Sampling projections in the uniform norm and optimal function recovery

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- Provide new relations between the Sampling, Gelfand and Kolmogorov numbers. Optimality of information: **linear vs standard**.
- Give new **discretization results**.
- Discuss **the bounds for the norms of projection operators** (related to the Kadets-Snobar theorem and Auerbach's/Novak's lemma).

Joint work with David Krieg, Mario Ullrich and Tino Ullrich

- *Sampling projections in the uniform norm*. arXiv: 2401.02220, 2024.
- *Sampling recovery in L_2 and other norms*. arXiv: 2305.07539, 2023.

+ **Work in progress with Felix Bartel and Tino Ullrich**

Information $\longrightarrow$ Recovery algorithm $\longrightarrow$ Reconstructed function

$N(f)$ $\qquad$ ψ $\qquad$ $f \approx \psi \circ N(f)$

- linear
  - Fourier coefficients
  - function values
- nonlinear

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  - non-adaptive
  - adaptive

$$\left\| f - \psi \circ N(f) \right\|_G \to \min$$

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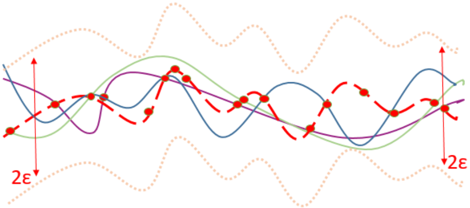
Given: $f(x_1), ..., f(x_n)$, $x_i \in D$, $i = 1, \dots, n$, for $f \in F \subset G$.

Estimate: the ***n*-th linear sampling number** of F in G

$$g_n^{\mathrm{lin}}(F, G) := \inf_{\substack{x_1,\dots,x_n \in D \\ \varphi_1,\dots,\varphi_n \in G}} \sup_{f \in F} \left\| f - \sum_{i=1}^{n} f(x_i)\, \varphi_i \right\|_G .$$



Red line – target, other lines – approximants

***n*-th Gelfand number** $c_n(F, G)$: **any** algorithm, **linear** information

***n*-th approximation number** $a_n(F, G)$: algorithm and information - **linear**

_n_-th Kolmogorov number

$$d_n(F, G) := \inf_{\substack{\boldsymbol{V_n \subset G} \\ \boldsymbol{\dim(V_n)=n}}} \sup_{f \in F} \inf_{g \in V_n} \|f - g\|_G$$

$c_n(F,G)$	$\leq$	$a_n(F,G)$	$\leq$	$g_n^{\mathrm{lin}}(F,G)$
lin. inf.		lin. inf.		**std.** inf.
arb. alg.		**lin.** alg.		lin. alg.

$$c_n(F,G) \quad \leq \quad a_n(F,G) \quad \leq \quad g_n^{\text{lin}}(F,G)$$

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$g_n^{\text{lin}}(F,G) \leq K(n) c_n(F,G)$ $\longrightarrow$ find the order of $K(n)$

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$g_n^{\mathrm{lin}}(F,G) \le K(n) c_n(F,G)$ $\longrightarrow$ find the order of $K(n)$

For every **convex and balanced** set F, it holds

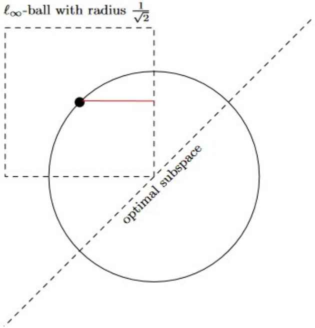
$$d_n(F, B(D)) \le c_n(F, B(D))$$

The goal: find $K(n)$ such that

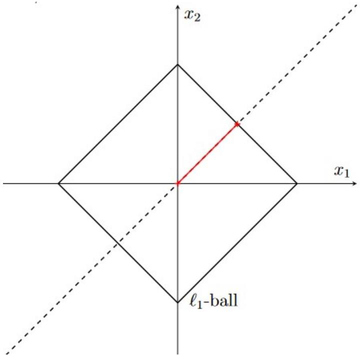
$$g_n^{\mathrm{lin}}(F, B(D)) \le K(n) d_n(F, B(D)) \le K(n) c_n(F, B(D))$$

Kolmogorov vs Gelfand numbers

Kolmogorov numbers
$d_1(B_2, \ell_\infty)$



Gelfand numbers
$c_1(B_1, \ell_2)$



How large are the gaps? What is the order of $K(n)$?

