Control point modifications of PH curves

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joint work with R.T. Farouki and M.L. Sampoli

OUTLINE

Motivation

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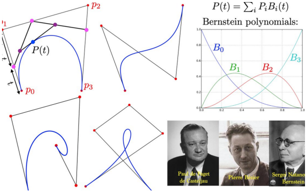
Examples & Closure



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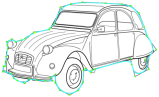

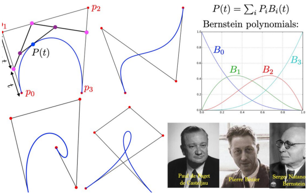

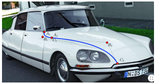

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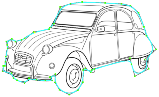
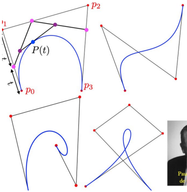
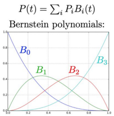
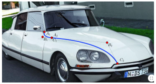
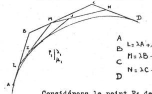








- The origins of the **control–polygon paradigm** for constructing and manipulating free–form curves can be traced to the pioneering ideas of Paul de Casteljau and Pierre Bézier
- De Casteljau's *courbes et surfaces à pôles*, based on using **pilot points** to design curves and surfaces



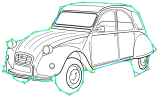
1.5.- Sous-Pôles d'une courbe

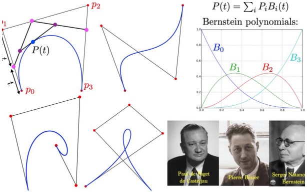
1.5.1.- Définition des sous-pôles. Considérons une cubique des pôles A, B, C, D. Nous avons vu précédemment que la construction par la méthode des barycentres donnent les différents points L, M, N, I, J, P.

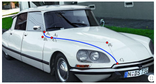
$L = \lambda A + \mu B$
$M = \lambda B + \mu C$
$N = \lambda C + \mu D$
$I = \lambda^2 A + 2\lambda\mu B + \mu^2 C$
$J = \lambda^2 B + 2\lambda\mu C + \mu^2 D$
$P = \lambda^3 A + 3\lambda^2\mu B + 3\lambda\mu^2 C + \mu^3 D$

Considérons le point P_1 de paramètres λ_1 et μ_1 (avec $\lambda_1 + \mu_1 = 1$) (λ_1 varie de 0 à 1 et μ_1 1 à 0, lorsque P va en D).

Cherchons les pôles de la cubique P_1 D. Cette courbe dérive de la cubique initiale AD par changement des paramètres.







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- The focus of the present study is to elucidate use of the control–polygon paradigm in the context of the **planar Pythagorean–Hodograph (PH) curves**

Motivation

- The **algebraic structure** of the PH curves facilitates an **exact computation** of various properties that otherwise necessitate numerical approximations:
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  - *ab initio* constructions of PH curves matching specified geometrical data, rather than *a posteriori* modification of existing PH curves **[Farouki, Jaklic, Jüttler, Kosinka, Giannelli, Manni, Pelosi, Pottmann, Sampoli, Sestini, Sir, Walton, ...]**

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    [Kim, 2017, Moon 2020]
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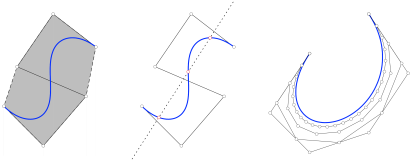
        - the convell hull
        - variation–diminishing properties
        - association of a unique curve with any given control polygon



Planar quintic PH curves

- **complex representation:** a planar PH quintic $\mathbf{r}(t)$ is generated from a quadratic pre–image polynomial $\mathbf{w}(t)$

$$\mathbf{w}(t) \;=\; \mathbf{w}_0\, b_0^2(t) + \mathbf{w}_1\, b_1^2(t) + \mathbf{w}_2\, b_2^2(t)$$

  - $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$: complex coefficients;
  - $b_i^n(t) \;=\; \binom{n}{i}(1-t)^{n-i}t^i\,, \quad i=0,\ldots,n$: Bernstein basis on $t \in [\,0,1\,]$

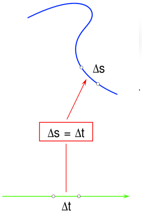
- by integrating the expression $\mathbf{r}'(t) = \mathbf{w}^2(t)$ yields the complex control points $\mathbf{p}_0,\ldots,\mathbf{p}_5$ of the Bézier form

$$\mathbf{r}(t) \;=\; \sum_{k=0}^{5} \mathbf{p}_k\, b_k^5(t)$$

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \tfrac{1}{5}\,\mathbf{w}_0^2\,,\\
\mathbf{p}_2 &= \mathbf{p}_1 + \tfrac{1}{5}\,\mathbf{w}_0\mathbf{w}_1\,,\\
\mathbf{p}_3 &= \mathbf{p}_2 + \tfrac{1}{5}\,\frac{2\,\mathbf{w}_1^2+\mathbf{w}_0\mathbf{w}_2}{3}\,,\\
\mathbf{p}_4 &= \mathbf{p}_3 + \tfrac{1}{5}\,\mathbf{w}_1\mathbf{w}_2\,,\\
\mathbf{p}_5 &= \mathbf{p}_4 + \tfrac{1}{5}\,\mathbf{w}_2^2\,,
\end{aligned}$$

where $\mathbf{p}_0$ is a freely–chosen integration constant.

Planar quintic PH curves: properties



- **Polynomial parametric speed**

$$\sigma(t) = |\mathbf{r}'(t)| = |\mathbf{w}(t)|^2$$

the derivative $\mathrm{d}s/\mathrm{d}t$ of arc length s with respect to the curve parameter t.

- The **curvature** may be expressed as

$$\kappa(t) \;=\; 2\,\frac{\mathsf{Im}(\overline{\mathbf{w}}(t)\mathbf{w}'(t))}{|\mathbf{w}(t)|^4}\;.$$

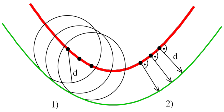
PH quintic:

- the numerator is the quadratic polynomial
  $2\,\mathsf{Im}(\overline{\mathbf{w}}_0\mathbf{w}_1)\,b_0^2(t) - \mathsf{Im}(\overline{\mathbf{w}}_2\mathbf{w}_0)\,b_1^2(t) + 2\,\mathsf{Im}(\overline{\mathbf{w}}_1\mathbf{w}_2)\,b_2^2(t)$
- (odd–multiplicity) real roots, if any, identify *inflections* of $\mathbf{r}(t)$ according with the sign of

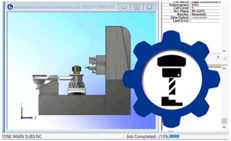
$$\Delta \;=\; \mathsf{Im}^2(\overline{\mathbf{w}}_2\mathbf{w}_0) - 4\,\mathsf{Im}(\overline{\mathbf{w}}_0\mathbf{w}_1)\,\mathsf{Im}(\overline{\mathbf{w}}_1\mathbf{w}_2)$$

  - two inflections for $\Delta > 0$
  - none if $\Delta < 0$
  - for $\Delta = 0$: double root, where $\kappa(t) = \kappa'(t) = 0$

Planar quintic PH curves: properties



- **rational offset curves** $\mathbf{r}_d(t) = \mathbf{r}(t) + d\mathbf{n}(t)$
  - defines center–line tool path, in order to cut a desired profile
  - defines tolerance zone characterizing allowed variations in part shape
  - defines erosion, dilation operators in mathematical morphology, image processing, geometrical smoothing procedures, etc.
- closed-form evaluation of **energy integral** $E = \int_0^1 \kappa^2 \mathrm{d}s$
- real–time CNC interpolators, rotation-minimizing frames, etc.



