \pi^*(T_M) \oplus \pi^*(T_M) = \pi^*(T_M) \otimes_{\mathbb{C}} \mathbb{C}^2.$$

We get a holomorphic metric g and an action of the quaternions on T_X:

$$g = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} i \cdot \mathbb{1} & 0 \\ 0 & -i \cdot \mathbb{1} \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad K = IJ.$$

The essential point here is that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$.

Theorem 2.3. *The operators I, J, K preserve the metric g and are parallel for the associated Levi-Civita connection precisely if the connections h_ϵ are all flat and symplectic.*

Thus a pre-Joyce structure on M defines a complex HK structure on $X = T_M$.

2.6. **Definition of a Joyce structure.** A Joyce structure is a pre-Joyce structure with some extra symmetries.

Definition 2.4. A Joyce structure on a complex manifold M consists of

(i) a pre-Joyce structure (ω, h) as above,

(ii) an integral affine structure $T_M^{\mathbb{Z}} \subset T_M$,

(iii) a $\mathbb{C}^*$-action on M,

satisfying the following conditions

(i) the connection h is invariant under translations by $T_M^{\mathbb{Z}} \subset T_M$ and hence descends to the bundle $T_M^{\#} = T_M/T_M^{\mathbb{Z}} \to M$ whose fibres are isomorphic to $(\mathbb{C}^*)^n$,

(ii) Let E be the generating vector field for the induced $\mathbb{C}^*$ action on X. Then

$$\mathrm{Lie}_E(g) = g, \quad \mathrm{Lie}_E(I) = 0, \quad \mathrm{Lie}_E(J \pm iK) = \mp(J \pm iK)$$

(iii) Let -1 denote the involution of X acting by -1 on the fibres of π. Then

$$(-1)^*(g) = -g, \quad (-1)^*(I) = I, \quad (-1)^*(J \pm iK) = -(J \pm iK),$$

In appropriate co-ordinates as above the conditions (i)-(iii) become

(i) $W(z_1, \cdots, z_n, \theta_1 + k_1, \cdots, \theta_n + k_n) = W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n)$,

(ii) $W(tz_1, \cdots, tz_n, \theta_1, \cdots, \theta_n) = t^{-1} W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n)$,

(iii) $W(z_1, \cdots, z_n, -\theta_1, \cdots, -\theta_n) = -W(z_1, \cdots, z_n, \theta_1, \cdots, \theta_n)$.