Novak '88

$$g_{n}^{\mathrm{lin}}(F, B(D)) \le (1+n)\, d_n(F, B(D))$$

The factor $(1+n)$ cannot be improved even for Hilbert spaces.

Kashin/Konyagin/Temlyakov '21

$$g_{9^n}^{\mathrm{lin}}(F, B(D)) \le 5\, d_n(F, B(D))$$

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With an oversampling bn, $b>1$ — YES — to $\sqrt{n}$!

Kashin/Konyagin/Temlyakov '21

$$g_{9^n}^{\mathrm{lin}}(F, B(D)) \leq 5\, d_n(F, B(D))$$

Theorem (KPUU '23)

For **any set** *D,* **any** *$F \subset B(D)$, and all $n \in \mathbb{N}$, it holds*

$$g_{2n}^{\text{lin}}(F, B(D)) \;\leq\; 58\sqrt{n}\, d_n(F, B(D)).$$

For **convex and balanced** *F, it holds*

$$g_{2n}^{\text{lin}}(F, B(D)) \;\leq\; 58\sqrt{n}\, c_n(F, B(D)).$$

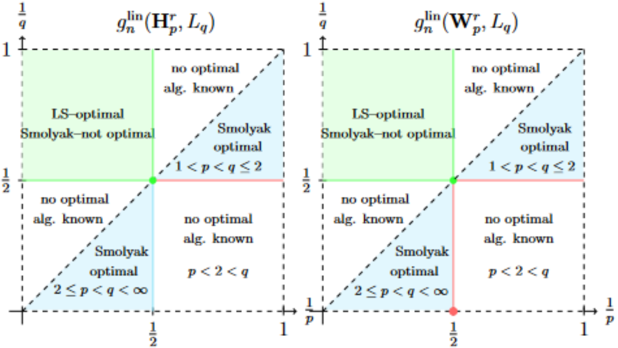
The factor $\sqrt{n}$ in general cannot be improved!

Note: Sharp bound without $\sqrt{n}$ in the Hilbert space setting satisfying certain conditions. (see [Geng, Wang '23] for the trigonometric system, [KPUU '23] for the general case)

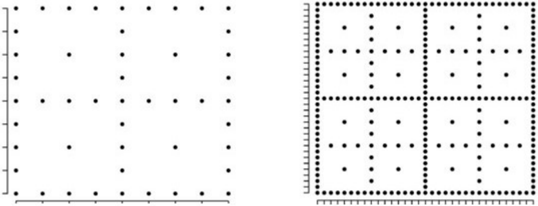
Nikol'skii-Besov $\mathbf{B}^{r}_{p,\infty} \equiv \mathbf{H}^{r}_{p}$ and Sobolev $\mathbf{W}^{r}_{p}$ spaces on the torus $\mathbb{T}^d$

$$\sup_{f \in F} \|f - A_m f\|_q \lesssim m^{-(r-t)} (\log m)^{(d-1)(r-t+\alpha)},$$

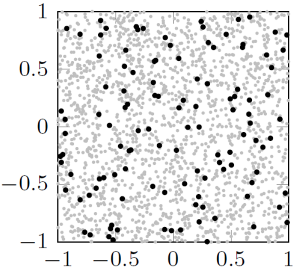
$$r > \max\{1/2, 1/p\}, \quad t := (1/p - 1/2)_+ + (1/2 - 1/q)_+, \quad \alpha = \begin{cases} 0 & F = \mathbf{W}^r_p, 1 \le q < \infty; \\ 1/2 & F = \mathbf{H}^r_p, 1 \le q < \infty;\ F = \mathbf{W}^r_p, q = \infty; \\ 1 & F = \mathbf{H}^r_p, q = \infty. \end{cases}$$



