Approximation properties of subdivision based isogeometric discretizations

Thomas Takacs

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences (ÖAW)
Linz, Austria

10th International Conference on Mathematical Methods for Curves and Surfaces, 26.-28.06.2024

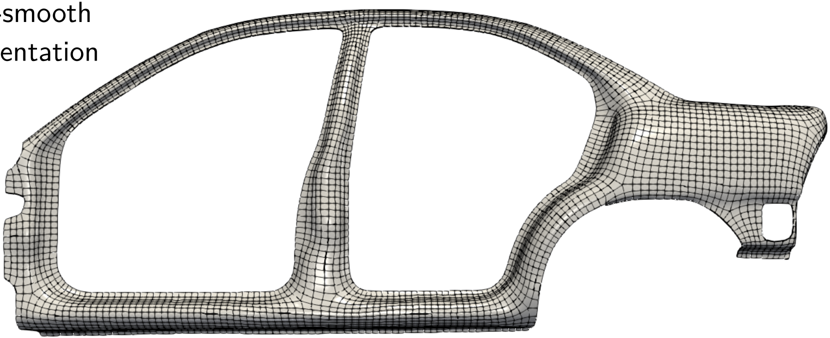
T. Approximation properties over self-similar meshes of curved finite elements and applications to subdivision based isogeometric analysis. ArXiv, 2023.

special thanks to

Roland Maier, Philipp Morgenstern, Stefan Takacs and Deepesh Toshniwal

Subdivision surfaces for numerical analysis

- start from mesh, reproduce B-splines
- geometrically flexible
- (at least) C^1-smooth
- infinite representation



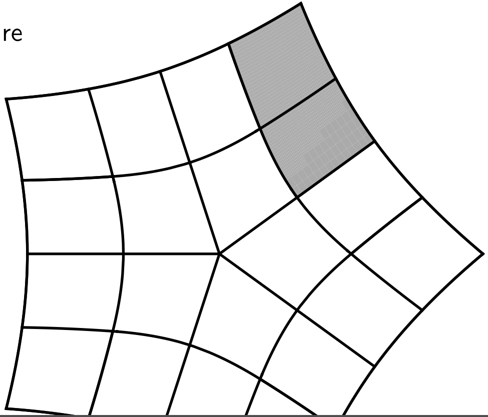
e.g. used for Kirchhoff–Love thin shell simulations

What we know . . .

- subdivision based discretizations converge suboptimally
  - L^∞-error: only $O(h^2)$
  - L^2-error: somewhere between $O(h^2)$ and $O(h^{p+1})$ (for degree p)

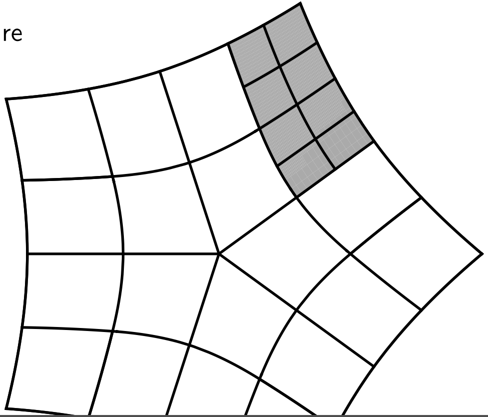
What we know . . .

- in regular regions the rates are optimal
  - both L^∞- and L^2-error: $O(h^{p+1})$
    cf. IGA and isoparametric FEM literature



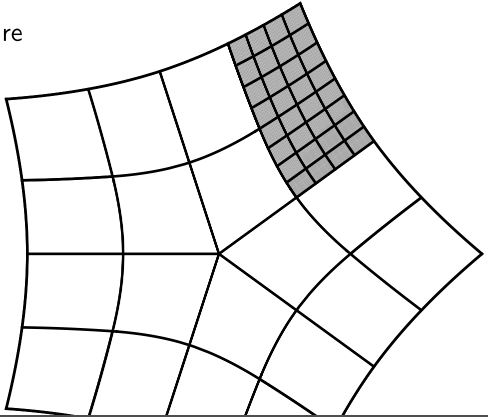
What we know . . .

