Optimal uniform approximation by planar parametric polynomials

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Overview

1. Functional case

2. Parametric case

3. Parametric Remez's algorithm

Functional case

- $f : [a, b] \to \mathbb{R}$ smooth function
- $\|\cdot\|_\infty$ max norm on $\mathcal{C}([a, b])$: $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$
- Find a polynomial $p^* \in \mathbb{P}_n$, such that

$$p^* = \operatorname{argmin} \inf_{p \in \mathbb{P}_n} \|f - p\|_\infty$$

- p^* is uniquely determined: the polynomial of the best uniform approximation (minimax polynomial)

Theorem (Chebyshev equioscillating theorem)

If $f \in \mathcal{C}([a,b])$ then p^* is the minimax polynomial of degree $\leq n$ if and only if there exist $n+2$ points on $[a,b]$, $a \leq x_0 < x_1 < \cdots < x_{n+1} \leq b$, such that

$$f(x_i) - p^*(x_i) = \sigma(-1)^i \|f - p^*\|_\infty, \quad i = 0, 1, \ldots, n+1, \quad \sigma \text{ is either } -1 \text{ or } 1.$$

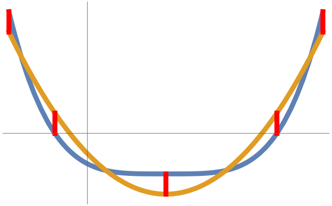


Figure: Function f (blue) and its minimax polynomial of degree 3 (orange).

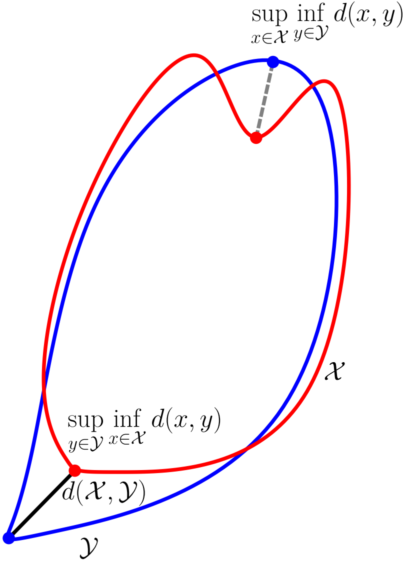
- Construction of p^*: Remez algorithm ([Remez 1934])

Parametric case

- The metric d on $\mathcal{C}([a,b])$: induced by $\|\cdot\|_\infty$: $d(f,g) = \|f-g\|_\infty$
- The metric d_H on the space of parametric curves: the Hausdorff distance of images
- Hausdorff distance between curves $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ is

$$d_H(\boldsymbol{c}_1, \boldsymbol{c}_2) = \max\left\{ \sup_{\boldsymbol{a}\in\boldsymbol{c}_1} \inf_{\boldsymbol{b}\in\boldsymbol{c}_2} d(\boldsymbol{a},\boldsymbol{b}), \sup_{\boldsymbol{b}\in\boldsymbol{c}_2} \inf_{\boldsymbol{a}\in\boldsymbol{c}_1} d(\boldsymbol{a},\boldsymbol{b}) \right\},$$

where d is the Euclidean distance


$\sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} d(x, y)$
$\mathcal{X}$
$\sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} d(x, y)$
$d(\mathcal{X}, \mathcal{Y})$
$\mathcal{Y}$

- Computation of d_H is difficult and expensive
- Even the discrete version is not effficient (quadratic complexity)
- Alternative upper bounds are used
- Parametric distance and normal distance

Parametric distance

- $\boldsymbol{f} : [a, b] \to \mathbb{R}^2$ is s given parametric curve
- $\boldsymbol{r} : [c, d] \to \mathbb{R}^2$ its (polynomial) approximation
- How to compare $\boldsymbol{f}(u)$ and $\boldsymbol{r}(t)$:

$$d_P(\boldsymbol{f}, \boldsymbol{p}) = \inf_{\rho} \max_{t \in [c,d]} \|(\boldsymbol{f} \circ \rho)(t) - \boldsymbol{r}(t)\|_2, \quad \rho : [c, d] \to [a, b], \ \textit{regular}$$

- d_P is the parametric distance ([Lyche and Mørken 1994])

