Optimal linear and non-linear dimensionality reduction
Theory and algorithms

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Taking place in St Malo, France, June 2026 (first or second week)

following Chamonix 1990, 1993, 1996; Saint-Malo 1999, 2002;
Avignon 2006, 2010;Paris 2014 ; Arcachon 2018, 2022

Save the date !

Dimensionality reduction - reduced modeling

Approximation of an unknown multivariate function $u \in V$ defined on some domain $\Omega \subset \mathbb{R}^d$ by simpler functions depending on finitely many parameters is used in various contexts.

Forward simulation : numerical computation of u solution to a given PDE.

Inverse problems : access to limited observation $\ell(u) = (\ell_1(u), \dots, \ell_m(u)) \in \mathbb{R}^m$.

Sampling : access to point values $u(x^1), \dots, u(x^m)$.

Here V is a Banach space equiped with norm $\|\cdot\| = \|\cdot\|_V$.

Prior information: $u \in \mathcal{K}$ compact class of V (smoothness class, family of solutions to parametrized PDEs...).

Computational strategies lead to approximation of u by $\tilde{u} \in V_n$ that can be described by $n \leq m$ parameters (or more generally $\mathcal{O}(n)$).

The set V_n can be a linear or nonlinear space

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The set V_n can be a linear or nonlinear space

Classical choice of approximation spaces.

Linear spaces:

- Algebraic polynomials: $V_n = \mathbb{P}_n$. When $d > 1$, we may set $V_n = \mathbb{P}_k$ with $n = \binom{k+d}{d}$.

- Span of the n first elements $\{e_1, \dots, e_n\}$ from a given basis $(e_k)_{k\geq 1}$ of V, for example trigonometric polynomials.

- Piecewise polynomials, splines or finite elements on a fixed partition of cardinality n.

Nonlinear spaces:

- Rational fractions: $V_n = \left\{ \frac{p}{q}\,;\, p, q \in \mathbb{P}_n \right\}$.

- Best n-term / sparse approximation in a basis $(e_k)_{k\geq 1}$: pick approximation from the set $V_n = \{\sum_{k\in E} c_k e_k \; : \; \#(E) \leq n\}$.

- splines, finite elements on meshes generated after n step of adaptive refinement (select and split an element in the current partition).

- Neural networks : functions $v : \mathbb{R}^d \to \mathbb{R}^m$ of the form

$$v = A_k \circ \sigma \circ A_{k-1} \circ \sigma \circ A_{k-2} \circ \cdots \circ \sigma \circ A_1,$$

where $A_j : \mathbb{R}^{d_j} \to \mathbb{R}^{d_{j+1}}$ is affine and σ is a nonlinear (rectifier) function applied componentwise, for example $\sigma(x) = RELU(x) = \max\{x, 0\}$. Here V_n is the set of such functions when the total number of parameters does not exceed n.

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Standard questions in approximation theory

For a given family $(V_n)_{n\geq 0}$ of such classical spaces, a standard question is the study of the best approximation error

$$e_n(u) = e_n(u)_V = \min_{v\in V_n} \|u - v\|.$$

Typically, one tries to understand which properties of u (smoothness, sparsity...) ensure a certain rate of decay $e_n(u) \leq Cn^{-r}$.

This allows to understand for a given prior model class $\mathcal{K}$ the asymptotic behaviour of $\text{dist}(\mathcal{K}, V_n)_V := \max_{u\in\mathcal{K}} e_n(u)$.

A related standard question is the construction of near best approximations: simple computational map $u \mapsto u_n \in V_n$ such that $\|u - u_n\| \leq Ce_n(u)$ for some fixed $C \geq 1$.

Optimal dimensionality reduction

A less standard question is that of an optimal choice of V_n.

For a given model class $\mathcal{K}$, can we find a linear or nonlinear space V_n that best approximate $\mathcal{K}$, that is, make $\text{dist}(\mathcal{K}, V_n)_V$ as small as possible.

Such spaces may in certain cases significantly differ from the classical examples listed above, and may have no simple description.

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The offline/online paradigm

We are interested in applications where we might need to search for many instances of $u \in \mathcal{K}$:

- Parametrized PDE's $\mathcal{P}(u, y) = 0$ giving rise to a solution map $y \mapsto u(y) \in V$ and manifold $\mathcal{K} := \{u(y) \ : \ y \in Y\}$.

