PLANAR QUINTIC PH CURVES

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NEW ALGEBRAIC AND GEOMETRIC CHARACTERIZATIONS

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JOINT WORK WITH

New algebraic and geometric characterizations of planar quintic Pythagorean-hodograph curves, CAGD, Vol. 108, 2024, 102256

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OUTLINE

1. Planar PH curves: from cubics to quintics

2. Algebraic characterization: known -vs- new

3. Geometric characterization: known -vs- new

PLANAR BÉZIER CURVES

degree: $n \in \mathbb{N}$

control polygon: $\{\, \boldsymbol{p}_k \in \mathbb{C} \,\}_{k=0}^{n} \quad \longrightarrow \quad \{\, \boldsymbol{e}_k = \boldsymbol{p}_{k+1} - \boldsymbol{p}_k \,\}_{k=0}^{n-1}$

Bernstein basis: $B_k^n(t) = \binom{n}{k} (1-t)^{n-k}\, t^k, \qquad k = 0, \ldots, n$

Bézier curve:

$$\begin{aligned} \boldsymbol{r} : \ \mathbb{R} &\longrightarrow \mathbb{C} \\ t &\longrightarrow \sum_{k=0}^{n} \boldsymbol{p}_k\, B_k^n(t) \end{aligned} \qquad \Rightarrow \qquad \boldsymbol{r}'(t) = n \sum_{k=0}^{n-1} \boldsymbol{e}_k\, B_k^{n-1}(t)$$

PYTHAGOREAN-HODOGRAPH CURVES

pythagorean-hodograph: $|\boldsymbol{r}'|$ is a real polynomial

Theorem (Kubota - 1972 / Lü - 1995 / Wang, Fang - 2009)

A regular Bézier curve $\boldsymbol{r}$ is a (primitive) PH curve if and only if

$$\boldsymbol{r}'(t) = \boldsymbol{w}(t)^2$$

for some complex polynomial $\boldsymbol{w}$ with $\boldsymbol{w}(t) \neq 0$ for $t \in \mathbb{R}$

CUBIC PH CURVES

$$\deg(\boldsymbol{r}') = 2 \iff \deg(\boldsymbol{w}) = 1$$

$$3 \sum_{k=0}^{2} \boldsymbol{e}_k \, B_k^2(t) = \boldsymbol{r}'(t) = \left(\boldsymbol{w}_0 \, (1-t) + \boldsymbol{w}_1 \, t \right)^2$$

$$= \boldsymbol{w}_0^2 \, B_0^2(t) + \boldsymbol{w}_0 \boldsymbol{w}_1 \, B_1^2(t) + \boldsymbol{w}_1^2 \, B_2^2(t)$$

Algebraic characterization (Farouki - 1994)

A regular cubic Bézier curve $\boldsymbol{r}$ is a PH curve if and only if

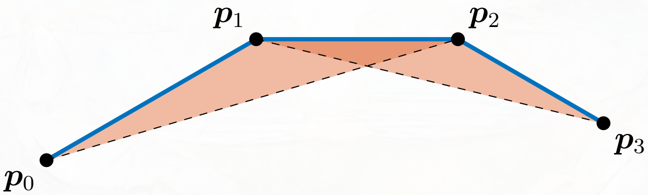
$$\boldsymbol{e}_0 \, \boldsymbol{e}_2 = \boldsymbol{e}_1^2$$

CUBIC PH CURVES

Geometric characterization (Farouki - 1994)

A regular cubic Bézier curve $\boldsymbol{r}$ is a PH curve if and only if

$$\triangle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2) \quad \text{is similar to} \quad \triangle(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3)$$