Normal distance

- Take the normal on $\boldsymbol{f}$ at $\boldsymbol{f}(u) = (x, y)$
- Find the closest intersection with $\boldsymbol{r}$
- Normal distance $d_N(\boldsymbol{f}, \boldsymbol{r})$ is the maximum of distances of such pair of points ([Degen 1994])

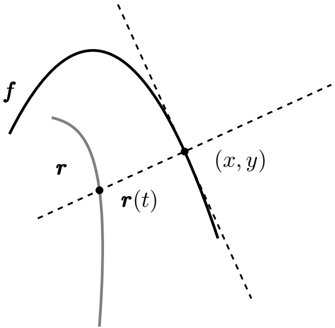


Figure: Normal distance between $\boldsymbol{f}(u) = (x, y)$. and $\boldsymbol{r}$

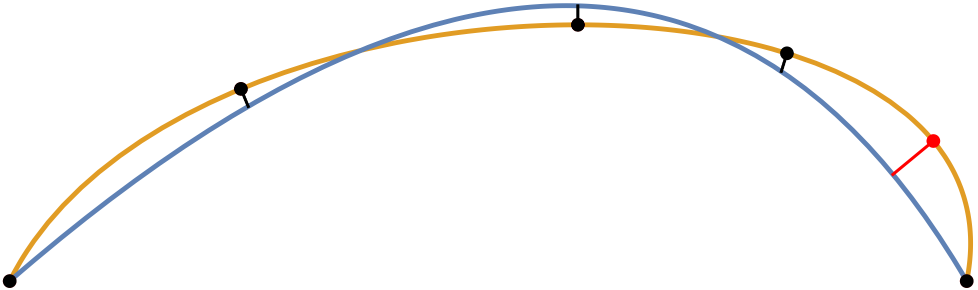
Parametric Remez's algorithm

- Given $\boldsymbol{f} : [a, b] \to \mathbb{R}^2$
- Require G^k interpolation at $\boldsymbol{f}(a)$ and $\boldsymbol{f}(b)$
- Find minimax parametric polynomial $\boldsymbol{p}$ of degree $\leq n$ (in normal distance d_N)
- Conjecture: $\boldsymbol{p}$ must equioscillate $\ell = 2(n-k) - 1$ times arround $\boldsymbol{f}$

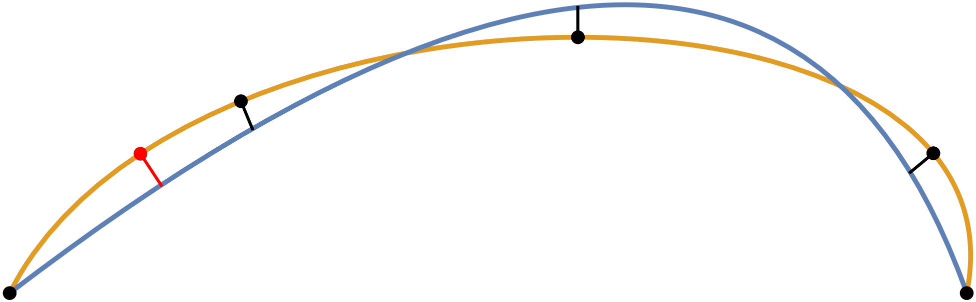
An algorithm:

- Choose $a < s_1 < s_2 < \cdots < s_\ell < b$
- Repeat:
  - Solve $\boldsymbol{p}(t_i) = \boldsymbol{f}(s_i) + (-1)^i d\, \textbf{\textit{unit\_normal}}(\boldsymbol{f}(s_i))$, $i = 1, 2, \ldots, \ell$, for $t_1, t_2, \ldots, t_\ell, d$ and $\boldsymbol{p}$ which fulfills G^k conditions
  - Find $s_{max} \in [a, b]$ for which $d_N(\boldsymbol{f}, \boldsymbol{p})$ is attained
  - Replace appropriate s_k by s_{max} to preserve oscillation
- Until equioscillation