Planar quintic PH curves

- **Complex control–polygon legs** of $\mathbf{r}(t)$:

$$\mathbf{L}_i \ = \ \mathbf{p}_i - \mathbf{p}_{i-1}\,, \quad i = 1, \ldots, 5, \qquad \boxed{\mathbf{L}_1 + \cdots + \mathbf{L}_5 = 1}$$

  - **canonical form** to simplify the construction and shape analysis:
    - invoke a translation/rotation/scaling transformation to eliminate all non–essential degrees of freedom:
    - $\mathbf{r}(0) = (0, 0)$ and $\mathbf{r}(1) = (1, 0)$

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- the control–polygon legs are related to the coefficients $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$

$$\left(\mathbf{w}_0^2, \mathbf{w}_0\mathbf{w}_1, \frac{2\,\mathbf{w}_1^2 + \mathbf{w}_0\mathbf{w}_2}{3}, \mathbf{w}_1\mathbf{w}_2, \mathbf{w}_2^2\right) = 5\,(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4, \mathbf{L}_5)\,.$$

Control Polygon PH–constraints

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  - degree $\frac{1}{2}(n-1)$ pre–image polynomial $\mathbf{w}(t)$
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- **1** for a **PH cubics**:
  - the simplest non–trivial PH curves, which are identified by

$$\mathbf{L}_2^2 = \mathbf{L}_1 \mathbf{L}_3$$

translated/scaled/rotated segments of a unique non–inflectional curve — *Tschirnhaus cubic* **[Farouki, 1990]**

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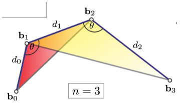

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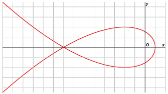

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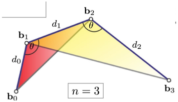

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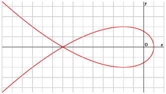

translated/scaled/rotated segments of a unique non–inflectional curve — *Tschirnhaus cubic* **[Farouki, 1990]**

- 2 for a **PH quintics**
  - are the lowest–order PH curves that are generally considered to be suitable for **free–form design** applications.

Control Polygon PH–constraints

Proposition [Farouki-1994]

Sufficient and necessary conditions for a quintic Bézier curve to be a PH curve is the satisfaction of

$$\mathbf{L}_1\mathbf{L}_4^2 = \mathbf{L}_5\mathbf{L}_2^2$$

and any one of the four equations

$$\begin{aligned}
3\,\mathbf{L}_1\,\mathbf{L}_2\,\mathbf{L}_3 - \mathbf{L}_1^2\,\mathbf{L}_4 - 2\,\mathbf{L}_2^3 &= 0\\
3\,\mathbf{L}_5\,\mathbf{L}_4\,\mathbf{L}_3 - \mathbf{L}_5^2\,\mathbf{L}_2 - 2\,\mathbf{L}_4^3 &= 0\\
3\,\mathbf{L}_1\,\mathbf{L}_4\,\mathbf{L}_3 - \mathbf{L}_5\,\mathbf{L}_1\,\mathbf{L}_2 - 2\,\mathbf{L}_2^2\,\mathbf{L}_4 &= 0\\
3\,\mathbf{L}_5\,\mathbf{L}_2\,\mathbf{L}_3 - \mathbf{L}_1\,\mathbf{L}_5\,\mathbf{L}_4 - 2\,\mathbf{L}_4^2\,\mathbf{L}_2 &= 0
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Canonical–form

$$\mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \mathbf{L}_4 + \mathbf{L}_5 = 1$$

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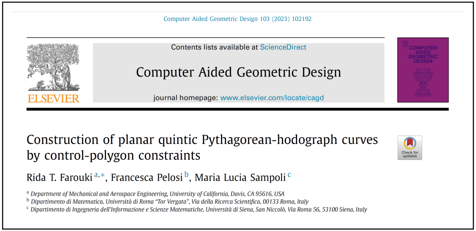
Control Polygon PH-construction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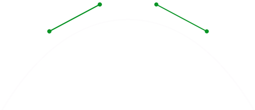

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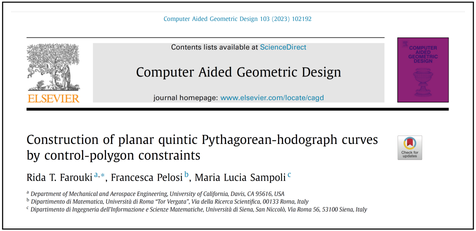
Control Polygon PH-construction



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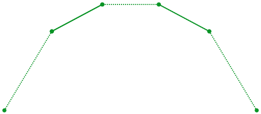

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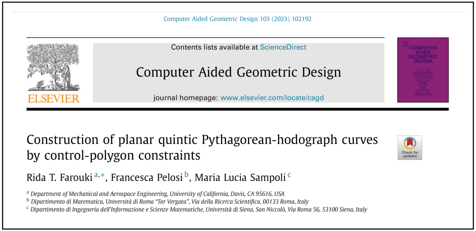
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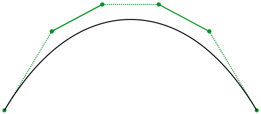
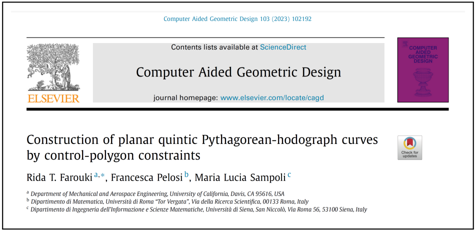


- A **canonical–form quintic PH curve** in complex form

  embodies **two free complex parameters** that must be chosen so as to ensure that its five control–polygon legs satisfy the **(1)-(2)** **constraints** that identify quintic PH curves
- Fixing **two control legs**, the remaining three can be **filled in** by a simple algorithm that requires only the solution of a **quadratic or quartic equation** with complex coefficients.

Control Polygon PH-construction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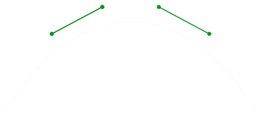

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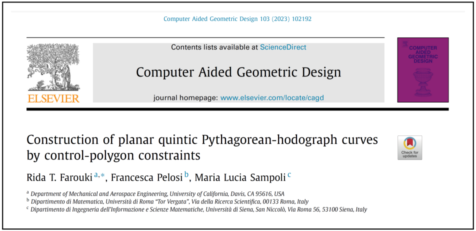

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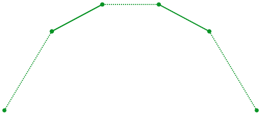

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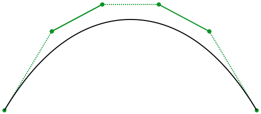

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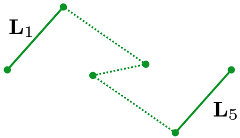
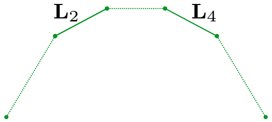
- Several examples illustrate how this approach can be employed in the **practical design** of planar PH quintics with desired shape features



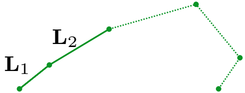
Hermite problem

4 distinct PH quintics



symmetric control polygon

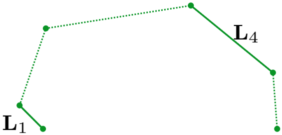
2 distinct PH curves



Assigned initial curvature

$$\kappa(0) = \frac{4}{5}\frac{(\mathbf{L}_1 \times \mathbf{L}_2)\cdot \mathbf{k}}{|\mathbf{L}_1|^3},$$

2 distinct PH quintics



$\mathbf{L}_1$, $\mathbf{L}_4$

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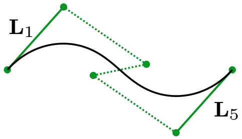

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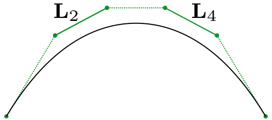

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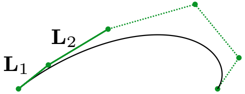

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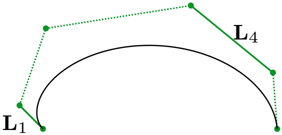

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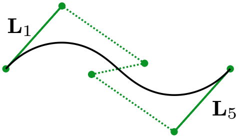

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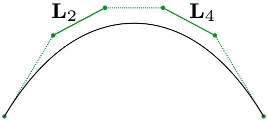

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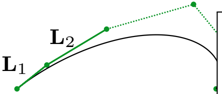

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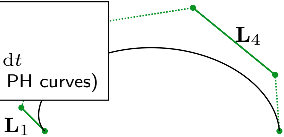
2 distinct PH quintics

the **good** PH has smallest
absolute rotation index
$R_{\text{abs}} = \frac{1}{2\pi}\int_0^1 |\kappa(t)|\,\sigma(t)\;\mathrm{d}t$
(exact evaluation for quintic PH curves)



$\mathbf{L}_1$, $\mathbf{L}_4$

4 distinct PH curves

Control Polygon PH-modification

- **a posteriori modification** of quintic PH curves:
  - intuitive approach of **displacing a subset of the control points**,

    $$\widetilde{\mathbf{p}}_k = \mathbf{p}_k + \boxed{\Delta \mathbf{p}_k}, \qquad k = 0, \ldots, 5$$

    the control polygon legs become

    $$\widetilde{\mathbf{L}}_k = \mathbf{L}_k + \boxed{\Delta \widetilde{\mathbf{L}}_k}, \quad k = 1, \ldots, 5,$$

    where $\Delta \widetilde{\mathbf{L}}_k := \Delta \mathbf{p}_k - \Delta \mathbf{p}_{k-1}$.
  - the control polygon legs $\widetilde{\mathbf{L}}_k$ must also satisfy the PH-constraints **(1)-(2)**