- Applications requiring multiple queries of the solution map: optimization/control (y is deterministic), or uncertainty quantification (y is random) or inverse problems (identify y from observations of $u(y)$).

- Or we may want to recover many instances of u in a model class $\mathcal{K}$, from their observations $z = (\ell_1(u), \dots, \ell_m(u))$.

Offline stage: design a space V_n of moderate dimension n that is optimally taylored to the class $\mathcal{K}$. This can be computationally intensive but it is only done once.

Online stage: for each required parameter instance y (or.data z), compute an approximation $u_n(y)$ (or u_n) in V_n, by a hopefully fast computation : n numbers to compute.

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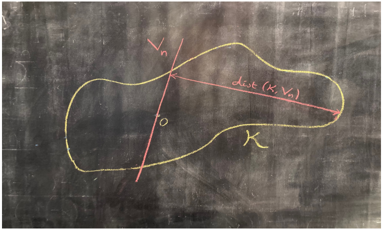
Kolmogorov n-widths are defined as

$$d_n = d_n(\mathcal{K})_V := \inf_{\dim(V_n)=n} \text{dist}(\mathcal{K}, V_n)_V, \qquad \text{dist}(\mathcal{K}, V_n)_V := \max_{u\in\mathcal{K}} \min_{v\in V_n} \|u - v\|_V,$$



The quantity $d_n(\mathcal{K})_V$ can be viewed as a benchmark/bottleneck for numerical methods applied to the elements from $\mathcal{K}$ that create approximations from linear spaces: interpolation, projection, least squares, Galerkin methods for solving PDEs...

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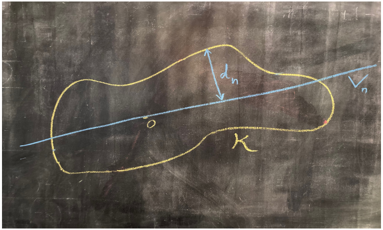

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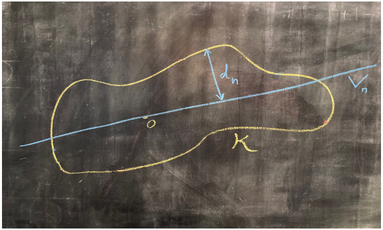

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Optimality in linear and nonlinear dimensionality reduction

Kolmogorov n-widths are the benchmark for approximation by linear spaces.

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with infimum over all linear recovery maps $R : \mathbb{R}^n \to V$ and continuous encoding maps $E : V \to \mathbb{R}^n$.

Similar quantities can be defined with other prescriptions on E and R.

R \ E	point values	linear	nonlinear
linear	r_n	a_n	d_n
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Resulting quantities

E \ R	point values	linear	nonlinear
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- $a_n = a_n(\mathcal{K})_V$ are the approximation numbers: $\inf \max_{u\in\mathcal{K}} \|u - Lu\|$, where the infimum is over all linear L of rank $\leq n$.

- Note that $a_n = d_n$ when V is a Hilbert space. Otherwise $a_n \geq d_n$.

- r_n and ρ_n are the sampling numbers: $\inf \max_{u\in\mathcal{K}} \|u - R(u(x^1),\dots,u(x^n))\|$, there the infimum is over all choices of sampling points $(x^1,\dots,x^n)$ and linear or continuous recovery maps.

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Some variants

One may relax these definitions by imposing that the encoder is only defined on $\mathcal{K}$ and not on the whole of V.

This allows us, for example, to define sampling numbers for spaces such as $V = L^p(\Omega)$, assuming that all functions $u \in \mathcal{K}$ are continuous, that is, $\mathcal{K} \subset \mathcal{C}(\Omega)$.

On the other hand, one may want to strengthen these definitions by asking that the encoding and recovery map have some stability.

This leads to the notion of stable widths: for $L \geq 1$, we define

$$\delta_{n,L} := \inf_{E,R} \max_{u\in\mathcal{K}} \|u - R(E(u))\|,$$

where the infimum is taken over all encoding and recovery maps such that

$$\|E(u) - E(v)\|_Z \leq L\|u - v\|_V \quad u, v \in V,$$

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For a given class $\mathcal{K}$ of interest, we would like to estimate the above quantities from above and below.

Estimate from above: compute $\max_{u\in\mathcal{K}} \|u - R(E(u))\|_V$ for particular admissible choices of E and R that are presumed to be near optimal.