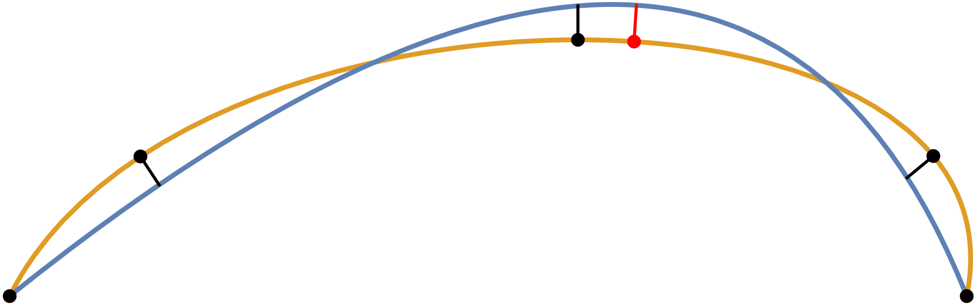
Quadratic G^0 example



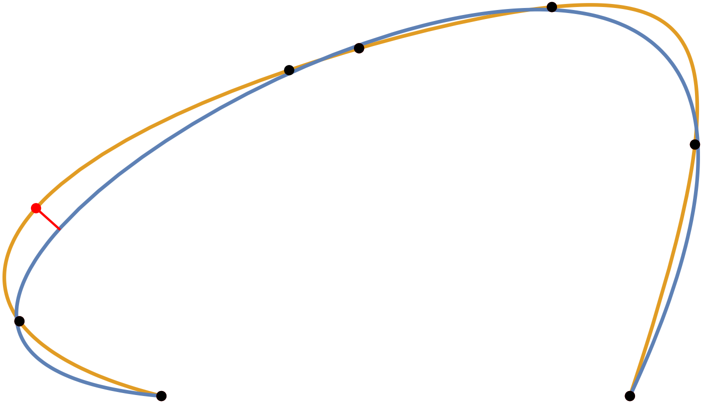
Quadratic G^0 example



Quadratic G^0 example



Cubic G^0 example



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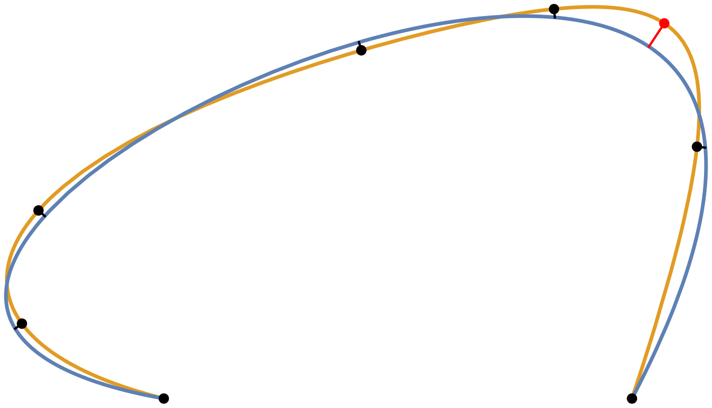

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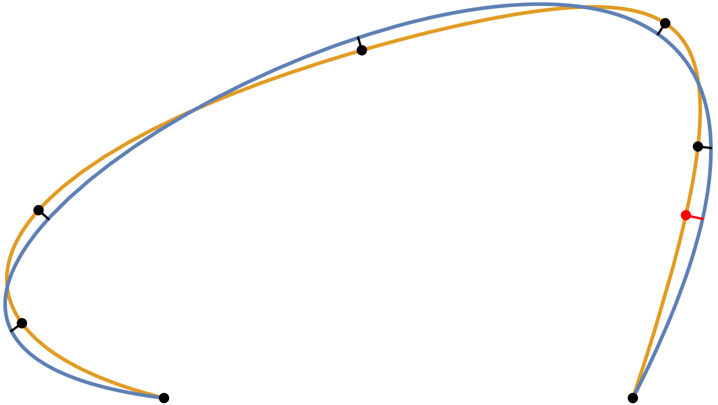