$\Rightarrow$ a system of equations that identify the **admissible** displacements $\boxed{\Delta \mathbf{p}_k}$

- for general PH:
  2 cubic constraints $\Rightarrow$ at least 2 non-zero displacements to obtain a different PH $\widetilde{\mathbf{r}}(t)$

Control Polygon PH-modification

- $\boxed{=2}$ **modified control points** $\Rightarrow$ finite number of modified PH quintics

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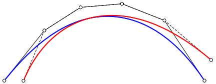
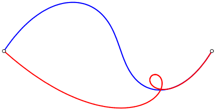
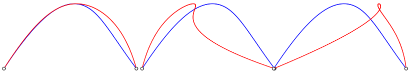


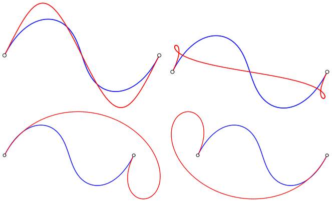
$\mathbf{p}_0,\ \mathbf{p}_5$
1 (good) solution



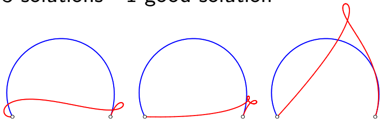
$\mathbf{p}_1,\ \mathbf{p}_2$
1 (unsatisfactory) solution



$\mathbf{p}_1,\ \mathbf{p}_4$
3 solutions - 1 good solution



$\mathbf{p}_2,\ \mathbf{p}_3$
4 solutions (1 good)



$\mathbf{p}_1,\ \mathbf{p}_3$
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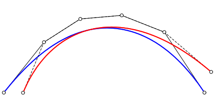
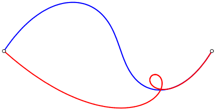
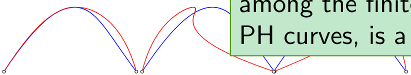


$\mathbf{p}_0,\ \mathbf{p}_5$
1 (good) solution

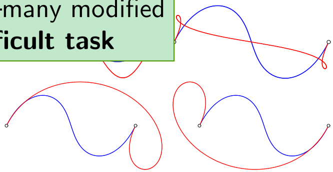


$\mathbf{p}_1,\ \mathbf{p}_2$

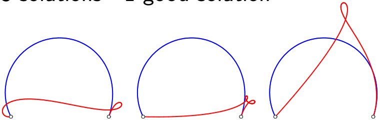
obtaining quintic PH curves with **predictably good shape** among the finitely–many modified PH curves, is a **difficult task**



$\mathbf{p}_1,\ \mathbf{p}_4$
3 solutions - 1 good solution



$\mathbf{p}_2,\ \mathbf{p}_3$
4 solutions (1 good)



$\mathbf{p}_1,\ \mathbf{p}_3$
3 (unsatisfactory) solutions

Control Polygon PH-modification

- > 2 **modified control points**
  - number of unknowns exceeds the number of constraints
  - infinitely–many modifications $\widetilde{\mathbf{r}}(t)$ are possible

$\Rightarrow$ exploit the excess freedoms in **optimizing a shape measure** for the modified curve

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- $\boxed{1}$ **fixed interior displacement** $\Delta\mathbf{p}_\ell$

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- $\mathbf{p}_0,\ \mathbf{p}_5$: fixed in canonical position
- 1 **fixed interior displacement** $\Delta\mathbf{p}_\ell$
- 3 **complex unknown displacement** $\Delta\mathbf{p}_i,\ \Delta\mathbf{p}_j,\ \Delta\mathbf{p}_k$ by

$$\min_{\Delta\mathbf{p}_i,\Delta\mathbf{p}_j,\Delta\mathbf{p}_k} F(\Delta\mathbf{p}_i,\Delta\mathbf{p}_j,\Delta\mathbf{p}_k)$$

$+$ subjects to PH constraints **(1)-(2)**

Control Polygon PH-modification

Penalty function:

- expect the shape changes localized to the vicinity of the modified control point $\mathbf{p}_\ell$;
- minimize the distance $\Delta\mathbf{r}$ between $\widetilde{\mathbf{r}}(t)$ and $\mathbf{r}(t)$ after imposing one displacement $\Delta\mathbf{r}(t) = \Delta\mathbf{p}_i b_i^5(t) + \Delta\mathbf{p}_j b_j^5(t) + \Delta\mathbf{p}_k b_k^5(t)$

$$\min_{\Delta\mathbf{p}_i, \Delta\mathbf{p}_j, \Delta\mathbf{p}_k} \int_0^1 |\Delta\mathbf{r}(t)|^2 \mathrm{dt}$$

considering the proportional expression:

Penalty function

$$F(\Delta\mathbf{p}_i, \Delta\mathbf{p}_j, \Delta\mathbf{p}_k) = C_{ii}|\Delta\mathbf{p}_i|^2 + C_{jj}|\Delta\mathbf{p}_j|^2 + C_{kk}|\Delta\mathbf{p}_k|^2 + 2\,\mathrm{Re}(C_{ij}\Delta\mathbf{p}_i\Delta\overline{\mathbf{p}}_j + C_{jk}\Delta\mathbf{p}_j\Delta\overline{\mathbf{p}}_k + C_{ki}\Delta\mathbf{p}_k\Delta\overline{\mathbf{p}}_i)$$

Control Polygon PH-modification

PH-constraints for the **modified** PH curve

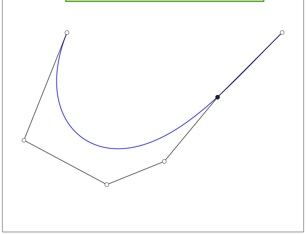
2 cubic complex PH–constraints in $\Delta \mathrm{L}_k = \Delta \mathrm{p}_k - \Delta \mathrm{p}_{k-1}$

$$2\,\mathbf{L}_4(\mathbf{L}_1 + \Delta\mathbf{L}_1)\Delta\mathbf{L}_4 - 2\,\mathbf{L}_2(\mathbf{L}_5 + \Delta\mathbf{L}_5)\Delta\mathbf{L}_2 + (\mathbf{L}_1 + \Delta\mathbf{L}_1)\Delta\mathbf{L}_4^2 - (\mathbf{L}_5 + \Delta\mathbf{L}_5)\Delta\mathbf{L}_2^2 + \mathbf{L}_4^2\Delta\mathbf{L}_1 - \mathbf{L}_2^2\Delta\mathbf{L}_5 = 0\,, \quad \textbf{(1)}$$

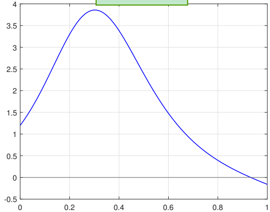
$$+ (3\,\mathbf{L}_2\mathbf{L}_3 - 2\,\mathbf{L}_1\mathbf{L}_4)\Delta\mathbf{L}_1 + 3\,(\mathbf{L}_1\mathbf{L}_3 - 2\,\mathbf{L}_2^2 + \mathbf{L}_3\Delta\mathbf{L}_1)\Delta\mathbf{L}_2 + 3\,\mathbf{L}_2(\mathbf{L}_1 + \Delta\mathbf{L}_1)\Delta\mathbf{L}_3 - \mathbf{L}_1(\mathbf{L}_1 + 2\,\Delta\mathbf{L}_1)\Delta\mathbf{L}_4 - (\mathbf{L}_4 + \Delta\mathbf{L}_4)(\Delta\mathbf{L}_1)^2 - 2\,(3\,\mathbf{L}_2 + \Delta\mathbf{L}_2)(\Delta\mathbf{L}_2)^2 + 3\,(\mathbf{L}_1 + \Delta\mathbf{L}_1)\Delta\mathbf{L}_2\Delta\mathbf{L}_3 = 0\,. \quad \textbf{(2)}$$

Example 1: data with inflection

PH - curve and CP



$\kappa-$ plot



$$\kappa(t) = 2\,\frac{\operatorname{Im}(\overline{\mathbf{w}}(t)\mathbf{w}'(t))}{|\mathbf{w}(t)|^4}$$

inflection near $\mathbf{r}(1)$

Example 1: data with inflection

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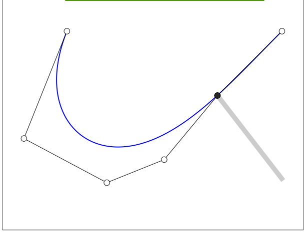

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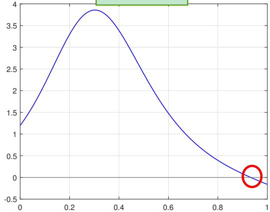

