Master project:
Numerical methods for nonlocal conservation laws

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1 Background and outline of the problem

Hyperbolic conservation laws are partial differential equations that can be written in the form

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

where $\partial_t = \frac{\partial}{\partial t}$ is time differentiation, $\nabla\cdot$ is divergence, $u = u(x,t)$ is the unknown (either a scalar- or vector-valued function) and f is a given (usually nonlinear) function, the *flux function*. The initial data is u_0 and the PDE is required to hold for $t > 0$, $x \in \mathbb{R}^d$. A frequently used "toy example" is (1) with $d = 1$ and $f(u) = uv(u)$, where e.g. $v(u) = 1 - u$. This equation can be used for modelling traffic, and $u(x,t)$ represents the density of cars at position x at time t.

A closely related problem is the *nonlocal* equation

$$\begin{aligned} \partial_t u + \nabla \cdot (uV(u)) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{2}$$

where now V is a *nonlocal operator*, say, $V(u)(x,t) = \int_{\mathbb{R}} \omega(y) v(u(x+y,t))\,dy$ for some *nonlocal kernel* $\omega : \mathbb{R} \to \mathbb{R}$ satisfying $\omega \geq 0$ and $\int_{\mathbb{R}} \omega(y)\,dy = 1$. If we choose, say, $\omega = \omega_\varepsilon$, where $\omega_\varepsilon(y) = \varepsilon^{-1}\omega_1(y/\varepsilon)$ and ω_1 is some fixed kernel, then formally speaking, $V(u) \to v(u)$ as $\varepsilon \to 0$. However, it is not at all obvious whether the corresponding solutions u_ε of (2) converge to a solution of (1) (with $f(u) = uv(u)$) as $\varepsilon \to 0$.

The question of whether solutions of (1) can be "approximated" by solutions of (2) in the way described above, is mostly open, but the first positive result was published just recently [2]. They let $v(u) = 1 - u$, $\omega_\varepsilon(y) = \varepsilon^{-1} e^{-y\varepsilon^{-1}}$ for $y \geq 0$ and $\omega_\varepsilon(y) = 0$ for $y < 0$. If u_0 is strictly positive then they find that $u_\varepsilon \to u$ as $\varepsilon \to 0$. The techniques they use rely on some standard convergence techniques (compactness via total variation bound and L^∞ bound, and identification of the limit), but with some twists like applying the Hardy–Littlewood rearrangement theorem and a reformulation of (2) to a relaxation-type system. A similar result was recently published by Coclite et al. [4], who used very different, but more standard, techniques. The fact that $u^\varepsilon \to u$ as $\varepsilon \to 0$ was more fully resolved in the recent paper [6].

2 Main goals of the project

The main goals of the project are to read and understand the recent works on this problem; to design a numerical method for (2), and to prove convergence of this numerical method as the resolution increases.

3 Required background

The project is suitable for anyone with a solid foundation in real analysis and PDEs. A more numerics oriented student might focus more on implementation and experimentation with various nonlocal differential equations, and less on the analysis.

References

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[6] Maria Colombo, Gianluca Crippa, Elio Marconi, and Laura V. Spinolo. Nonlocal traffic models with general kernels: singular limit, entropy admissibility, and convergence rate. *Preprint* (2022). Available from `https://arXiv.org/abs/2206.03949`.

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