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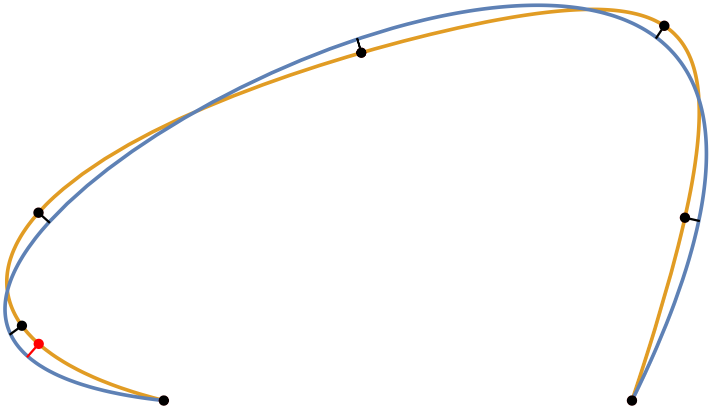

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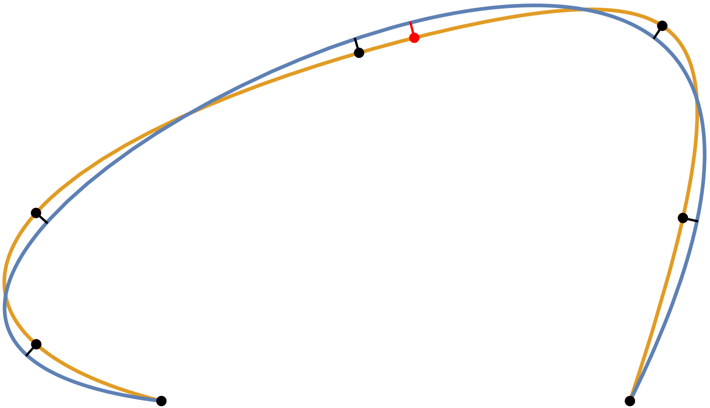

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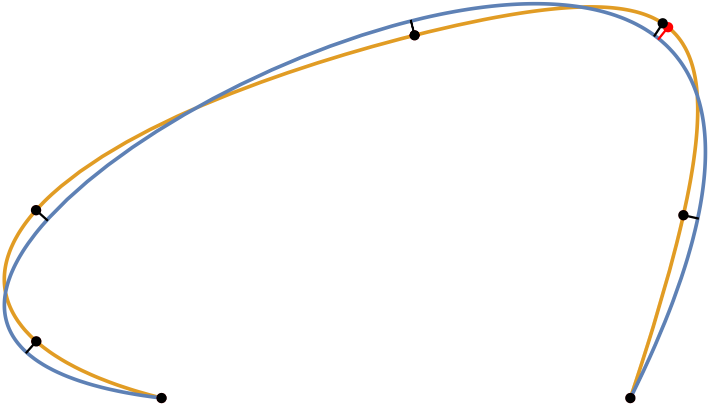

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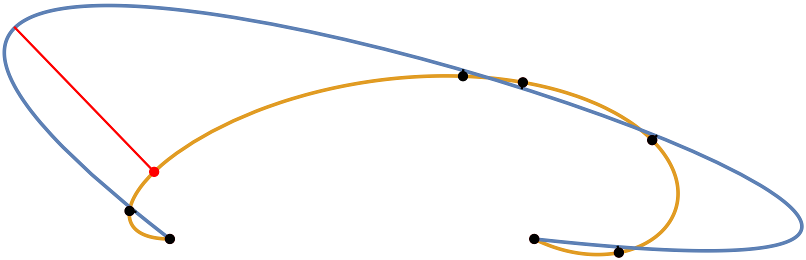

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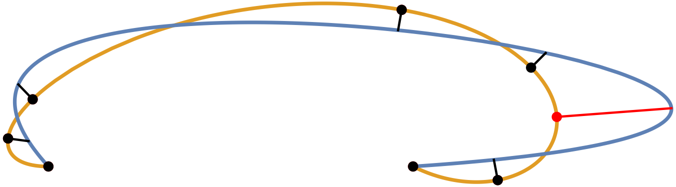

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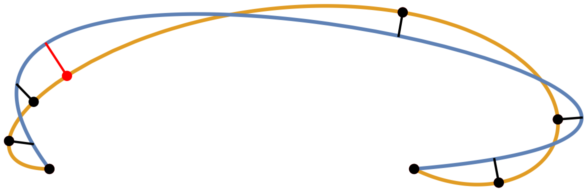

Cubic G^0 example

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