# Master class
# Equilibrium states in semigroup theory, K-theory and number theory

## November 4-6, 2019

*The master class is part of the project Pure Mathematics in Norway, 2018-2022, supported by the Trond Mohn Foundation*

# Program

**Monday, November 4**

Niels Henrik Abels hus, room 108

09:30–10:30 **Marcelo Laca**: *Equilibrium states of $C^*$-algebras from number theory. I*

10:30–11:00 Coffee/tea

11:00–12:00 **Xin Li**: *K-theory for semigroup $C^*$-algebras. I*

12:00–14:00 Lunch at Abels utsikt (NHA hus, room 1259)

Niels Henrik Abels hus, room 1120

14:00–15:00 **Alina Vdovina**: *$C^*$-algebras coming from cube complexes and buildings. I*

15:15–15:45 **Jim Tao**: *A twisted local index formula for curved noncommutative two tori*

**Tuesday, November 5**

Georg Sverdrups hus, Undervisningsrom 2

09:30–10:30 **Xin Li**: *K-theory for semigroup $C^*$-algebras. II*

10:30–11:00 Coffee/tea

11:00–12:00 **Alina Vdovina**: *$C^*$-algebras coming from cube complexes and buildings. II*

12:00–14:00 Lunch at Abels utsikt (NHA hus, room 1259)

14:00–15:00 **Marcelo Laca**: *Equilibrium states of $C^*$-algebras from number theory. II*

15:15–16:00 **Johannes Christensen**: *KMS states on the crossed product $C^*$-algebra of a homeomorphism*

18:00–21:00 Dinner at Sudøst restaurant

**Wednesday, November 6**

Georg Sverdrups hus, Auditorium 2

09:30–10:30 **Alina Vdovina**: *$C^*$-algebras coming from cube complexes and buildings. III*

10:30–11:00 Coffee/tea

11:00–12:00 **Marcelo Laca**: *Equilibrium states of $C^*$-algebras from number theory. III*

12:00–14:00 Lunch at Abels utsikt (NHA hus, room 1259)

Georg Sverdrups hus, Undervisningsrom 2

14:00–15:00 **Xin Li**: *K-theory for semigroup $C^*$-algebras. III*

15:15–16:00 **Chris Bruce**: *$C^*$-algebras from actions of congruence monoids*

# Abstracts

**Marcelo Laca**: *Equilibrium states of C*-algebras from number theory*

Lecture 1: *Quantum statistical mechanical systems and their equilibrium states*

A brief introduction to KMS states of C*-dynamical systems, with examples drawn from matrix algebras, Cuntz algebras, their Toeplitz extensions, and C*-algebras of semigroups. Motivation for systems from number theory, the free Riemann gas and connections between multiplicative number theory and quantum statistical mechanics, early work of B. Julia and of D. Spector. The 'model with interaction' of Bost and Connes, exhibiting a phase transition with spontaneous symmetry breaking and the Riemann zeta function as partition function.

Lecture 2: *Generalized Gibbs states and low-temperature phase transitions*

The simplex of $\mathrm{KMS}_\beta$ states as a function of the inverse temperature $\beta \in \mathbb{R}$. Arbitrary phase transitions at arbitrary inverse temperatures, work of O. Bratteli, G. Elliott, R. Herman and A. Kishimoto. Ground states and $\mathrm{KMS}_\infty$ states. Groupoid C*-algebras and dynamics from cocycles. Quasi-invariant measures and isotropy. Renault's characterization of KMS states and Neshveyev's generalization to nonprincipal groupoids. $\mathrm{KMS}_\infty$ states and the boundary groupoid. Liouville operators. A closer look at partition functions. Persistence of type I pure phases. C*-dynamical invariants.

Lecture 3: *Systems from semigroups of algebraic integers*

Brief introduction to algebraic number fields: algebraic integers, ideals and unique factorization, completions, finite adeles and ideles. Systems arising from $ax + b$ semigroup of rings of algebraic integers and congruence monoids. Computation of equilibrium states. The noncommutative parameter space $\bigoplus_{\chi\in \mathrm{Cl}} C^*(J_\chi \rtimes R^*)$ and groups of linear toral automorphisms. Ergodic invariant measures, primitive ideals, and partition functions. The generalized Furstenberg question. Connections with $K$-theory.

**Xin Li**: *K-theory for semigroup C*-algebras*

Lecture 1: *An introduction to semigroup C*-algebras*

Lecture 2: *Computing K-theory for crossed products of certain zero-dimensional dynamical systems*

Lecture 3: *A K-theory formula for semigroup C*-algebras and its applications*

**Alina Vdovina**: *C*-algebras coming from cube complexes and buildings*

We will give an elementary introduction to the theory of cube complexes and buildings. Then we show how explicit geometric structures help in studying higher-rank graph C*-algebras and their K-theory.

**Chris Bruce**: *C*-algebras from actions of congruence monoids*

The left regular C*-algebra of the full $ax+b$-semigroup over a ring of integers in a number field was first studied by Cuntz, Deninger, and Laca. These C*-algebras exhibited several new and interesting phenomena, and their investigation led to several new developments, particularly in the theory of semigroup C*-algebras. From a number-theoretic point of view,

it is natural to consider semigroup C*-algebras of a more general class of semigroups that are constructed from actions of congruence monoids on rings of algebraic integers. In this talk, I will explain this more general construction, and then discuss how to use K-theoretic invariants to recover number-theoretic data from these semigroup C*-algebras. (Joint work with Xin Li.)

**Johannes Christensen**: *KMS states on the crossed product $C^*$-algebra of a homeomorphism*

Assume that $X$ is a locally compact second countable Hausdorff space, and that $\phi : X \to X$ is a homeomorphism. The crossed product C* -algebra $C(X) \rtimes_\phi \mathbb{Z}$ of $(X, \phi)$ is the universal C*-algebra generated by a copy of $C(X)$ and a unitary $U$ satisfying

$$UfU^* = f \circ \phi^{-1} \text{ for } f \in C(X).$$

These C*-algebras have a canonical class of 1-parameter groups: Given any continuous function $F : X \to \mathbb{R}$ the condition:

$$\alpha_t^F(fU) = fUe^{-itF} = e^{-itF\circ\phi^{-1}}fU$$

for $f \in C(X)$ defines a continuous 1-parameter group. In this talk I am going to discuss the problem of describing the KMS states for the C*-dynamical system $(C(X) \rtimes_\phi \mathbb{Z}, \alpha^F)$, and among other things I am going to present some preliminary results on the existence of KMS states for such systems.

**Jim Tao**: *A twisted local index formula for curved noncommutative two tori*

We consider the Dirac operator of a general metric in the canonical conformal class on the noncommutative two torus, twisted by an idempotent (representing the K-theory class of a general noncommutative vector bundle), and derive a local formula for the Fredholm index of the twisted Dirac operator. Our approach is based on the McKean-Singer index formula, and explicit heat expansion calculations by making use of Connes' pseudodifferential calculus. As a technical tool, a new rearrangement lemma is proved to handle challenges posed by the noncommutativity of the algebra and the presence of an idempotent in the calculations in addition to a conformal factor.