Smolyak points



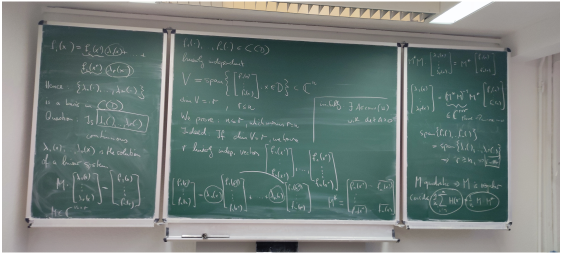
Subsampled random points



Lemma (KPUU '23)

Let D be a set and $V_n \subset B(D)$, $\dim V_n = n$. Then there exist $2n$ points $x_1, \dots, x_{2n} \in D$ such that, for all $f \in V_n$, we have

$$\|f\|_\infty \le 58\sqrt{n}\Big(\frac{1}{2n}\sum_{k=1}^{2n}|f(x_k)|^2\Big)^{1/2} \le 58\sqrt{n}\max_{k=1,\dots,2n}|f(x_k)|$$



Part of the proof.

Based on [Kiefer/Wolfowitz '60] + constructive subsampling.

- Given n linearly independent complex-valued functions $f_n \in C(D)$, for a fixed $x \in D$ solve

$$A := \max_{(x_k)_{k=1}^N,\,(\lambda_k)_{k=1}^N} \det(B^* B),$$

$$B := \begin{pmatrix} \sqrt{\lambda_1} f_1(x_1) & \sqrt{\lambda_1} f_2(x_1) & \cdots & \sqrt{\lambda_1} f_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\lambda_N} f_1(x_N) & \sqrt{\lambda_N} f_2(x_N) & \cdots & \sqrt{\lambda_N} f_n(x_N) \end{pmatrix}$$

with $N = n^2 + 1$ nodes $(x_k)_{k=1}^N$ and weights $(\lambda_k)_{k=1}^N$.

- Write the singular value decomposition

$$A = VDV^*, \qquad A^{-1/2} = VD^{-1/2}V^*.$$

- Put

$$G(x) := A^{-1/2}F(x), \qquad F := (f_1, \dots, f_n)^\top.$$

We get, that for the components of $G(x) = (g_1, \dots, g_n)^\top$ form an ONB in V_n w.r.t. the atomic measure $\varrho = \sum_{i=1}^N \lambda_i \delta_{x_i}$, and it holds

$$\sup_{f \in V_n \setminus \{0\}} \frac{\|f\|_\infty}{\|f\|_{L_2(\varrho)}} = \sup_{x \in D} \left(\sum_{i=1}^n |g_i(x)|^2 \right)^{1/2} \leq \sqrt{n}.$$

Theorem (KPUU '23)

Let D be a set and $V_n \subset B(D)$, $\dim V_n = n$. Then there are $2n$ points $x_i \in D$ and functions $\varphi_i \in V_n$, such that $P\colon B(D) \to V_n$ with

$$Pf = \sum_{i=1}^{2n} f(x_i)\,\varphi_i \quad \text{is a projection with} \quad \|P\| \le 58\sqrt{n}.$$

Theorem is **sharp** in the following sense:

- Using $m = \mathcal{O}(n)$ samples, the norm bound $C\sqrt{n}$ cannot be replaced with a lower-order term.
- Using only $m = n$ samples, the norm bound $C\sqrt{n}$ has to be replaced by a linear term in n (*Novak's/Auerbach's lemma*).

Thank you for your attention!

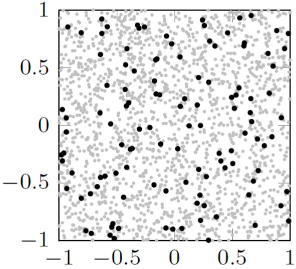
Krieg/M. Ullrich '19 (Parts I, II), **Kämmerer/T. Ullrich/Volkmer '21**, **M. Ullrich '20**, **Moeller/T. Ullrich '21**, **Nagel/Schäfer/T. Ullrich '21** **Dolbeault/Krieg/M. Ullrich '22**, **Krieg/P./M. Ullrich/T. Ullrich '23**

- ***F*** RKHS

$$g_{cn}^{\mathrm{lin}}(F, L_2) \leq \sqrt{\frac{1}{n}\sum_{k\geq n} d_k(F, L_2)^2}$$