Estimate from below is more tricky. We discuss further two possible ways.

For certains classes $\mathcal{K}$ and spaces V, all these quantities behave similar as $n \to \infty$.

Example : $V = L^\infty(I)$ where $I = [0,1] \subset \mathbb{R}$ and

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Upper bound: use piecewise linear reconstruction from n equispaced points.

General rate $n^{-\frac{s-t}{d}}$ for $V = W^{t,p}(Q)$ and $\mathcal{K} = \mathcal{U}(W^{s,p}(Q))$, with $Q = [0,1]^d$.

Thus for such prior classes, nonlinear approaches are not beneficial.

For other classes discussed further, one may have $\delta_n << d_n$.

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Lower bounds: using continuity

Continuity of the encoder E essential in the definition of the nonlinear widths δ_n. Indeed, if we drop this assumption, then we would have

$$\delta_n(\mathcal{K}) = \inf_D \sup_{u\in\mathcal{K}} \inf_{x\in\mathbb{R}^n} \|u - D(x)\|,$$

which is the minimal distance of $\mathcal{K}$ to n-dimensional parametrized manifolds.

This quantity is 0 even for $n = 1$: space filling continuous curves.

Assuming continuity of E, we obtain lower bounds using Borsuk-Ulam theorem: if $W \subset V$ is any $n+1$ dimensional space, then any continuous map E from the n-sphere $S_n = \partial B_W$ to $\mathbb{R}^n$ admits a point $u^* \in S_n$ such that $E(u^*) = E(-u^*)$.

Thus, if $\mathcal{K}$ contains a rescaled ball rB_W of an $n+1$-dimensional space W, there exists $u^*, -u^* \in \mathcal{K}$ such that $R(E(u^*)) = R(E(-u^*))$ for any recovery map, and thus

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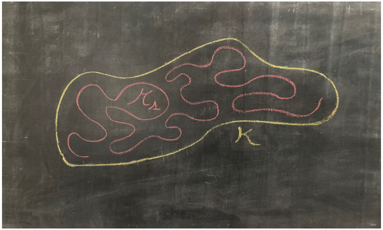

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$$\delta_n(\mathcal{K}) = \inf_D \sup_{u\in\mathcal{K}} \inf_{x\in\mathbb{R}^n} \|u - D(x)\|,$$

which is the minimal distance of $\mathcal{K}$ to n-dimensional parametrized manifolds.

This quantity is 0 even for $n = 1$: space filling continuous curves.

Assuming continuity of E, we obtain lower bounds using Borsuk-Ulam theorem: if $W \subset V$ is any $n+1$ dimensional space, then any continuous map E from the n-sphere $S_n = \partial B_W$ to $\mathbb{R}^n$ admits a point $u^* \in S_n$ such that $E(u^*) = E(-u^*)$.

Thus, if $\mathcal{K}$ contains a rescaled ball rB_W of an $n+1$-dimensional space W, there exists $u^*, -u^* \in \mathcal{K}$ such that $R(E(u^*)) = R(E(-u^*))$ for any recovery map, and thus

$$\max_{u\in\mathcal{K}} \|u - R(E(u))\|_V \geq r.$$

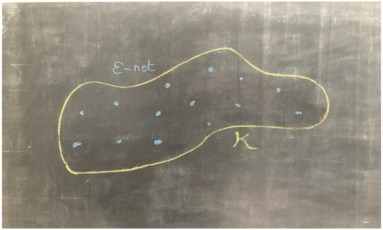
It follows that

$$\delta_n \geq b_n,$$

where the Bernstein n-width $b_n = b_n(\mathcal{K})_V$ is defined as the largest $r \geq 0$ such that there exists $W \subset V$ of dimension $n+1$ with $rB_W \subset \mathcal{K}$.

For $V = L^\infty(I)$ and $\mathcal{K} = \mathcal{U}(\mathrm{Lip}(I))$ one finds that $b_n \geq \frac{1}{2n}$.

Define the entropy numbers $\varepsilon_n = \varepsilon_n(\mathcal{K})_V$ as the smallest ε such that $\mathcal{K}$ can be covered by 2^n balls of radius ε.



Related to lossy coding : Elements of $\mathcal{K}$ can be encoded with n bits up to precision ε_n.