- in regular regions the rates are optimal
  - both L^∞- and L^2-error: $O(h^{p+1})$
    cf. IGA and isoparametric FEM literature



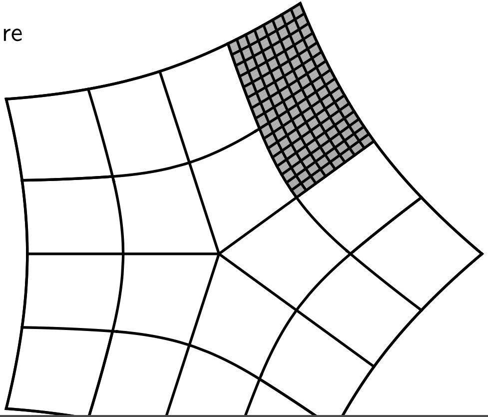
What we know . . .

- in regular regions the rates are optimal
  - both L^∞- and L^2-error: $O(h^{p+1})$
    cf. IGA and isoparametric FEM literature



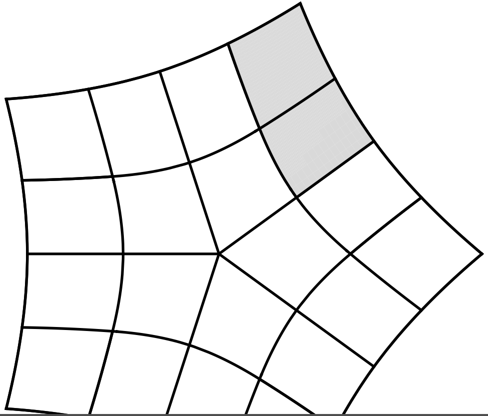
What we know ...

- in regular regions the rates are optimal
  - both L^∞- and L^2-error: $O(h^{p+1})$
    cf. IGA and isoparametric FEM literature



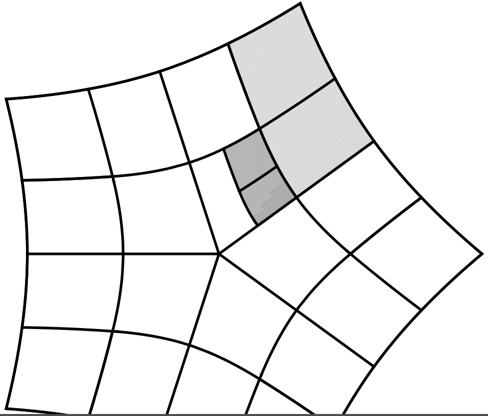
What we know ...

- any point away from an EP is inside a regular region for some level ℓ fine enough



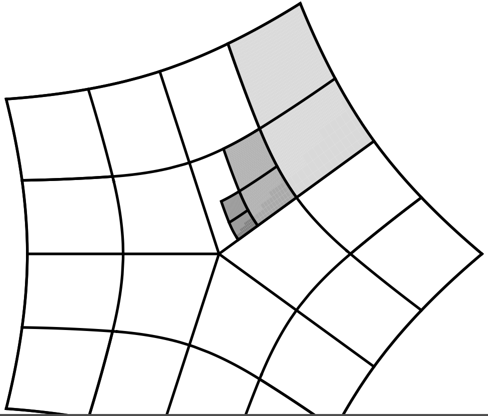
What we know ...

- any point away from an EP is inside a regular region for some level ℓ fine enough

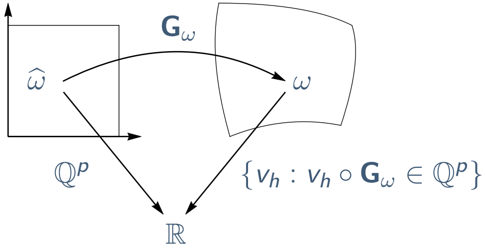


What we know . . .

- any point away from an EP is inside a regular region for some level ℓ fine enough



Sufficient for optimal convergence



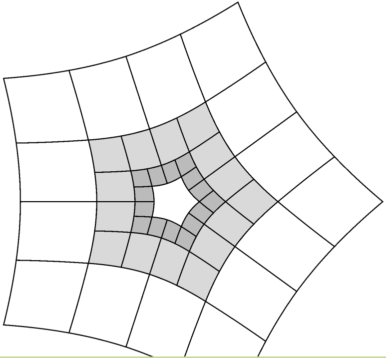
- polynomial reproduction in ω +
- shape regularity of ω

  or
- polynomial reproduction in $\widehat{\omega}$ +
- element mappings converge to affine mappings fast enough

What we don't know ...