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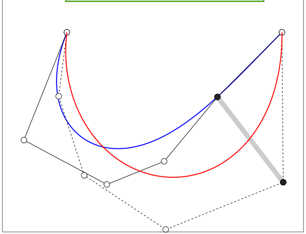

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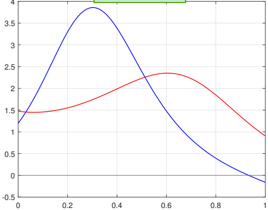


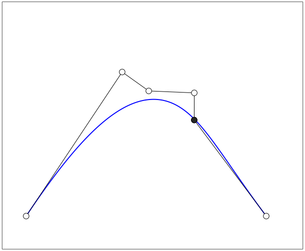
$$\kappa(t) = 2\frac{\mathrm{Im}(\overline{\mathbf{w}}(t)\mathbf{w}'(t))}{|\mathbf{w}(t)|^4}$$

inflection near $\mathbf{r}(1)$

$\Delta\mathbf{p}_4 = 0.31 - 0.39\mathrm{i}$
$|\Delta\mathbf{p}_4| = 0.5$
Eq.(1)-(2)$= 1\mathrm{e}-16$

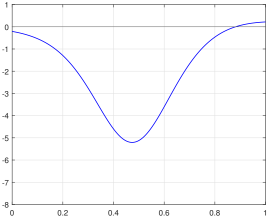
Example 2: data with inflection

Large $|\Delta \mathbf{p}_4|$ $(= 0.5)$



inflection near $\mathbf{r}(1)$

$\Delta \mathbf{p}_4 = 0.352 + 0.354i$
$|\Delta \mathbf{p}_4| = 0.5$



Curvature

Example 2: data with inflection

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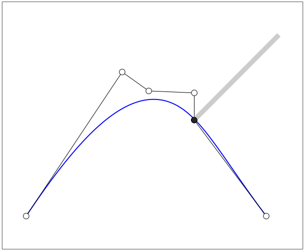

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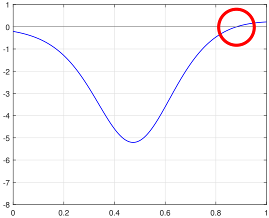

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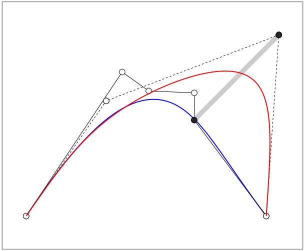

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Eq.(1)-(2)$= 1e - 10$
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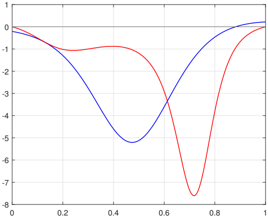
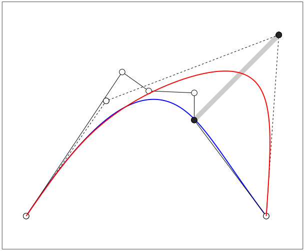


Curvature

Example 2: data with inflection

Large $|\Delta \mathbf{p}_4|$ $(= 0.5)$



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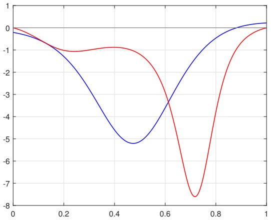

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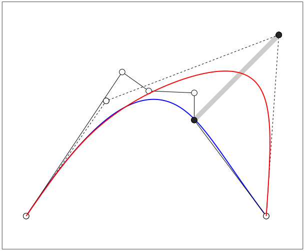

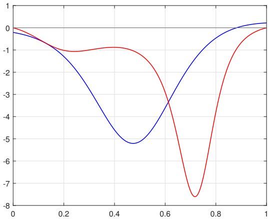

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$\Rightarrow$ **sequence of smaller steps,**
modified PH used as input,
in a predictor-corrector scheme,
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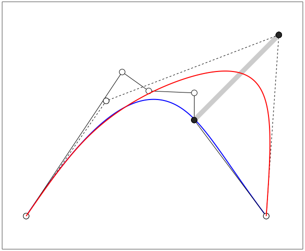

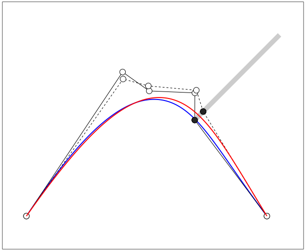

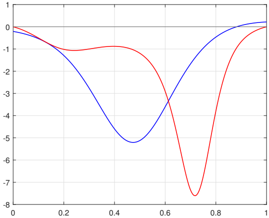

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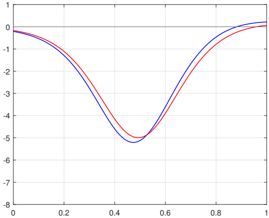

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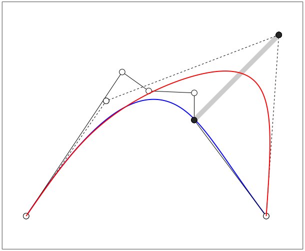

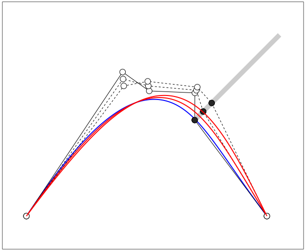

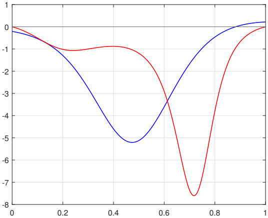

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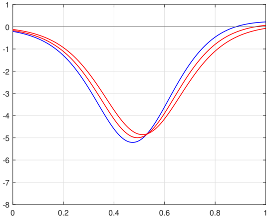

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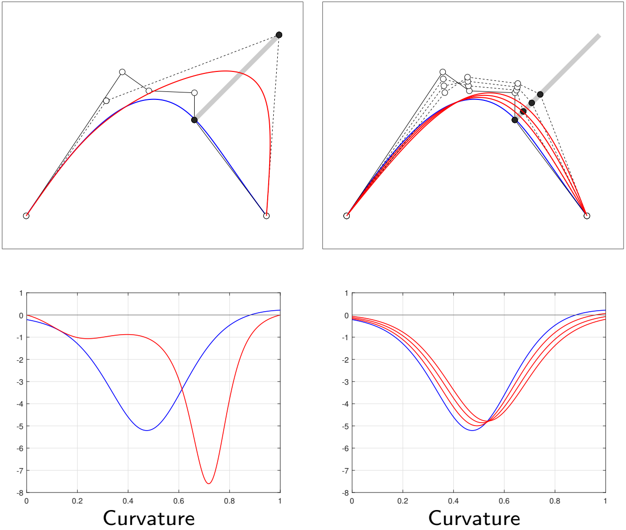

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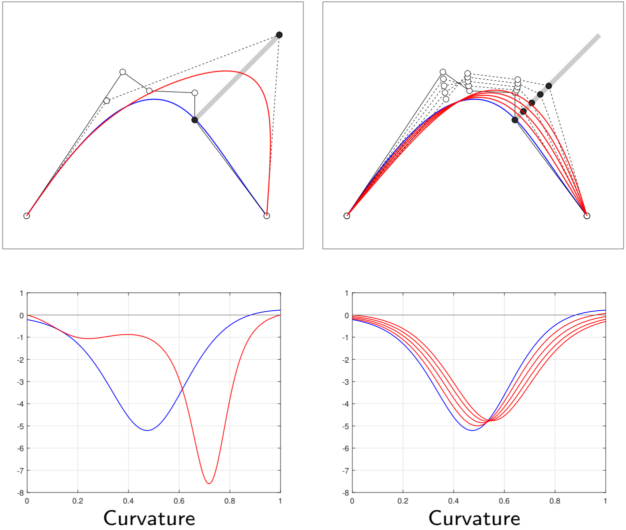

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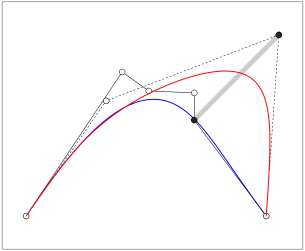

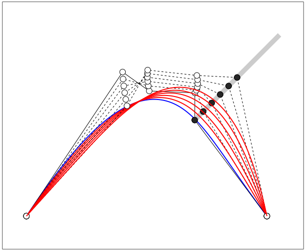

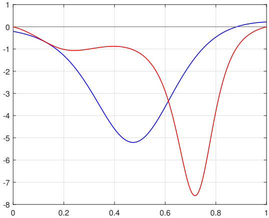

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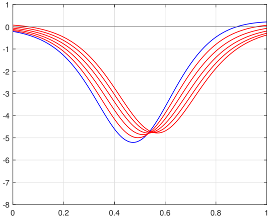