QUINTIC PH CURVES

$$\deg(\boldsymbol{r}') \ = \ 4 \quad \Longleftrightarrow \quad \deg(\boldsymbol{w}) \ = \ 2$$

$$5 \sum_{k=0}^{4} \boldsymbol{e}_k \, B_k^4(t) \ = \ \boldsymbol{r}'(t) \ = \ \left(\boldsymbol{w}_0(1-t)^2 \ + \ 2\boldsymbol{w}_1(1-t)t \ + \ \boldsymbol{w}_2 t^2 \right)^2$$

$$\Downarrow$$

$$5\boldsymbol{e}_0 = \boldsymbol{w}_0^2, \quad 5\boldsymbol{e}_1 = \boldsymbol{w}_0\boldsymbol{w}_1, \quad 5\boldsymbol{e}_2 = \frac{2\boldsymbol{w}_1^2 + \boldsymbol{w}_0\boldsymbol{w}_2}{3}, \quad 5\boldsymbol{e}_3 = \boldsymbol{w}_1\boldsymbol{w}_2, \quad 5\boldsymbol{e}_4 = \boldsymbol{w}_2^2$$

KNOWN ALGEBRAIC CHARACTERIZATION

Theorem (Farouki - 1994)

A regular quintic Bézier curve $\boldsymbol{r}$ is a PH curve if and only if its control edges satisfy

$$\boldsymbol{e}_0\,\boldsymbol{e}_3^2 \;=\; \boldsymbol{e}_1^2\,\boldsymbol{e}_4$$

and are consistent with the six constraints

$$\begin{aligned}
3\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_0^2\boldsymbol{e}_3 - 2\boldsymbol{e}_1^3 &= 0\\
3\boldsymbol{e}_4\boldsymbol{e}_3\boldsymbol{e}_2 - \boldsymbol{e}_4^2\boldsymbol{e}_1 - 2\boldsymbol{e}_3^3 &= 0\\
3\boldsymbol{e}_0\boldsymbol{e}_3\boldsymbol{e}_2 - \boldsymbol{e}_4\boldsymbol{e}_0\boldsymbol{e}_1 - 2\boldsymbol{e}_1^2\boldsymbol{e}_3 &= 0\\
3\boldsymbol{e}_4\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_0\boldsymbol{e}_4\boldsymbol{e}_3 - 2\boldsymbol{e}_3^2\boldsymbol{e}_1 &= 0\\
9\boldsymbol{e}_0\boldsymbol{e}_2^2 - 6\boldsymbol{e}_1^2\boldsymbol{e}_2 - 2\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_0^2\boldsymbol{e}_4 &= 0\\
9\boldsymbol{e}_4\boldsymbol{e}_2^2 - 6\boldsymbol{e}_3^2\boldsymbol{e}_2 - 2\boldsymbol{e}_4\boldsymbol{e}_3\boldsymbol{e}_1 - \boldsymbol{e}_4^2\boldsymbol{e}_0 &= 0
\end{aligned}$$

A KEY OBSERVATION

Since $\boldsymbol{w}_0\boldsymbol{w}_2 \neq 0$,

$$5\sum_{k=0}^{4} \boldsymbol{e}_k\, B_k^4(t) = \boldsymbol{r}'(t) = \left(\boldsymbol{w}_0(1-t)^2 + 2\boldsymbol{w}_1(1-t)t + \boldsymbol{w}_2 t^2 \right)^2$$

$$= 5\boldsymbol{k}\left(\boldsymbol{u}(1-t)^2 + 2\boldsymbol{v}(1-t)t + \frac{1}{\boldsymbol{u}}t^2 \right)^2$$

$$\Downarrow$$

$$\boldsymbol{e}_0 = \boldsymbol{k}\boldsymbol{u}^2, \quad \boldsymbol{e}_1 = \boldsymbol{k}\boldsymbol{u}\boldsymbol{v}, \quad \boldsymbol{e}_2 = \boldsymbol{k}\frac{2\boldsymbol{v}^2+1}{3}, \quad \boldsymbol{e}_3 = \boldsymbol{k}\frac{\boldsymbol{v}}{\boldsymbol{u}}, \quad \boldsymbol{e}_4 = \boldsymbol{k}\frac{1}{\boldsymbol{u}^2}$$

WHAT'S THE "CORE" OF THE MATTER?