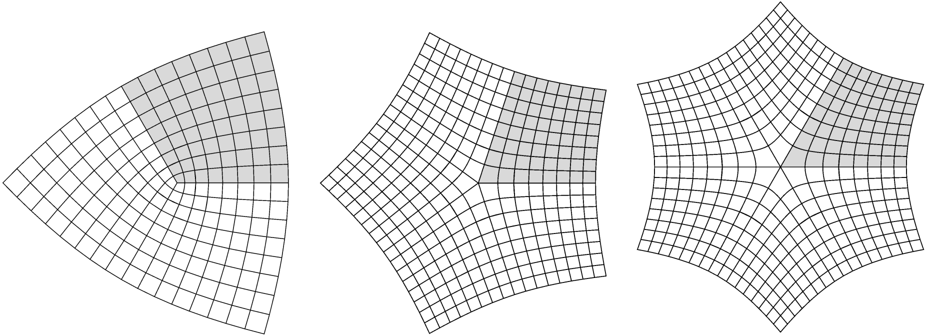
- why is approximation power reduced?
  - overconstraining the space?
  - insufficient shrinking?
  - something else?

Does polynomial reproduction in physical domain determine speed of convergence?

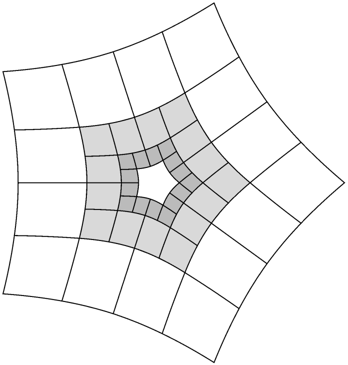


Subdivision Isogeometric Analysis: A Todo List; Dietz, Peters, Reif, Sabin, Youngquist, *Dolomites Research Notes on Approximation* 15(5), 2023.

Characteristic rings



Characteristic rings of subdivision surfaces



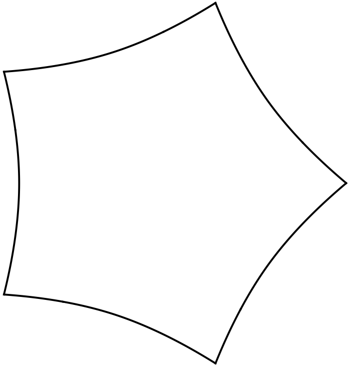
- domain

$$\Omega = \bigcup_{\ell=0}^{\infty} \Omega^\ell$$

$$\Omega^\ell = \lambda^\ell \Omega^0$$

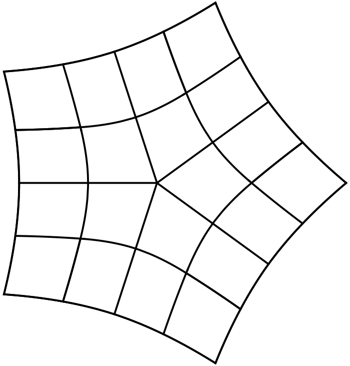
- for Doo–Sabin: $\lambda = 1/2$
- for Catmull–Clark
  - valence 3: $\lambda \approx 0.410097$
  - valence 5: $\lambda \approx 0.549988$
  - valence 6: $\lambda \approx 0.579682$

What do we study?



- function $\varphi \in H^{p+1}(\Omega)$

What do we study?

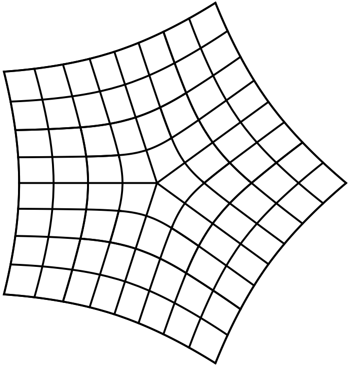


- function $\varphi \in H^{p+1}(\Omega)$
- discretization space V_ℓ:

  mapped, p.w. polynomials of bi-degree (p, p)
  finite cap with Coons-patches
- ϱ observed convergence rate for:

$$\inf_{\varphi_\ell \in V_\ell} \|\varphi - \varphi_\ell\|_{L^2(\Omega)} \eqsim \varrho(\ell) \|\varphi\|_{H^{p+1}(\Omega)}$$

What do we study?