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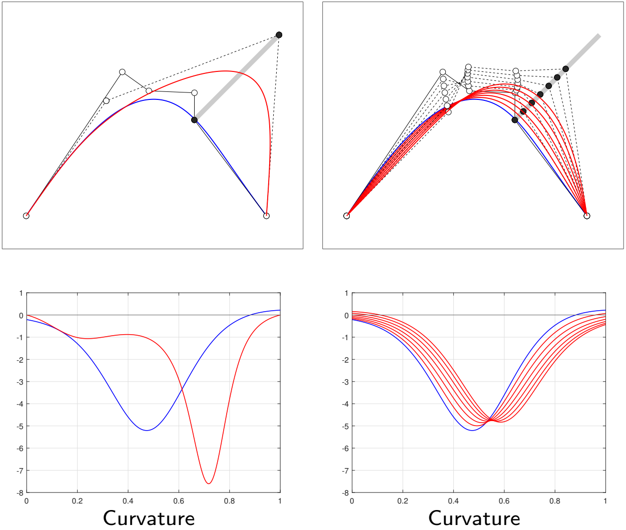

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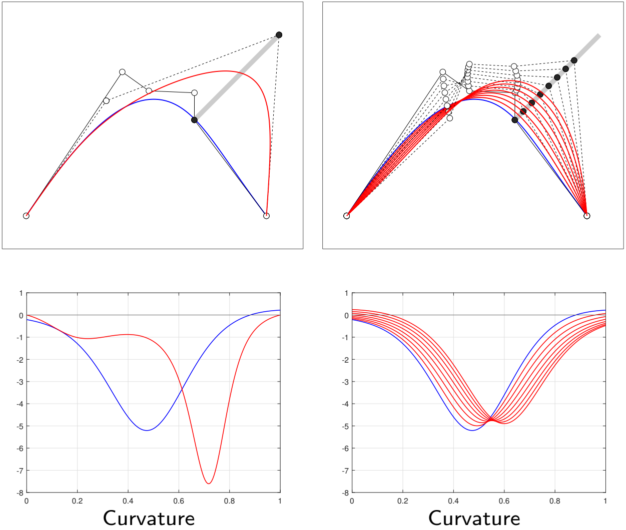

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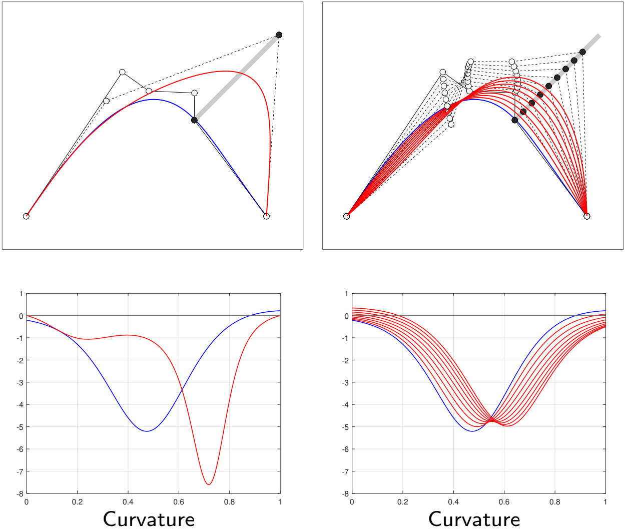

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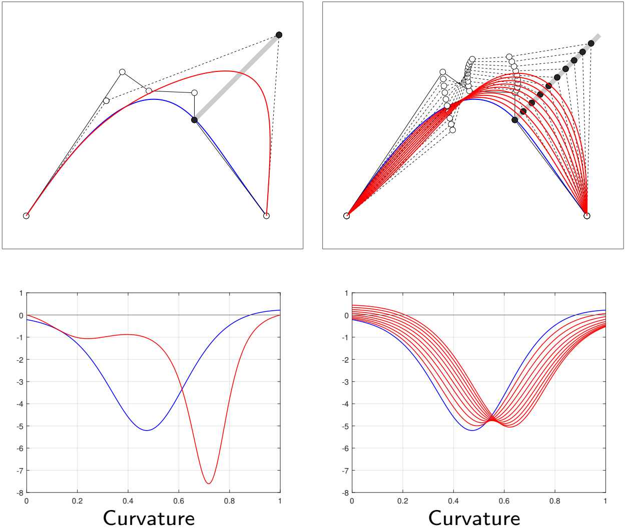

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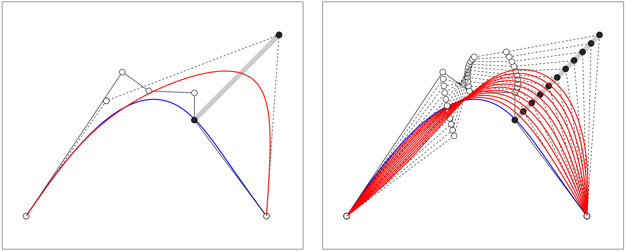

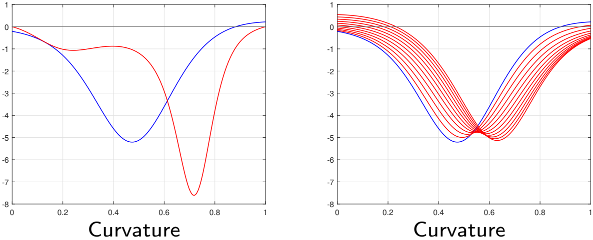

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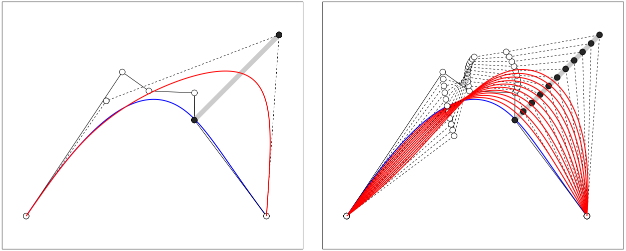
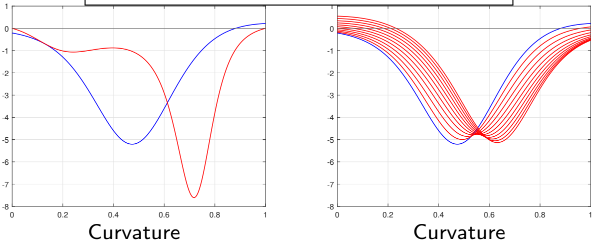


Smooth transition, regular control polygons



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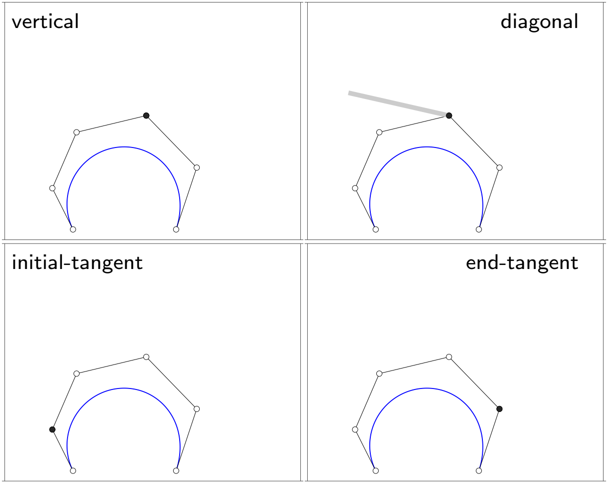

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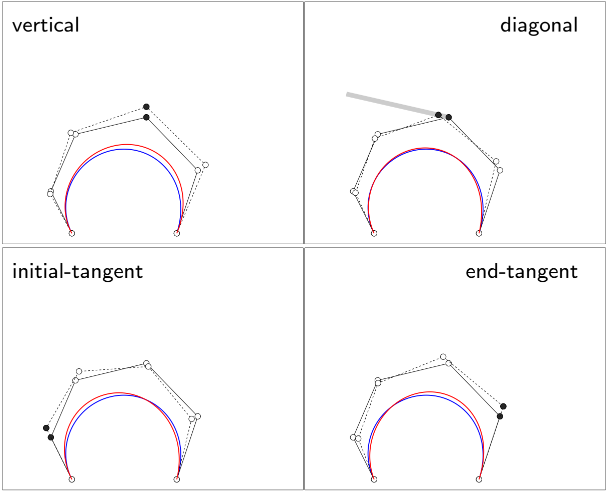