$$\boldsymbol{e}_0 = \boldsymbol{k}\boldsymbol{u}^2, \qquad \boldsymbol{e}_1 = \boldsymbol{k}\boldsymbol{u}\boldsymbol{v}, \qquad \boldsymbol{e}_2 = \boldsymbol{k}\frac{2\boldsymbol{v}^2+1}{3}, \qquad \boldsymbol{e}_3 = \boldsymbol{k}\frac{\boldsymbol{v}}{\boldsymbol{u}}, \qquad \boldsymbol{e}_4 = \boldsymbol{k}\frac{1}{\boldsymbol{u}^2}$$

$$\Downarrow$$

$$\boldsymbol{k}^2 = \boldsymbol{e}_0\,\boldsymbol{e}_4$$

which root?

WHAT'S THE "CORE" OF THE MATTER?

$$\boldsymbol{e}_0 = \boldsymbol{k}\boldsymbol{u}^2, \quad \boldsymbol{e}_1 = \boldsymbol{k}\boldsymbol{u}\boldsymbol{v}, \quad \boldsymbol{e}_2 = \boldsymbol{k}\frac{2\boldsymbol{v}^2+1}{3}, \quad \boldsymbol{e}_3 = \boldsymbol{k}\frac{\boldsymbol{v}}{\boldsymbol{u}}, \quad \boldsymbol{e}_4 = \boldsymbol{k}\frac{1}{\boldsymbol{u}^2}$$

$$\Downarrow$$

$$\boldsymbol{k} = \frac{\boldsymbol{e}_0\boldsymbol{e}_3}{\boldsymbol{e}_1} = \frac{\boldsymbol{e}_1\boldsymbol{e}_4}{\boldsymbol{e}_3}$$

undefined for $\boldsymbol{e}_1, \boldsymbol{e}_3 = 0$

WHAT'S THE "CORE" OF THE MATTER?

$$\boldsymbol{e}_0 = \boldsymbol{k}\boldsymbol{u}^2, \quad \boldsymbol{e}_1 = \boldsymbol{k}\boldsymbol{u}\boldsymbol{v}, \quad \boldsymbol{e}_2 = \boldsymbol{k}\frac{2\boldsymbol{v}^2 + 1}{3}, \quad \boldsymbol{e}_3 = \boldsymbol{k}\frac{\boldsymbol{v}}{\boldsymbol{u}}, \quad \boldsymbol{e}_4 = \boldsymbol{k}\frac{1}{\boldsymbol{u}^2}$$

$$\Downarrow$$

$$\boldsymbol{k} := 3\boldsymbol{e}_2 - \left(\frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0} + \frac{\boldsymbol{e}_3^2}{\boldsymbol{e}_4} \right)$$

kern, defined for every regular quintics

NEW ALGEBRAIC CHARACTERIZATION

Theorem (Hormann, Romani, V. - 2024)

A regular quintic Bézier curve $\boldsymbol{r}$ is a PH curve if and only if its control edges satisfy

$$\begin{aligned} \boldsymbol{e}_0\boldsymbol{e}_4 &= \boldsymbol{k}^2 \\ \boldsymbol{e}_0\boldsymbol{e}_4 &= 3\boldsymbol{k}\boldsymbol{e}_2 - 2\boldsymbol{e}_1\boldsymbol{e}_3 \end{aligned}$$

KNOWN GEOMETRIC CHARACTERIZATION

Theorem (Farouki - 1994)

A regular quintic Bézier curve $\boldsymbol{r}$ is a PH curve if and only if

$$\frac{|\boldsymbol{e}_1|}{|\boldsymbol{e}_3|} = \sqrt{\frac{|\boldsymbol{e}_0|}{|\boldsymbol{e}_4|}}$$

$$\arg\frac{\boldsymbol{e}_0}{\boldsymbol{e}_1} + \arg\frac{\boldsymbol{e}_3}{\boldsymbol{e}_4} = \arg\frac{\boldsymbol{e}_1}{\boldsymbol{e}_2} + \arg\frac{\boldsymbol{e}_2}{\boldsymbol{e}_3}$$

