Counting Stable Spherical Bundles on a K3 Surface

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Background/Motivation

Theorem (-, Lin, Zhao) Let S be a complex K3 surface of **Picard number one**. Assume that there is a fully faithful functor $\mathrm{D}^b(S) \to \mathrm{D}^b(S')$ for a smooth irreducible surface S'. Then S' is birational to a Fourier–Mukai partner of S.

To complete this result in higher Picard number cases, the only missing step is:

- For every semi-rigid/spherical object E, there exists a Bridgeland stability condition σ such that E is σ-stable. $(\star)$

$(\star)$ is true when the K3 surface S is of Picard number one thanks to theorem of Bayer–Bridgeland stating that $\mathrm{Stab}(S)$ is contractible.

Definition and Notion

Let S be a complex K3 surface, an object $E \in D^b(S)$ is called spherical if

$$\mathrm{RHom}(E, E) = \mathbb{C} \oplus \mathbb{C}[-2].$$

Let σ be a stability condition, the width of an object E is defined to be $w_E(\sigma) = w(\sigma, E) := \phi^+_\sigma(E) - \phi^-_\sigma(E)$.

Bayer–Bridgeland show that when S is of **Picard number one**, for every spherical (or semi-rigid object) E, the stability manifold $\mathrm{Stab}(S)$ contracts to $w_E^{-1}(0)$, which is closed with the same dimension of $\mathrm{Stab}(S)$. In particular, E is σ-stable for every σ in the inner points of $w_E^{-1}(0)$.

Questions by Huybrechts

We notice that even the classical version of $(\star)$ is unknown. Four exercise questions are posted at the end of Chapter 16 Huybretchs's book 'Lectures on K3 Surfaces'.

1. E spherical object $\implies E^{\perp} \neq \emptyset$?
2. Let E be a spherical bundle $\implies \exists H \in \mathrm{Amp}(S)$ such that E is p_H-stable?
   Is there a way to 'count' spherical vector bundles with a given Mukai vector v?
3. Find two non-isomorphic K3 surfaces X and Y over a field K with $D^b(X) \simeq D^b(Y)$ over K and $X_{\overline{K}} \simeq Y_{\overline{K}}$.
4. Let X be a K3 surface. Then $\mathrm{Stab}(X)$ is connected and simply-connected. ($\implies \mathrm{Aut}_s(D^b(X))/\mathbb{Z}[2] \simeq \pi_1[\widetilde{O}(N(X)) \setminus D_0]$.)

Goal: To get 40/100 (pass marks in the UK system).

Notations and classical results

Let S be a smooth project K3 surface over $\mathbb{C}$. We recap the concept of Mukai vector/pairing.

- $\widetilde{H}(S) = H^0(S,\mathbb{Z}) \oplus \mathrm{NS}(S) \oplus H^4(S,\mathbb{Z})$.
- For $E \in \mathrm{Coh}(S)$, the Mukai vector of E is

$$v(E) = (\mathrm{rk}(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E) + \mathrm{rk}(E)) \in \widetilde{H}(S).$$

- Mukai pairing: $v_i = (r_i, D_i, s_i) \in \widetilde{H}(S)$, where $i = 1, 2$:

$$\langle v_1, v_2 \rangle := -r_1 s_2 - r_2 s_1 + D_1 D_2.$$

- Rieman–Roch: $-\chi(E,F) = \langle v(E), v(F) \rangle$.
- E spherical $\implies \langle v(E), v(E) \rangle = -2$.

Existence result

We call $v \in \widetilde{H}(S)$ **spherical** if $\langle v, v \rangle = -2$ and $\mathrm{rk}(v) > 0$.
Theorem(Kuleshov) $\forall$ spherical Mukai vector v, $\exists$ bundle E on S with $v(E) = v$.

- $\mathcal{O}_S(D)$ is spherical.
- If $C \simeq \mathbf{P}^1$ is a (-2)-curve on S, then $\mathcal{O}_C(n)$ is spherical. But for a given $v = (0, D, s)$ with $D^2 = -2$, D may not be effective, so there is no coherent sheaf with Mukai vector v.