- ***F*** general

$$g_{cn}^{\mathrm{lin}}(F, L_2) \leq \frac{1}{\sqrt{n}}\sum_{k\geq n} \frac{d_k(F, L_2)}{\sqrt{k}}$$



If $d_n(F, L_2) \asymp n^{-\alpha}(\log n)^\gamma$ with $\alpha > 1/2$, $\gamma \in \mathbb{R}$, then

$$g_n^{\mathrm{lin}}(F, L_2) \asymp d_n(F, L_2). \quad \text{[DKU '23]}$$

If $d_n(F, L_2) \asymp n^{-1/2}(\log n)^\gamma$ with $\gamma < -1$, then

$$g_n^{\mathrm{lin}}(F, L_2) \lesssim \mathbf{\log n} \cdot d_n(F, L_2). \quad \text{[KPUU '23]}$$

Temlyakov '21, Bartel/Schäfer/T. Ullrich '23

$$g^{\text{lin}}_{bn}(F, L_2) \leq \frac{Bb^3}{(b-1)^3} \cdot d_n(F, B(D))$$

Oversampling factor $b > 1$, $B > 0$ absolute constant.

P./T. Ullrich '22, Krieg/P./M. Ullrich/T. Ullrich '23

$$g^{\text{lin}}_{2n}(F, B(D)) \leq 58\sqrt{n}\, d_n(F, B(D))$$

$$g^{\text{lin}}_{4n}(F, L_p(\mu)) \leq 83\, n^{(1/2-1/p)_+}\, d_n(F, B_\mu(D)), \qquad 1 \leq p \leq \infty.$$

Let $H \subset L_2(\mu)$ be a RKHS on a set D with a **bounded** kernel

$$K(x,y) = \sum_{k=1}^{\infty} \sigma_k^2 \, b_k(x) \overline{b_k(y)},$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$; $\{b_k\}_{k=1}^{\infty}$ is orthonormal in $L_2(\mu)$.

The set $\{\sigma_k b_k\}_{k=1}^{\infty}$ is an orthonormal basis in H. [Steinwart/Scovel '12]

$$V_n = \text{span}\{b_1, \ldots, b_n\}$$

$$\Lambda(V_n, B(D)) \; = \sup_{f \in V_n \setminus \{0\}} \|f\|_\infty / \|f\|_2$$

If $\Lambda(V_n, B(D)) := \sup_{f \in V_n \setminus \{0\}} \frac{\|f\|_\infty}{\|f\|_2} = \left\| \left(\sum_{k=1}^n |b_k|^2 \right)^{1/2} \right\|_\infty \lesssim \sqrt{n},$

where $\{b_k\}_{k=1}^\infty$ is any ONB in $L_2(\mu)$ and μ is a finite measure, then

$$g_{cn}^{\text{lin}}(H, B(D)) \le \sqrt{bB \cdot \mu(D)} \cdot c_n(H, B(D)).$$

Linear sampling algorithms are (up to constants) **as powerful as** arbitrary non-linear algorithms using general linear measurements.

We do not require any decay of the Gelfand width c_n !

Can also be applied for: the Haar or the Walsh systems, certain wavelets,
the Chebychev polynomials, the spherical harmonics, ...

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$$\|f\|_\infty \le 58\sqrt{n}\Big(\frac{1}{2n}\sum_{k=1}^{2n}|f(x_k)|^2\Big)^{1/2} \le 58\sqrt{n}\max_{k=1,\ldots,2n}|f(x_k)|$$

- There **exists a probability measure** μ on D: $\forall f \in V_n \subset B(D)$

$$\|f\|_\infty \le \sqrt{n+\varepsilon}\cdot\|f\|_2 \qquad (1)$$

  If $V_n \subset C(D)$ for some compact topological space D, then $\varepsilon = 0$.

  Based on [Kiefer/Wolfowitz '60]

- There **exists a point set** $X \subset D$, $|X| \le 2n$: $\forall f \in V_n$

$$\|f\|_2 \le 57\left(\frac{1}{|X|}\sum_{x\in X}|f(x)|^2\right)^{1/2} \qquad (2)$$

  [Bartel/Schaefer/T. Ullrich '22]