Carl's inequality : for all $s > 0$ one has

$$(n+1)^s \varepsilon_n \leq C_s \sup_{m=0,\dots,n} (m+1)^s d_m, \quad n \geq 0$$

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$$d_n \lesssim n^{-s}, \quad n \geq 0 \implies \varepsilon_n \lesssim n^{-s}, \quad n \geq 0.$$

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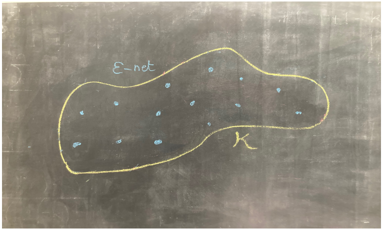

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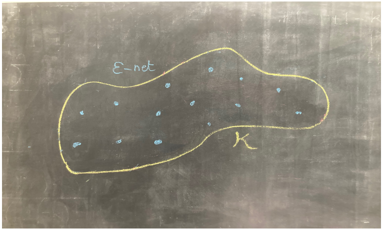

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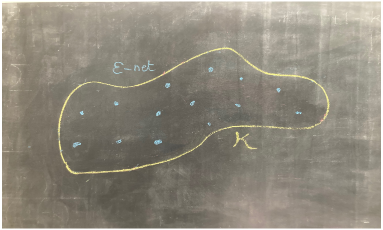

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Approximation by neural networks

The family V_n of all neural networks $v : \mathbb{R}^d \to \mathbb{R}^m$ described by at most n parameters is an instance of nonlinear approximation.

For a given target function u, the search of an approximation $u_n \in V_n$ is usually done by solving an optimization problem using a large training set of points

$$u_n = \mathrm{argmin}_{v \in V_n} \sum_{i=1}^{M} |u(x^i) - v(x^i)|^2.$$

This is a non-convex optimization problem in the parameter space, often solved by stochastic gradient algorithms.

Approximation results by Yarotzki, Shen-Yang-Zhang (2020) for $d = 1$ and $m = 1$: neural networks approximation of functions in $\mathrm{Lip}(I)$ converge in L^∞ with rate n^{-2} !

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The power of point value information for linear recovery

Recent results (M. Ullrich, T. Ullrich, Nagel, Krieg, Dolbeault...) reveal that when $V = L^2$, point value evaluation are enough for optimal rates of linear recovery.

Under mild assumptions on the $\mathcal{K}$, linear sampling number behave as well as n-widths.

$$a_n = d_n \lesssim n^{-s} \iff r_n \lesssim n^{-s}.$$

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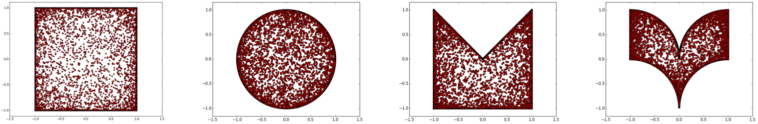

Optimal densities when V_n are bivariate polynomials of fixed total degree $k = 9$.

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Point values are generally uneffective for optimal nonlinear recovery: $\delta_n << \rho_n$ for certain classes $\mathcal{K}$.

In DNN refered to as theory to practice gap (Adcock, Grohs, Voigtlaender...)

Achieving the accuracy of nonlinear spaces V_n may requires $m >> n$ point evaluations.

Cohen-DeVore-Petrova-Wojtaszczyk (2021) : when V is a Hilbert space, linear measurements are enough for optimal rates of stable nonlinear recovery.

Stable sensing numbers decay similar to stable manifold widths and entropy numbers.

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Conversely, in Hilbert spaces, we establish a direct comparison: with $L = 2$ and $c = 26$,

$$s_{cn,L}(\mathcal{K})_V \leq 3\varepsilon_n(\mathcal{K})_V,$$

1. Consider $\mathcal{N}$ an ε_n-net of $\mathcal{K}$ with $\#(\mathcal{N}) = 2^n$.
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3. This gives an exact recovery map R that is 2-Lipschitz from $P_W\mathcal{N}$ to $\mathcal{N}$.
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This recovery procedure is not computationally feasible.

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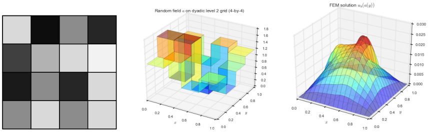
An instructive example : consider the steady-state elliptic diffusion equation

$$-\mathrm{div}(a\nabla u) = f, \quad \text{on} \quad \Omega \subset \mathbb{R}^2, \quad u_{\partial\Omega} = 0,$$

with fixed f, and piecewise constant diffusion function $a = a(y)$ having value $\overline{a} + y_j$ on subdomain Ω_j, where $y = (y_1, \ldots, y_d) \in Y = [-b, b]^d$, where $0 < b < a$.