Theorem(Siedel–Thomas): Let $E \in D^b(S)$ be spherical, one may define a spherical twist $\mathsf{T}_E : D^b(S) \to D^b(S)$ which is an equivalence on $D^b(S)$. On the level of objects, $\mathsf{T}_E(F)$ is defined as

$$E \otimes \mathrm{RHom}(E, F) \xrightarrow{ev} F \to \mathsf{T}_E(F) \xrightarrow{+}$$

Examples of spherical sheaves

Let D be an effective divisor such that $\mathcal{O}_S(D)$ is globally generated. Then there is the short exact sequence

$$0 \to \mathcal{K}_D \to \mathcal{O}_S \otimes \mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S(D)) \xrightarrow{ev} \mathcal{O}_S(D) \to 0.$$

It follows that $\mathcal{K}_D = \mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_S(D))[-1]$ is a spherical bundle.
Let $D \neq 0$ be a divisor such that neither $\pm D$ is effective.

$$0 \to \mathcal{O}_S(D) \to \mathcal{E}_D \to \mathcal{O}_S \otimes \mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S(D)[1]) \to 0.$$

Then $\mathcal{E}_D = \mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_S(D))$ is a spherical bundle.
Let $C \simeq \mathbf{P}^1$ be a (-2)-curve, then

$$0 \to \mathcal{O}_C(-2) \to \mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2)) \to \mathcal{O}_S \to 0.$$

The spherical sheaf $\mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2))$ is non-torsion and non-torsion-free.

Remark on $E^\perp$

Theorem: Let E be an n-spherical object in $D^b(X)$, where X is an n-dimensional smooth variety over $\mathbb{C}$, then $E^\perp$ is non-empty.

- When X is a K3 surface of Picard number one, this has been proved by Bayer using tools and results on the contractibility of Stab(X).
- When $n = 2$, we expect that

$$\overline{D^b(X)/\langle E^\perp, E\rangle} \simeq D^b_{\mathrm{sing}}(\mathrm{Spec}\mathbb{C}[x]/(x^2)).$$

  E.g. $E = \mathcal{O}_C(-1)$ where $C \simeq \mathbf{P}^1$ is a (-2)-curve.
- Another corollary is that the power of a spherical twist T_E^k (when $k \neq 0$) is not in $\mathbb{Z}[1] \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$.

Subtotal: 25/100

Stabilities

For the rest of the talk, we focus on the second question.
Let $H \in \mathrm{Amp}(S)$. The **slope** of F with respect to H is:

$$\mu_H(F) := \frac{H\mathrm{ch}_1(F)}{\mathrm{rk}(F)}.$$

Slope stability: A torsion-free sheaf F is called μ_H-(semi)**stable** if $\forall 0 \neq E \subsetneq F$, $rk(E) < rk(F)$, we have

$$\mu_H(E) < (\leq) \mu_H(F).$$

Reduced Hilbert polynomial: $p_{H,E}(n) := \chi(E(nH))/\mathrm{rk}(E)$.
Gieseker stability: A sheaf F is called p_H-(semi)**stable** if $\forall 0 \neq E \subsetneq F$, we have

$$p_{H,E}(n) < (\leq) p_{H,E}(n),$$

for $n \gg 0$.

- μ_H-stable $\Longrightarrow$ p_H-stable $\Longrightarrow$ p_H-semistable $\Longrightarrow$ μ_H-semistable
- E is μ_H-stable $\Longrightarrow$ $\mu_{H\pm\epsilon D}$-stable for $0 < \epsilon \ll 1$.
- If E is a p_H-stable spherical bundle, then E is the unique p_H-semistable with Mukai vector $v(E)$.

Example: Let $H_0 \in$Amp(S) and $D \neq$ be a divisor such that $H_0D = 0$. Then $\mathcal{E}_D$ is μ_{H_-}-stable, μ_{H_+}-unstable; μ_{H_0}-semistable, p_{H_0}-stable.

Example: Let S be a generic elliptic K3 surface with a section. We may consider the spherical bundle $B := \mathsf{T}_{\mathcal{O}_S(E-F)}(\mathsf{T}_{\mathcal{O}_S(F-E)}(\mathcal{O}_S(2E-2F)))$, which is with Mukai vector $v(B) = (113, 82E - 82F, -119)$. Then B is p_{3E+F}-stable, but B is not slope semistable with respect to any other (up to a scalar) polarization. The spherical bundle B is not p_H-stable with respect to any polarization other than $3E + F$ up to a scalar.

Theorem(Mukai): Let S be a K3 surface with Picard number one, then every spherical bundle is p_H-stable.