$$3|\boldsymbol{e}_0||\boldsymbol{e}_1||\boldsymbol{e}_2|\cos\arg\frac{\boldsymbol{e}_1}{\boldsymbol{e}_2} = |\boldsymbol{e}_0|^2|\boldsymbol{e}_3|\cos\arg\frac{\boldsymbol{e}_3}{\boldsymbol{e}_4} + 2|\boldsymbol{e}_1|^3\cos\arg\frac{\boldsymbol{e}_0}{\boldsymbol{e}_1}$$

$$3|\boldsymbol{e}_0||\boldsymbol{e}_1||\boldsymbol{e}_2|\sin\arg\frac{\boldsymbol{e}_1}{\boldsymbol{e}_2} = |\boldsymbol{e}_0|^2|\boldsymbol{e}_3|\sin\arg\frac{\boldsymbol{e}_3}{\boldsymbol{e}_4} + 2|\boldsymbol{e}_1|^3\sin\arg\frac{\boldsymbol{e}_0}{\boldsymbol{e}_1}$$

KNOWN GEOMETRIC CHARACTERIZATION

- $\boldsymbol{q}_1 \in \overline{\boldsymbol{p}_0\boldsymbol{p}_1}$, $\boldsymbol{q}_4 \in \overline{\boldsymbol{p}_4\boldsymbol{p}_5}$:

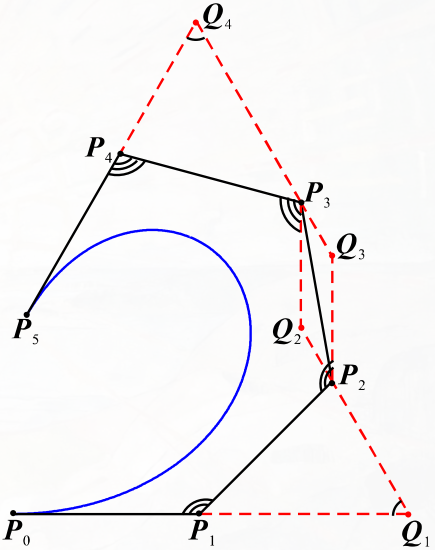
$$\overline{\boldsymbol{q}_1\boldsymbol{p}_2} \parallel \overline{\boldsymbol{p}_3\boldsymbol{q}_4}$$

$$\angle(\boldsymbol{p}_0, \boldsymbol{q}_1, \boldsymbol{p}_2) = \angle(\boldsymbol{p}_3, \boldsymbol{q}_4, \boldsymbol{p}_5)$$

- $\boldsymbol{q}_2 \in \overline{\boldsymbol{q}_1\boldsymbol{p}_2}$, $\boldsymbol{q}_3 \in \overline{\boldsymbol{q}_4\boldsymbol{p}_3}$:

$$\angle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2) = \angle(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{q}_3)$$

$$\angle(\boldsymbol{q}_2, \boldsymbol{p}_3, \boldsymbol{p}_4) = \angle(\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5)$$



KNOWN GEOMETRIC CHARACTERIZATION

Theorem (Fang, Wang - 2018)

A regular quintic Bézier curve $\boldsymbol{r}$ is a PH curve if and only if the following conditions hold

$\square(\boldsymbol{p}_2, \boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{q}_2)$ is a parallelogram

$\triangle(\boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{p}_2)$ is similar to $\triangle(\boldsymbol{p}_3, \boldsymbol{q}_4, \boldsymbol{p}_4)$

$$2|\boldsymbol{e}_1|^2 = 3|\boldsymbol{e}_0||\boldsymbol{p}_2 - \boldsymbol{q}_3|$$

$$2|\boldsymbol{e}_3|^2 = 3|\boldsymbol{e}_4||\boldsymbol{q}_2 - \boldsymbol{p}_3|$$

$$|\boldsymbol{e}_0||\boldsymbol{e}_4| = 9|\boldsymbol{p}_2 - \boldsymbol{q}_2|^2$$