How large is the Kolmogorov n-width of $\mathcal{K} = \{u(y) \ : \ y \in Y\} \subset V = H^1(\Omega)$?

Solutions $u(y)$ are bounded in H^s iff $s < 3/2$ and $d_n(\mathcal{U}(H^s))_{H^1} \sim n^{-(s-1)/2} \gtrsim n^{-1/4}$.

In fact $d_n(\mathcal{K})_{H^1} \lesssim exp(-cn^{1/d})$. This follows from the holomorphy of the map $y \mapsto u(y)$ that we can approximate by truncated power series

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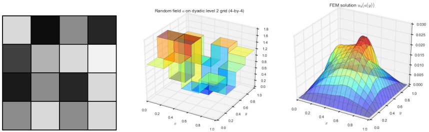

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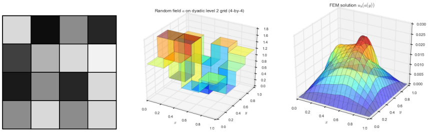

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Estimating n-width of solution manifolds

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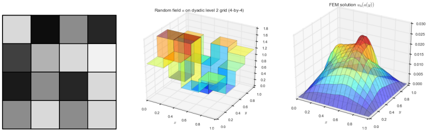

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Failure of linear reduced modeling

Linear reduced modeling for parametrized hyperbolic PDEs suffers from a slow decay of Kolmogorov n-width.

Simple example : consider the univariate linear transport equation

$$\partial_t u + a\partial_x u = 0,$$

with constant velocity $a \in \mathbb{R}$ and initial condition $u_0 = u(x,0) = \chi_{[0,1]}(x)$.

Parametrize the solution by the velocity $a \in [a_{\min}, a_{\max}]$ and consider the solution manifold at final time $T = 1$,

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It is easily checked that

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A multivariate class

More generally consider in $Q = [0,1]^d$ and with $s \geq 1$,

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Can be made a compact set of $L^2(Q)$ by imposing a uniform $\mathcal{C}^s$ bound on the local parametrizations of Ω.

One can prove that $d_n(\mathcal{K}_s)_{L^2} \sim n^{-\frac{1}{2d}}$ regardless of how large is s.

On the other hand,

$$s_n(\mathcal{K}_s)_{L^2} \sim \delta_n(\mathcal{K}_s)_{L^2} \sim \varepsilon_n(\mathcal{K}_s)_{L^2} \sim n^{-\frac{s}{2(d-1)}}.$$

Open problem: achievable by simple linear measurements and recovery strategies ?

Remark: sampling numbers seem to have intermediate rate $\rho_n \sim n^{-\frac{s}{2(d-1)+2s}}$.

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Open problem: achievable by simple linear measurements and recovery strategies ?

Remark: sampling numbers seem to have intermediate rate $\rho_n \sim n^{-\frac{s}{2(d-1)+2s}}$.

A multivariate class

More generally consider in $Q=[0,1]^d$ and with $s \geq 1$,

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For classes $\mathcal{K}$ such as solution manifolds of parametrized PDEs in Hilbert spaces: investing some offline computation of a near optimal approximation space V_n can be highly beneficial for fast online solvers, compared conventional approximation methods (finite elements, splines).

The reduced basis approach (Maday, Patera,...): $V_n = \mathrm{span}\{u^1, \dots, u^n\}$, with $u^i \in \mathcal{K}$.

Greedy selection: given V_{k-1} pick next u^k such that

$$\|u^k - P_{V_{k-1}} u^k\| = \max_{u \in \mathcal{K}} \|u - P_{V_{k-1}} u\|_V,$$

or in practice $\|u^k - P_{V_{k-1}} u^k\| \geq \gamma \max_{u \in \mathcal{K}} \|u - P_{V_{k-1}} u\|_V$ for fixed $\gamma \in]0, 1[$.