Example: never-stable spherical bundle

Question: Let E be a spherical bundle $\implies$ Does there exists $H \in$Amp(S) such that E is p_H-stable?
Recall the example of

$$0 \to \mathcal{O}_C(-2) \to \mathcal{F}_C \to \mathcal{O}_S \to 0.$$

The torsion part $\mathcal{O}_C(-2)$ always destabilizes $\mathcal{F}_C$. To get a spherical vector bundle that is never stable, we may apply the spherical twist $\mathsf{T}_m := \mathsf{T}_{\mathcal{O}_C(-mG)}[-1]$ on $\mathcal{F}_C$ for some ample divisor G and $m \gg 0$. In particular, we have the short exact sequence of spherical bundles.

$$0 \to \mathsf{T}_m(\mathcal{O}_C(-2)) \to \mathsf{T}_m(\mathcal{F}_C) \to \mathsf{T}_m(\mathcal{O}_S) \to 0.$$

Example (continued)

By a direct computation, we have

$$\mu(\mathrm{T}_m(\mathcal{O}_C(-2))) - \mu(T_m(\mathcal{O}_S)) \sim G - \frac{m^2G^2+2}{2m^2GC-2m}C.$$

To let $\mathrm{T}_m(\mathcal{O}_C(-2))$ always destabilize $\mathrm{T}_m(\mathcal{F}_C)$ when $m \gg 0$, we need to choose G so that $G - \frac{G^2}{2GC}C$ is effective.
This relies on $\overline{NE}(S)$. For example, if S is of Picard number 2 with $\overline{NE}(S)$ spanned by $[C]$ and the class of an elliptic curve, then $G - \frac{G^2}{2GC}C$ is never effective.

If S is of Picard number 2 with $\overline{NE}(S)$ spanned by two (-2)-classes, then there exists G such that $G - \frac{G^2}{2GC}C$ is effective. In particular, when $m \gg 0$, the spherical bundle $\mathrm{T}_m(\mathcal{F}_C)$ is never μ_H-semistable for any $H \in$Amp(S).

Example (remark)

- By Theorem of Kovács, there exists a K3 surface S of Picard number 2 with $\overline{NE}(S)$ spanned by two (-2)-classes.
- That is the only construction of we know so far.
- In the case that S is an elliptic K3 surface admitting a section $C \simeq \mathbf{P}^1$ of Picard number two, we would like to know if there exist spherical bundles that is never μ_H-semistable.

Subtotal: 35/100

Example: infinitely many spherical bundles

Recall the second part of the question: is there a way to count spherical vector bundles with a given Mukai vector?
(Firstly, we wonder if there are only finitely many spherical vector bundles with a given Mukai vector?)

By Torelli Theorem, there exists a projective K3 surface S such that $\mathrm{NS}(S) = \mathbb{Z}\mathrm{h} \oplus \mathbb{Z}\mathrm{e} \oplus \mathbb{Z}\mathrm{f}$ such that: $\mathrm{h}^2 = 12$, $\mathrm{e}^2 = -6, \mathrm{f}^2 = -30$, and $\mathrm{h}, \mathrm{e}, \mathrm{f}$ orthogonal to each other.
By Theorem of Kovács, the cone

$$\overline{NE}(S) = Nef(S) = \{\alpha \in NS_{\mathbb{Q}}(S) | \alpha^2 \geq 0, \alpha \mathrm{h} \geq 0\}$$

is circular. For a spherical vector v with rank greater than 1, it is 'likely' that there are infinitely many walls for v. It follows that there are infinitely many spherical bundles with Mukai vector v.

More precisely, we may choose $v = (2, \mathrm{f}, -7)$.
Then for every pair of positive integer solutions (a, b) to the Pell equation $2x^2 - y^2 = 1$, we may consider the divisors $D_a = a\mathrm{h} + b\mathrm{e} + \mathrm{f}$ and $E_a = -a\mathrm{h} - b\mathrm{e}$.
Note that

$$(D_a - E_a)^2 = 48a^2 - 24b^2 - 30 = -6.$$

By the description for $\overline{NE}(S)$, neither $D_a - E_a$ or $E_a - D_a$ is effective.
It follows that $RHom(\mathcal{O}_S(D_a), \mathcal{O}_S(E_a)) = \mathbb{C}[-1]$.
So $V_a = \mathrm{T}_{\mathcal{O}_S(E_a)}(\mathcal{O}_S(D_a))$ is a rank 2 spherical bundle that fits into the short exact sequence

$$0 \to \mathcal{O}_S(D_a) \to V_a \to \mathcal{O}_S(E_a) \to 0.$$

The Mukai vector of V_a is given as

$$v(V_a) = (1, D_a, \tfrac{1}{2}D_a^2 + 1) + (1, E_a, \tfrac{1}{2}E_a^2 + 1) = (2, f, -7).$$

Example (remark)

- Each V_a is μ_{H_a}-stable where $H_a := 7a\mathrm{h} + 7b\mathrm{e} + 3\mathrm{f}$.
- When $\mathrm{Nef}(S)$ is circular (in the case that S is of Picard number two, that is, the two boundary rays of $\mathrm{Nef}(S)$ are irrational), we expect there always exists spherical vector v so that there are infinitely many spherical bundles with Mukai vector v.
- The naive counting does not make sense in the above cases, but we do not know if one can put certain weights on spherical bundles to solve this issue.