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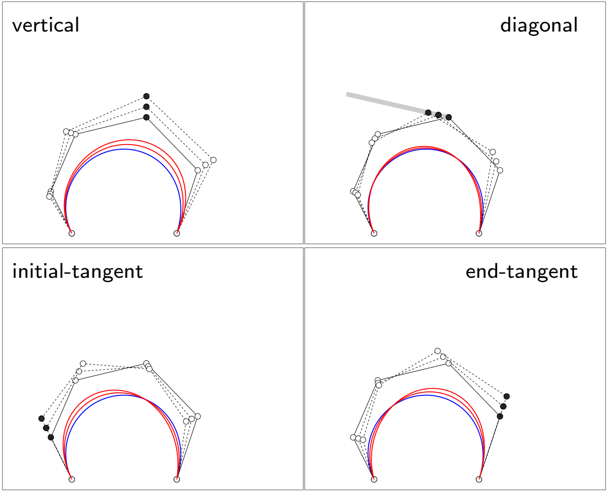

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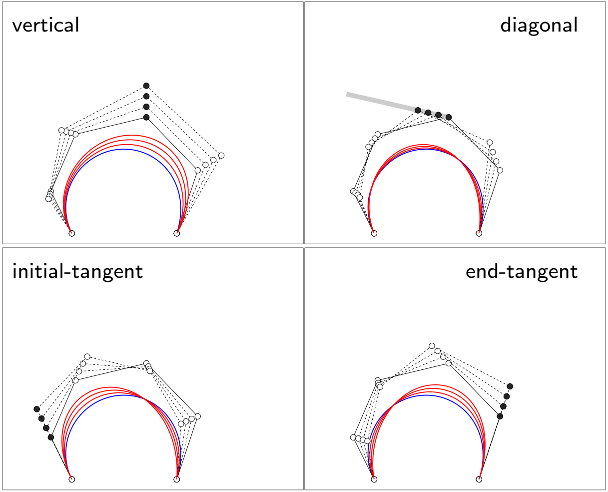

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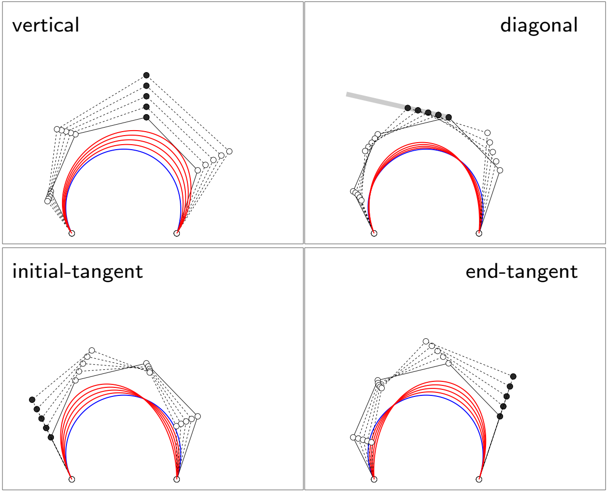

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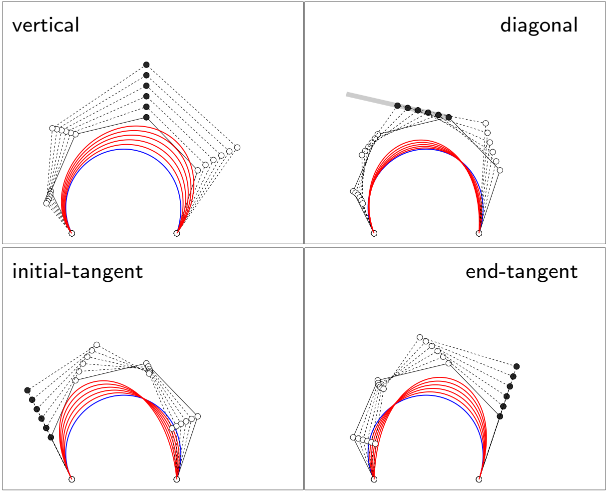

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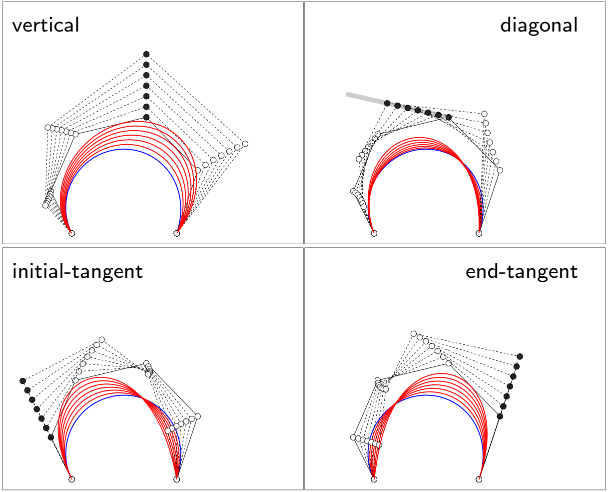

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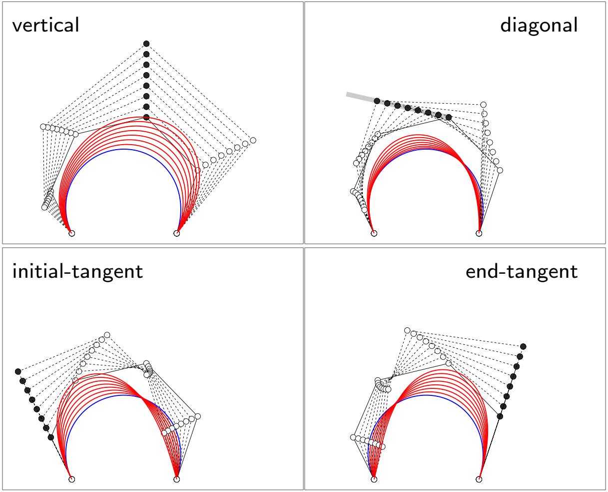

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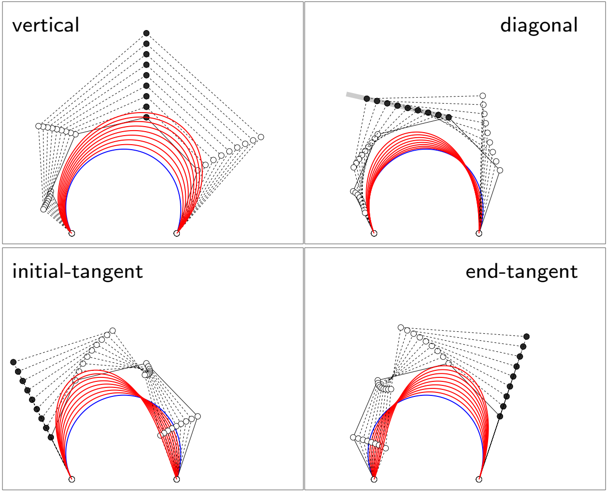

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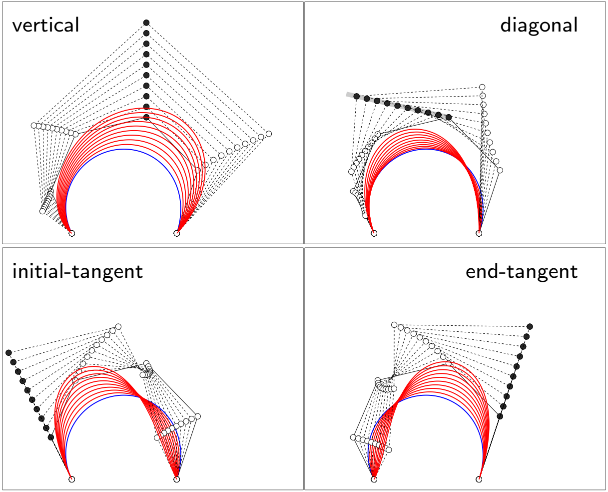

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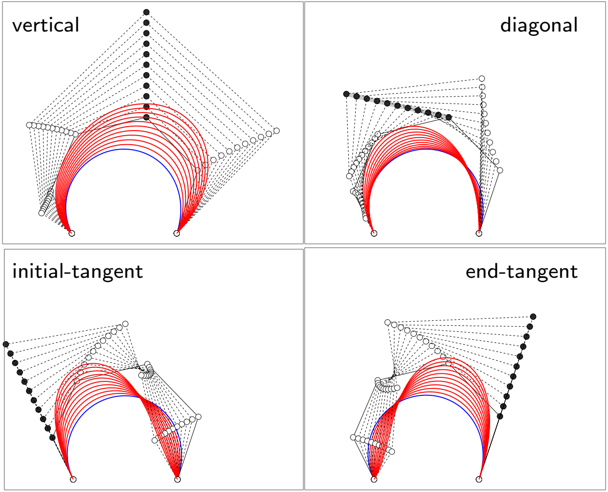