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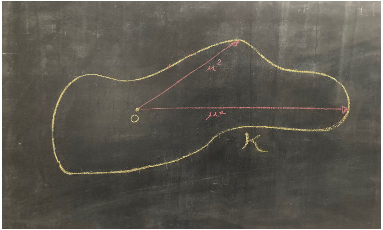

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Can be thought as a learning problem over $\mathcal{K}$. DNN auto-encoders ?

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One of the objective of reduced modeling is the fast access to approximations of the solutions to general PDE's $\mathcal{P}(u) = 0$.

As a basic example, consider the elliptic problem: find $u \in V$ such that

$$a(u, v) = \ell(v), \quad v \in V,$$

in a Hilbert space V, under the standard Lax-Milgram assumptions. Equivalently

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Approximation in a linear reduced model V_n by Galerkin: find $u_n \in V_n$ such that

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Computational time: dense $n \times n$ linear system with moderate n.

Assembling time: computation of matrix elements $a(\phi_j, \phi_j)$ can be the dominant part.

Linear reduced models in forward simulation

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How can we adapt these approaches/results to nonlinear reduced models V_n ?

A systematic approach: for a fixed norm $\|\cdot\|_Z$, minimize the residual

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PINN's methods : use deep neural networks for V_n and the ℓ^2 norm on a sufficiently large training point sets $\sum_i |\mathcal{P}(v)(x^i)|^2$.

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All these approaches amount in solving non-convex optimization problems that can be computationally untractable, even for moderate values of n.

Polynomially mapped manifold (Haasdonk, Farhat, Willcox..): for some fixed $k \geq 1$ consider reduced models of the form

$$V_n := \Big\{ \sum_{|\nu|\leq k} x^\nu \varphi_\nu \ : \ (x_1, \dots, x_n) \in \mathbb{R}^n \Big\}.$$

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or equivalently $P_W u$ where $W = \mathrm{span}\{\omega_1, \dots, \omega_m\}$ is the measurement space.

Recovery in a linear reduced model space $V_n \subset V$

Can we recover up to the best approximation error $e_n(u) = \|u - P_{V_n} u\|$?

Best fit (least square) estimator : $\tilde{u} := \mathrm{argmin}\{\|P_W(u - v)\| \ : \ v \in V_n\}$

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Extensions : V Banach space, nonlinear measurement functionals.

The constant μ is sometimes difficult to estimate.

Compressed sensing example: V_n space of n-sparse vectors in $V = \mathbb{R}^N$ with $N >> n$.

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Cohen-Dolbeault-Mula-Somacal (2022): introduce the stability constant

$$\mu = \mu(V_n, W) := \max_{v_1, v_2 \in V_n} \frac{\|v_1 - v_2\|}{\|P_W(v_1 - v_2)\|},$$

Then

$$\|u - \tilde{u}\| \leq (1 + 2\mu) e_n(u)_V.$$

Extensions : V Banach space, nonlinear measurement functionals.

The constant μ is sometimes difficult to estimate.

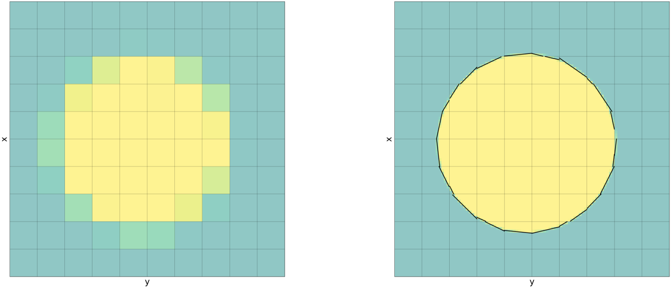
Compressed sensing example: V_n space of n-sparse vectors in $V = \mathbb{R}^N$ with $N >> n$.

Control of μ is equivalent to the so-called null-space property.

Application: shape recovery from local averages

Recover a characteristic function from the class $\mathcal{K}_s$ from cell-averages of sidelength h.

For simplicity, consider here dimension $d = 2$.

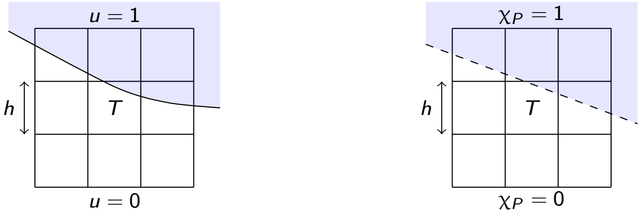


Linear reconstruction by piecewise constants (left) give L^2 rate $\sqrt{h} \sim n^{-1/4}$.