Subtotal: 38/100

A finiteness result

Theorem: Let S be a smooth projective surface over $\mathbb{C}$ with $Nef(S)$ being **rational polyhedral**. Then for every class $v = (rk, \ell, s) \in \widetilde{H}(S, \mathbb{Z})$ with $rk > 0$ and (rk, ℓ) being primitive, there are only finitely many numerical walls of v. In particular, there are finitely many walls and chambers for the moduli space $M^s_H(v)$ in $Amp(S)$.

- In particular, when S is a K3 surface with $\mathrm{Nef}(S)$ rational polyhedral, then for every spherical vector v, it makes sense to count all stable spherical bundles with Mukai vector v.

$$\begin{aligned}\mathrm{H}_S(v) :=& \#\{E \mid v(E) = v; \exists H \in Amp(S),\ E \text{ is } \mu_H\text{-stable.}\} < +\infty. \\ =& \#\{\text{chambers of } M^s(v)\}.\end{aligned}$$

In general, $\mathrm{H}_S(v) < \#\{E \mid v(E) = v,\ E \text{ is a vector bundle.}\} =: \mathrm{H}'_S(v)$, which we do not know (but we expect) to be finite when $\mathrm{Nef}(S)$ is rational polyhedral.

Subtotal: 39/100

Reduce to numerical counting

Prop: Let S be a K3 surface, v be a spherical vector. Then $W \subset Amp(S)$ is an actual wall of v if and only if it is a numerical wall of v, in other words, $\exists$ a spherical w satisfying

1. $rk(w) < rk(v)$;
2. $\langle v, w \rangle < 0$;
3. $W = (\mu(v) - \mu(w))^{\perp}$.

Cor: The counting $\mathrm{H}_S(v)$ only depends on $Nef(S)$ and $(-,-)_{\text{intersection}}$.
Let S' be another K3 surface. Assume that there exists an injective group homomorphism $f\colon NS(S') \to NS(S)$ preserving the intersection numbers and $f(Nef(S')) \subseteq Nef(S)$. Then $\mathrm{H}_{S'}((rk, \ell, s)) \leq \mathrm{H}_S((rk, f(\ell), s))$.

E.g. Assume that S is an elliptic K3 surface admitting a section $C \simeq \mathbf{P}^1$, then $\mathrm{H}_S \geq \mathrm{H}_{S'}$ where S' is such an elliptic curve of Picard number two.

Case study

For the rest of the talk, we discuss $\mathsf{H}_S(v)$ for the generic elliptic K3 surface. Let S be an elliptic K3 surface with a section $C \simeq \mathbf{P}^1$. Denote by e the divisor class of elliptic fiber and σ the section.

- $NS(S) = \mathbb{Z}e \oplus \mathbb{Z}\sigma$;
- $e^2 = 0 \quad e.\sigma = 1 \quad \sigma^2 = -2.$
- $Nef(S) = \mathbb{R}_{\geq 0}.e + \mathbb{R}_{\geq 0}.(2e + \sigma)$; $\overline{NE}(S) = \mathbb{R}_{\geq 0}.e + \mathbb{R}_{\geq 0}.\sigma$.
- A spherical vector $v = (r, n\sigma + me, s)$ satisfies

$$-2 = \langle v, v \rangle = -2n^2 + 2nm - 2rs$$

For any given (r, n) satisfying $\gcd(r, n) = 1$, $\exists! m \mod r$. The last integer s is determined by (r, n, m).
As F is μ_H-stable $\iff$ $F(D)$, $F^\vee$ is stable.
We only need to compute $\mathsf{H}(r, n) := \mathsf{H}((r, n\sigma + me, s))$ for $\gcd(r, n) = 1$ and $n \in [0, r/2]$.