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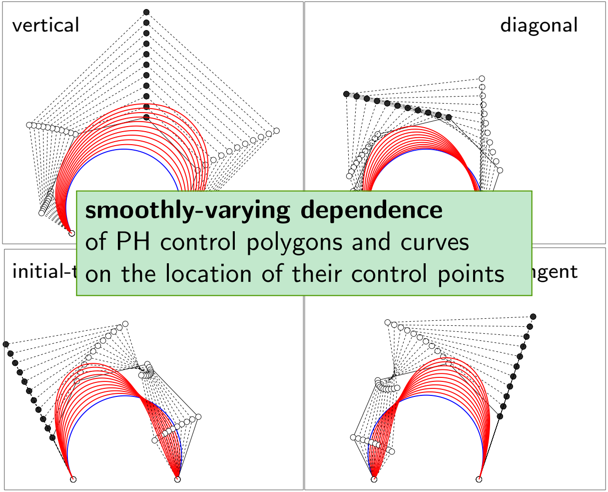

smoothly-varying dependence
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Example 4: inflectional data

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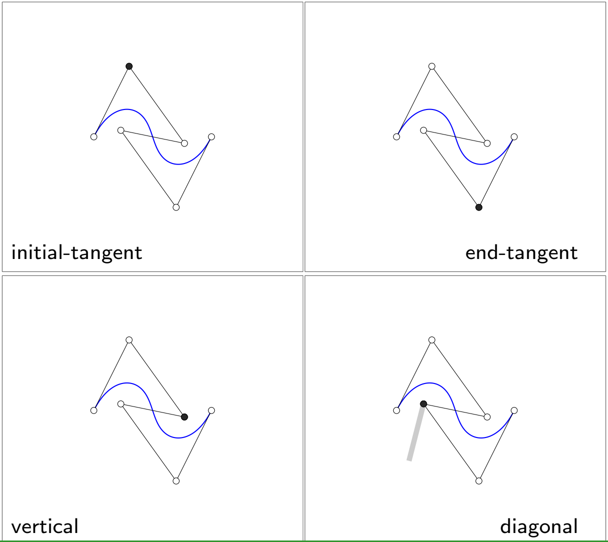

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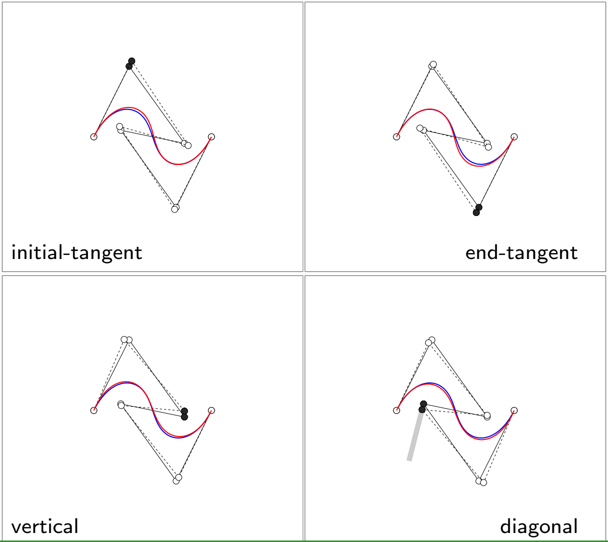

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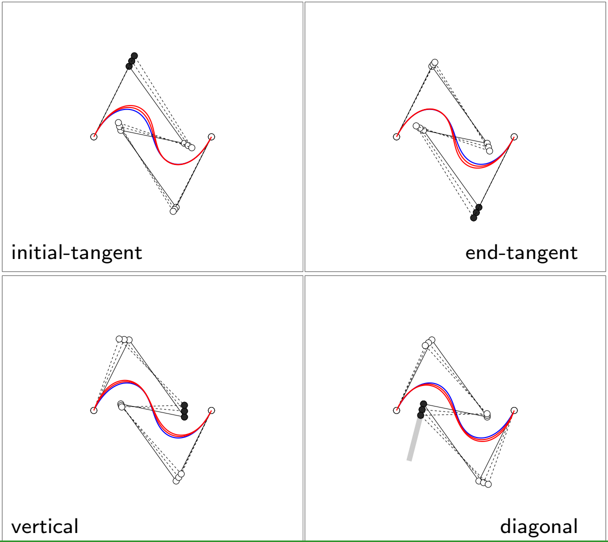

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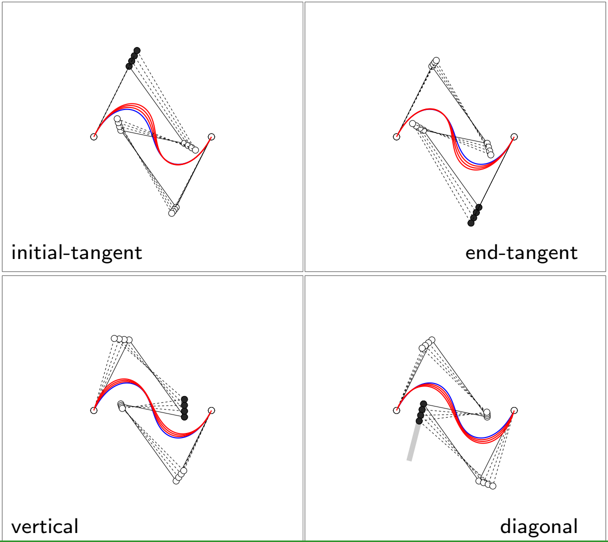

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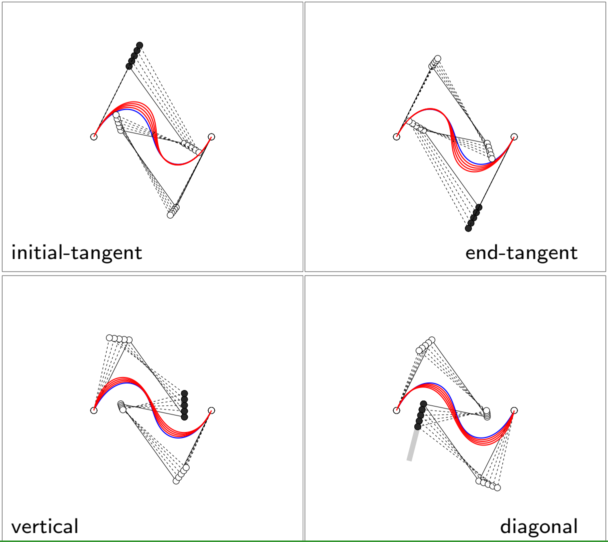

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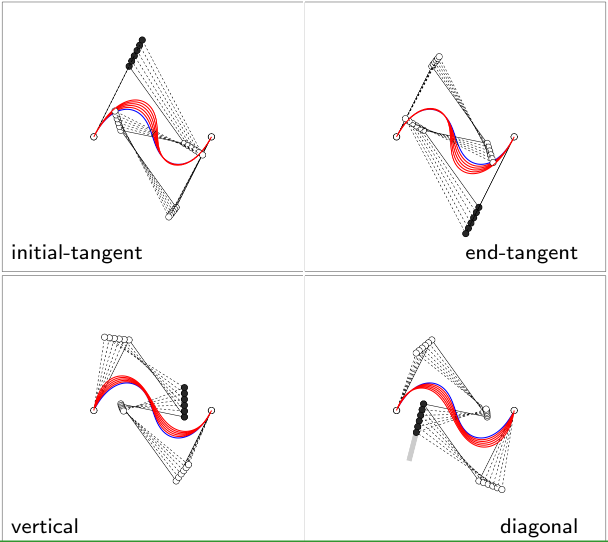

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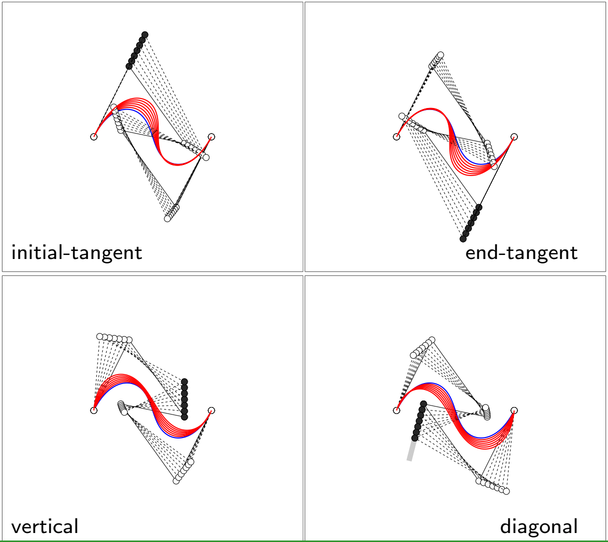