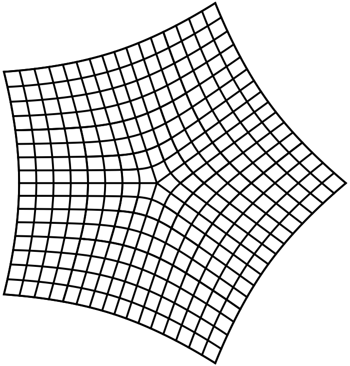


- function $\varphi \in H^{p+1}(\Omega)$
- discretization space V_ℓ:

  mapped, p.w. polynomials of bi-degree (p,p)
  finite cap with Coons-patches
- ϱ observed convergence rate for:

$$\inf_{\varphi_\ell \in V_\ell} \|\varphi - \varphi_\ell\|_{L^2(\Omega)} \eqsim \varrho(\ell) \|\varphi\|_{H^{p+1}(\Omega)}$$

What do we study?



- function $\varphi \in H^{p+1}(\Omega)$
- discretization space V_ℓ:

  mapped, p.w. polynomials of bi-degree (p, p)
  finite cap with Coons-patches
- ϱ observed convergence rate for:

$$\inf_{\varphi_\ell \in V_\ell} \|\varphi - \varphi_\ell\|_{L^2(\Omega)} \eqsim \varrho(\ell) \|\varphi\|_{H^{p+1}(\Omega)}$$

Subdivision based approaches

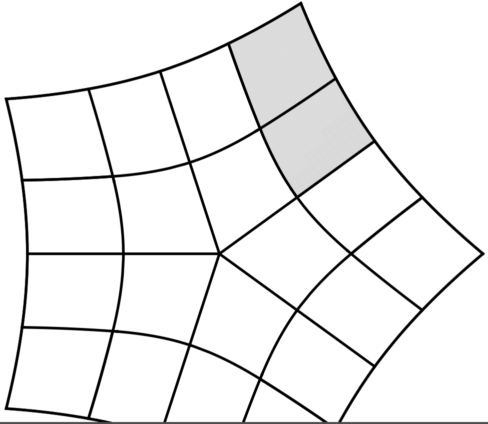
ω^ℓ . . . element on level ℓ
λ . . . scaling factor of the subdivision scheme

$$\omega^\ell = \lambda^\ell \omega^0$$

κ . . . reproduction degree in physical coordinates

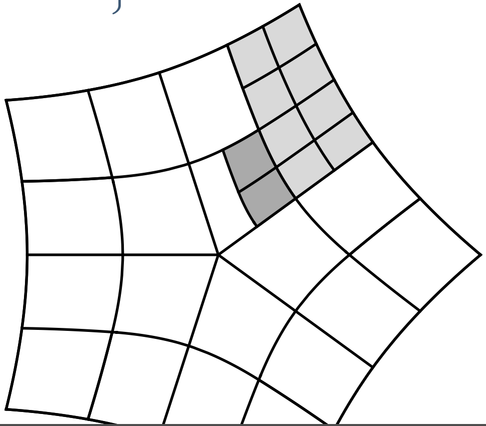
Approximating a function

$$\inf_{\varphi_0 \in V_0} \|x^{\kappa+1} - \varphi_0\|_{L^\infty} = \max\left\{ \underline{C}_0 \right\}$$



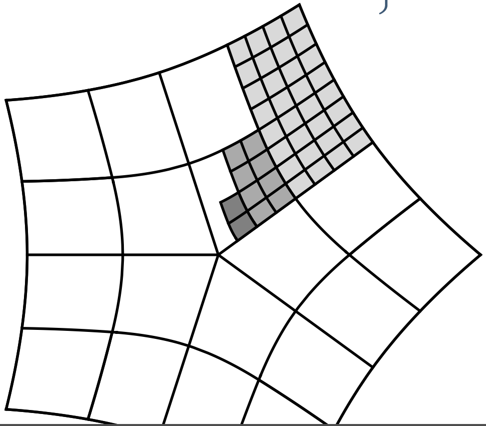
Approximating a function

$$\inf_{\varphi_1 \in V_1} \|x^{\kappa+1} - \varphi_1\|_{L^\infty} \geq \max \left\{ \left(\tfrac{1}{2}\right)^{p+1} \underline{c}_0 \, , \; \lambda^{\kappa+1} \underline{C}_0 \right\}$$



Approximating a function

$$\inf_{\varphi_2 \in V_2} \|x^{\kappa+1} - \varphi_2\|_{L^\infty} \geq \max\left\{ \left(\tfrac{1}{2}\right)^{2(p+1)} \underline{c}_0 \,,\; \left(\tfrac{1}{2}\right)^{p+1} \lambda^{\kappa+1} \underline{c}_0 \,,\; \lambda^{2(\kappa+1)} \underline{C}_0 \right\}$$