Numbers

$\mathsf{H}(r, n) := \mathsf{H}((r, n\sigma + me, s))$

- $\mathsf{H}(1, 0) = 1$;
- $\mathsf{H}(2, 1) = 2$ given by the extensions $\mathcal{O}_S(e) \to F_1 \to \mathcal{O}(\sigma - e)$ and $\mathcal{O}_S(\sigma - e) \to F_2 \to \mathcal{O}(e)$.
- $\mathsf{H}(3, 1) = 3$; $\mathsf{H}(4, 1) = 4$; $\mathsf{H}(5, 1) = 5$;
- $\mathsf{H}(5, 2) = 6$; the only case that is greater than the rank.
- $\mathsf{H}(7, 1) = \mathsf{H}(7, 2) = \mathsf{H}(7, 3) = 7$;
- $\mathsf{H}(8, 1) = 8$; $\mathsf{H}(8, 3) = 6$; $\mathsf{H}(9, 1) = 9$; $\mathsf{H}(9, 2) = 7$;$\mathsf{H}(9, 4) = 8$;

Proposition: $\mathsf{H}(r, 1) = r$.

Numbers

Results via the help of a computer.

- $\mathsf{H}(21, 2) = 13$; $\mathsf{H}(21, 8) = 11$;
- $\mathsf{H}(93, 2) = 49$; $\mathsf{H}(93, 4) = 30$; $\mathsf{H}(93, 10) = 26$;
- $\mathsf{H}(811, 92) = 48$; $\mathsf{H}(811, 93) = 47$; $\mathsf{H}(811, 94) = 46$;
- $\mathsf{H}(3, 1) = 3$; $\mathsf{H}(5, 2) = 6$; $\mathsf{H}(8, 3) = 6$; $\mathsf{H}(13, 5) = 11$; $\mathsf{H}(23, 8) = 11$; $\mathsf{H}(34, 13) = 18$; $\mathsf{H}(55, 21) = 18$; $\mathsf{H}(89, 34) = 27$; $\mathsf{H}(144, 55) = 27$; $\mathsf{H}(233, 89) = 38$.

Let $b_0 = b_1 = 1$ and $b_{n+1} = b_n + b_{n-1}$ be the Fibonacci sequence. Then $\mathsf{H}(b_{n+1}, b_{n-1}) = \lfloor \frac{n}{2} \rfloor^2 + 2$ when $n \geq 3$.

For a 'random' (R, N), it is expected that $\mathsf{H}(R, N) \sim (\ln R)^2$.

Asymptotic Behavior

We have the following conjecture on the asymptotic behavior of the counting numbers.

$$\min_{rk(v)=R}\{\mathsf{H}_S(v)\} \sim (\ln R)^2.$$
$$\text{Average}_{rk(v)=R}\{\mathsf{H}_S(v)\} \sim (\ln R)^2.$$

We can only prove a weaker version on the average estimation:

Proposition: For every $\alpha > 0$,

$$\frac{\phi(R)}{R}(\ln R)^2 \lesssim \text{Average}_{rk(v)=R}\{\mathsf{H}_S(v)\} \lesssim R^\alpha.$$

Here $\phi(m) = \sum_{\gcd(m,n)=1, 1\leq n\leq m} 1$ is the Euler totient function.

For a given spherical vector $v = (R, N\sigma + me, s)$ and $\gcd(R, r) = 1$, the number of actual walls with respect to spherical vectors w in the form of $(r, n\sigma + *e, *)$ is:

$$\#\left\{t \in \mathbb{Z}_{\geq 1} : (Rn - rN) | R^2 + r^2 - tRr > (Rn - rN)^2\right\}$$

This reduces the conjecture on the average estimation to an analytic number theory question.

$$A_R := \{R^2 + r^2 - tRr \in \mathbb{Z} \cap [1, R^2] \mid t \in \mathbb{Z}, 1 \leq r \leq R, \gcd(r, R) = 1\}.$$

$$\mathrm{G}(R) := \sum_{a \in A_m} \tau(a).$$

Here $\tau(m) := \sum_{d|m} 1$ is the divisor function.
Question: $\mathrm{G}(R) \sim (\ln R)(\# A_R)$?

Final remarks

A few remarks on $\tau(m)$.

- $\sum_{m \le R} \tau(m) = R \ln R + (2\gamma - 1)R + O(\sqrt{R})$.
- $\sum_{R \le m \le R + R^\alpha} \tau(m) \sim R^\alpha \ln R$.

We still do not know the answer to many questions even in this elliptic K3 surface case.

- Does there $\exists$ a spherical bundle never semistable?
- $\mathrm{H}'_S(v) := \#\{E \mid v(E) = v, \ E \text{ is a vector bundle.}\}$
  Q: $\mathrm{H}'_S(v) < +\infty$ finite or even $\mathrm{H}'_S(v) < c\,\mathrm{H}_S(v)$ for some constant $c > 0$.
- Question $(\star)$ at the beginning: Let E be a spherical object. Does there $\exists$ a stability condition σ so that E is σ-stable?

Total: 40/100

Thank you!