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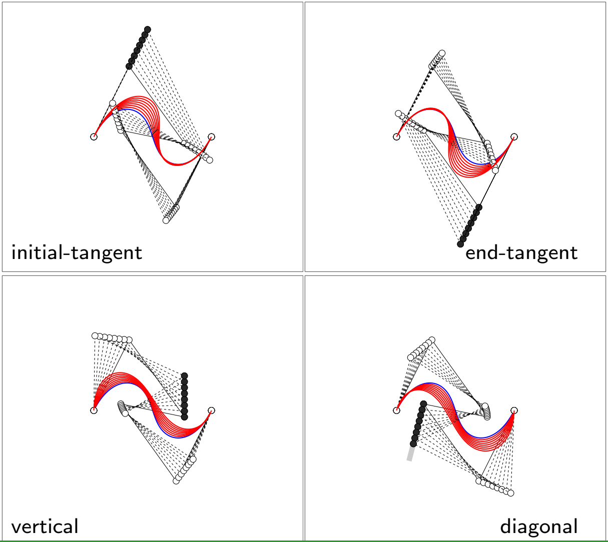

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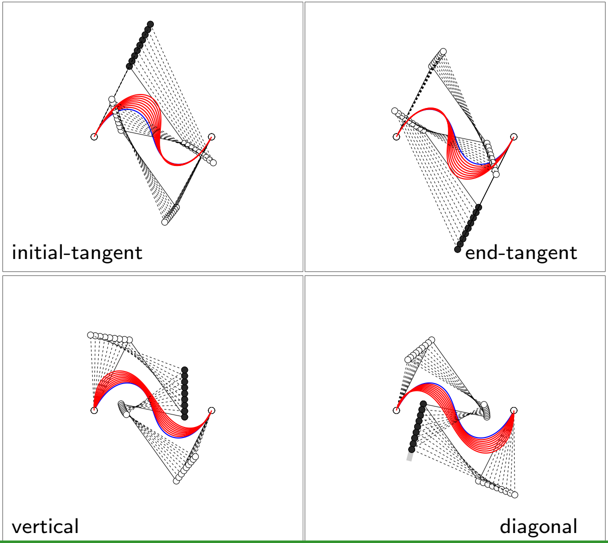

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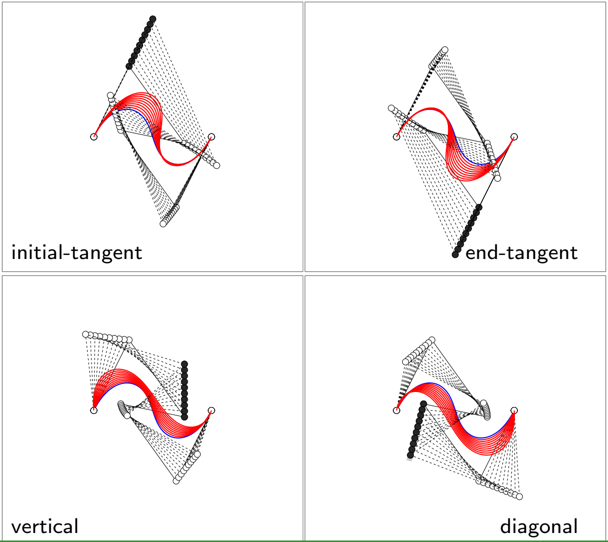

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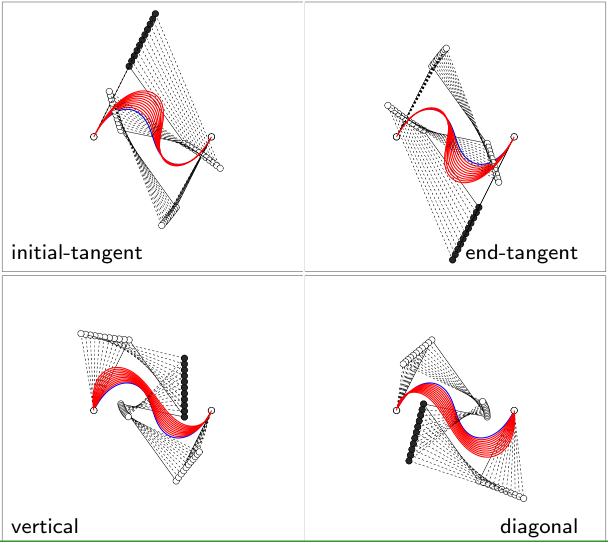

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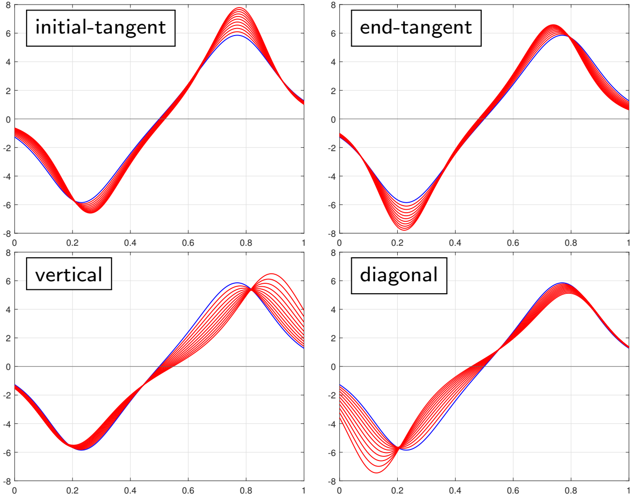
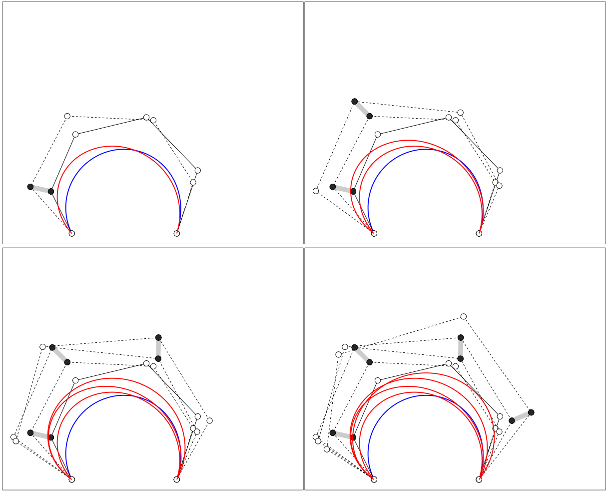


Smooth transition of the curvature plot

Example 5: sequential displacements

- All interior control points are sequentially modified ($|\Delta \mathbf{p}_i| = 0.2$)



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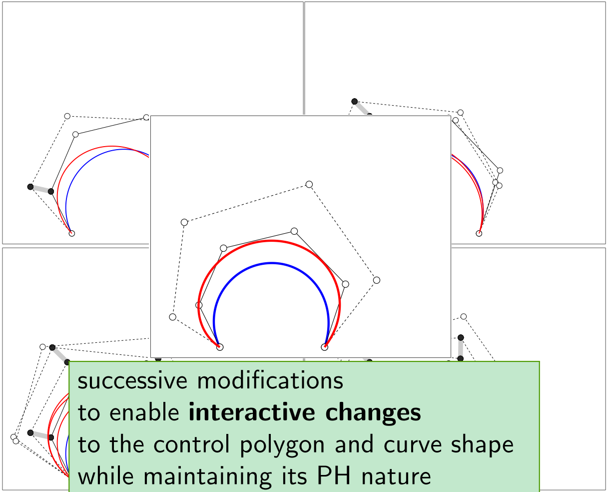

successive modifications
to enable **interactive changes**
to the control polygon and curve shape
while maintaining its PH nature

Summarizing: we have presented ...

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Summarizing: we have presented ...

- practical and efficient means for the **modification of planar PH quintics through the control points**:
  - the **displacement of a single interior control point** is considered
  - the remaining interior control points are exploited as **free parameters** to minimize a modified/original PH curves **distance**
  - subject to satisfaction of the PH constraints
- it works
  best for **displacements of reasonable magnitude**
- larger modifications may also be achieved incrementally
  - using **successive applications**
  - the output of each step serving as input for the next step
- modifications to **more** than one of the interior control points can be **sequentially** implemented:
  - the optimization process is sufficiently **fast** (0.015 sec.) to admit **real–time user modification and display** of planar quintic PH curves

⇒ **a rich set of "neighboring" PH quintics** that have the same end points

... spatial case?

- Although it seems natural to seek a generalization of the methodology to **spatial PH curves**, this is not a trivial task
  - **no system of control–polygon constraints** for the spatial PH quintics is currently known.
  - Moreover in the quaternion representation, the spatial PH quintic interpolants to given first–order Hermite data comprise a **two–parameter family** rather than a discrete set as in the planar case.

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Thanks for the attention!!

... from Arcachon

Happy Birthday Tom!!