Doo–Sabin subdivision

$p = 2$
$\lambda = 1/2$
$h = 1/2^{\ell}$

	L^∞-rate	L^2-rate	H^1-rate
DS, valence $= 4$	h^3	h^3	h^2
DS, valence $\neq 4$	h^2	$\sqrt{1 - \log_2(h)}\ h^3$	$\sqrt{1 - \log_2(h)}\ h^2$

Catmull–Clark subdivision

$p = 3$
$\lambda \approx 0.410097$ for valence 3
$\lambda \approx 0.549988$ for valence 5
$\lambda \approx 0.579682$ for valence 6
$h = \max(1/2, \lambda)^\ell$

	L^∞-rate	L^2-rate	H^1-rate
CC, valence 3	$h^{2.57193} \sim 2^{-2.57193}$	$h^{3.85789} \sim 2^{-3.85789}$	$h^{2.57193} \sim 2^{-2.57193}$
CC, valence 4	$h^4 \sim 2^{-4}$	$h^4 \sim 2^{-4}$	$h^3 \sim 2^{-3}$
CC, valence 5	$h^2 \sim 2^{-1.72505}$	$h^3 \sim 2^{-2.58758}$	$h^2 \sim 2^{-1.72505}$
CC, valence 6	$h^2 \sim 2^{-1.57333}$	$h^3 \sim 2^{-2.35999}$	$h^2 \sim 2^{-1.57333}$

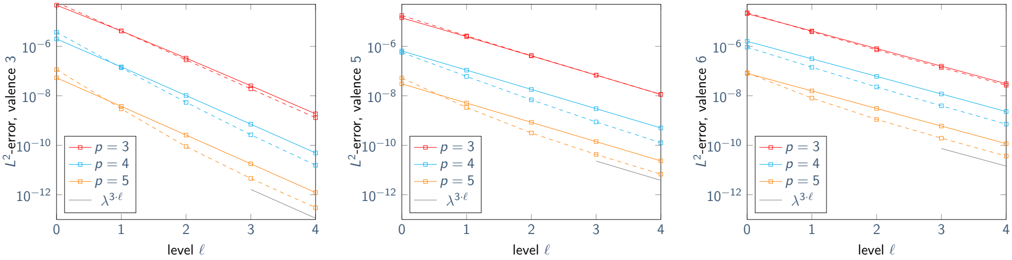
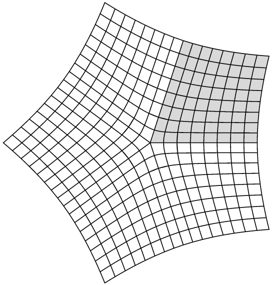
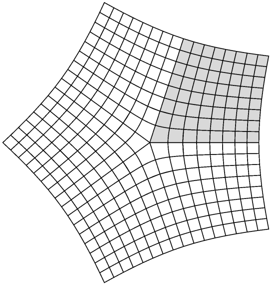


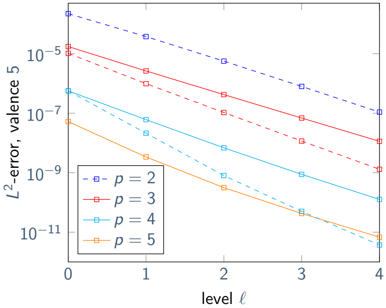
Figure: Convergence rates for L^2-approximation on characteristic rings of Catmull–Clark subdivision for valence three (left), five (center) and six (right), function $\varphi(x, y) = x^2 + y^2$ (solid lines) and $\varphi(x, y) = \sin(x)\cos(y + 1)$ (dashed lines).

All rates tend to λ^3, where $\lambda^3 \sim (1/2)^{3.8579}$ for valence three, $\lambda^3 \sim (1/2)^{2.5876}$ for valence five and $\lambda^3 \sim (1/2)^{2.35999}$ for valence six, respectively.

Comparison between Doo–Sabin and Catmull–Clark subdivision







errors for Doo–Sabin shown as dashed lines; for Catmull–Clark as solid lines

Summary and conclusions

- can prove lower bounds for approximation error
- reasonable explanation for suboptimality of subdivision
- also applicable to scaled boundary parametrizations
- smoothness near vertices may cause additional reduction in approximation power