Nonlinear reconstruction by piecewise-linear interfaces.

Expect improved L^2 rate $h \sim n^{-1/2}$ for $s \geq 2$.

On each cell T, approximate u by $\tilde{u}|_T = \chi_P|_T$, where P is a half-plane computed from the average values of u on a 3×3 stencil S composed of T and 8 neighboring cells.



Subcell resolution (1d Harten 1992, 2d Arandiga-Cohen-Donat-Dyn-Matei 2003).

Volume of fluid, ELVIRA (Pilliod-Puckett 1997, Zaleski 1998).

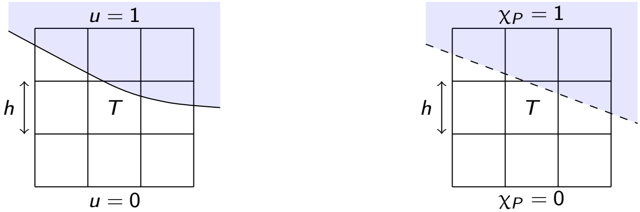
With $V_2 = \{\chi_P|_T,\ P \text{ half-plane}\}$, the local approximation error of $u \in \mathcal{K}_2$ in cells containing the interface $\partial\Omega$ is bounded by

$$\min_{v \in V_2} \|u - v\|_{L^1(T)} \leq Ch^3.$$

Half plane computed by least-squares fitting of the averages $\ell_j(u)$ for $j = 1, \dots, 9$.

A local approach

On each cell T, approximate u by $\tilde{u}|_T = \chi_P|_T$, where P is a half-plane computed from the average values of u on a 3×3 stencil S composed of T and 8 neighboring cells.



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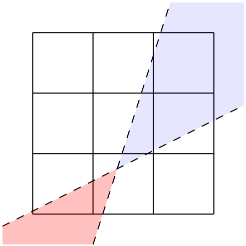
Stability



The optimal constant can be proved to be

$$\mu = \mu(V_2, W) := \max_{P,Q \text{ half-planes}} \frac{\|\chi_P - \chi_Q\|_{L^1(S)}}{\sum_{j=1}^{9} |\ell_j(\chi_P - \chi_Q)|} = \frac{3}{2}$$

This leads to the global second order reconstruction L^1 bound: for $u \in \mathcal{K}_2$

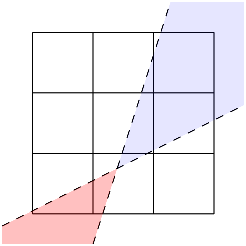
$$\|u - \tilde{u}\|_{L^1} \leq Ch^2,$$

and for the L^2 norm,

$$\|u - \tilde{u}\|_{L^2} \leq Ch = Cn^{-1/2}.$$

which is still not the optimal rate $n^{-\frac{s}{2(d-1)}} = n^{-1}$.

Stability



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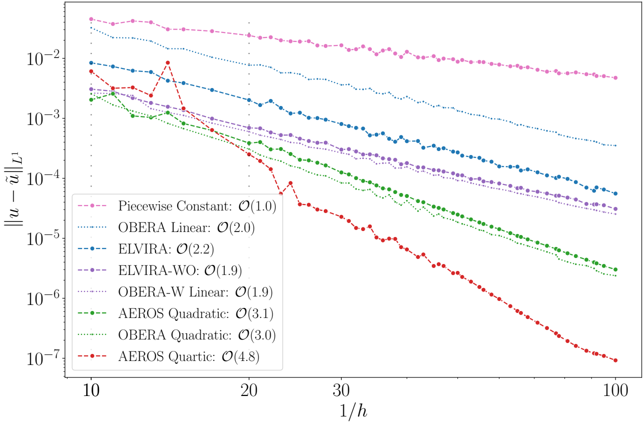
Cohen-Mula-Somacal (2024): higher order reconstructions by curved interfaces, treatment of corners...



Convergence for different reconstruction models (estimated rates in parenthesis).

Conclusions

A standard idea: reduce complexity of solving PDE's and inverse problems searching the approximation within a finite n-dimensional space.

A less standard idea: optimize the choice of the n-dimensional space. Theory is well settled. Provably optimal model reduction algorithms are available.

The nonlinear perspective: theoretical pillars are available. Provably optimal nonlinear model reduction algorithms are still lacking.

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Thanks !