NEW GEOMETRIC CHARACTERIZATION

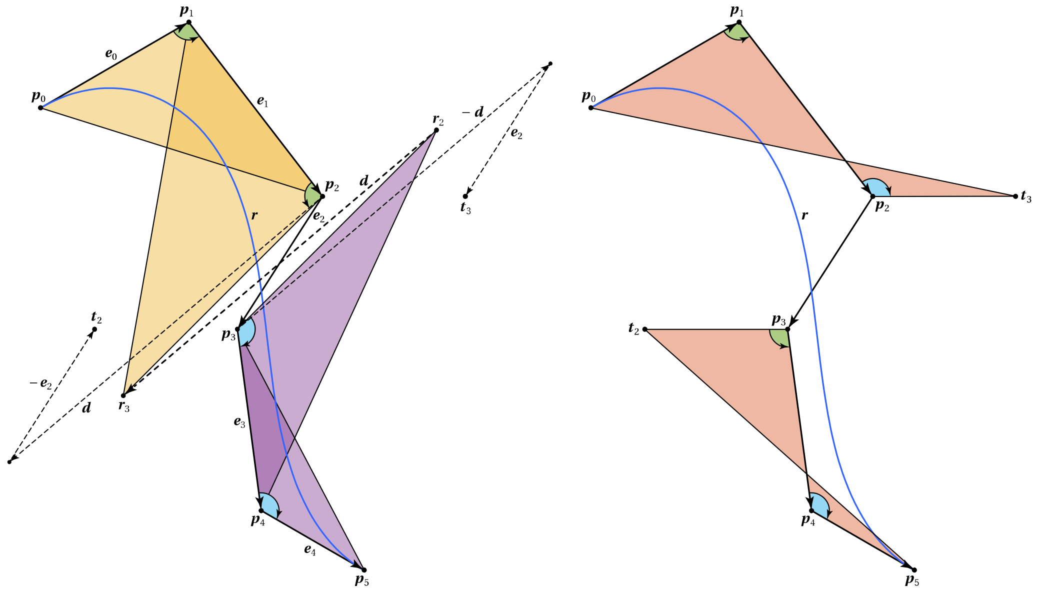
$$\boldsymbol{t}_2 := \boldsymbol{p}_3 - \boldsymbol{k} \quad \text{and} \quad \boldsymbol{t}_3 := \boldsymbol{p}_2 + \boldsymbol{k}$$

Theorem (Hormann, Romani, V. - 2024)

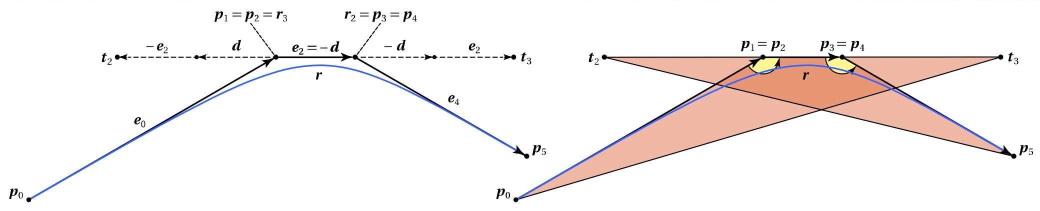
A regular quintic Bézier curve is a PH curve if and only if

$$\square(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{t}_3) \quad \text{is similar to} \quad \square(\boldsymbol{t}_2, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5)$$

NEW GEOMETRIC CHARACTERIZATION



NEW GEOMETRIC CHARACTERIZATION



WRAP-UP

- ✓ New algebraic characterization of planar quintic PH curves:
  - minimal formulation with two equations
  - encompasses all special cases
  - structure similar to the one of the cubic case

- ✓ New geometric characterization of planar quintic PH curves:
  - only two auxiliary points and one similarity condition
  - no algebraic conditions needed
  - structure similar to the one of the cubic case

K. Hormann, L. Romani, A. Viscardi, *New algebraic and geometric characterizations of planar quintic Pythagorean-hodograph curves*, CAGD, Vol. 108, 2024, 102256

THANK YOU