C*-algebras associated with algebraic actions

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Topic: Actions by endomorphisms on a compact abelian group H.
Most typical examples:

$$H = \mathbb{T}^n \qquad H = (\mathbb{Z}/n)^{\mathbb{N}} \qquad H = \varprojlim_{z \mapsto z^n} \mathbb{T}$$

We consider an endomorphism α of H satisfying

- α is surjective
- Ker α is finite
- $\bigcup_n \operatorname{Ker} \alpha^n$ is dense in H.

α preserves Haar measure on H and therefore induces an isometry s_α on $L^2 H$. Also $C(H)$ act as multiplication operators on $L^2 H$.

Definition We denote by $\mathcal{A}[\alpha]$ the sub-C*-algebra of $\mathcal{L}(L^2 H)$ generated by $C(H)$ together with s_α.

Remark. $\mathcal{A}[\alpha]$ is not the crossed product of $C(H)$ by α.

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The C*-algebra $\mathcal{A}[\alpha]$ contains as a natural subalgebra the C*-algebra $\mathcal{B}[\alpha]$ generated by $C(H)$ together with all range projections $s_\alpha^n s_\alpha^{*\,n}$. This subalgebra is of UHF- or Bunce-Deddens type and is simple with a unique trace. It can also be described as a crossed product $\overline{H} \rtimes \hat{H}$, where $\hat{H}$ denotes the dual group and $\overline{H}$ an α-adic completion of $\hat{H}$.
Moreover $\mathcal{A}[\alpha]$ is a crossed product $\mathcal{B}[\alpha] \rtimes \mathbb{N}$ by the action of α.

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Theorem (Cuntz-Vershik) $\mathcal{A}[\alpha]$ is simple and purely infinite. It can be described as a universal C*-algebra with a natural set of generators and relations.

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Theorem (Cuntz-Vershik) $\mathcal{A}[\alpha]$ is simple and purely infinite. It can be described as a universal C*-algebra with a natural set of generators and relations.
K-theory The K-theory of $\mathcal{A}[\alpha]$ fits into an exact sequence of the form

$$\begin{array}{ccccc} K_* C^*(\hat{H}) & \xrightarrow{1-b(\alpha)} & K_* C^*(\hat{H}) & \longrightarrow & K_* \mathcal{A}[\alpha] \\ & \nwarrow & & \swarrow & \\ & & \longleftarrow & & \end{array}$$

where the map $b(\alpha)$ satisfies the equation $b(\alpha)\alpha_* = N(\alpha)$ with $N(\alpha) = |\mathrm{Ker}\,\alpha|$.

The next question concerns the case where a single endomorphism of H is replaced by a (countable) family of endomorphisms. An especially interesting case arises from the ring R of algebraic integers in a number field K. Here we consider the additive group R and its dual group $H = \hat{R} \cong \mathbb{T}^n$ and the endomorphisms determined by the elements of the multiplicative semigroup $R^\times$ of R. Again $C(H)$ acts by multiplication on $L^2H \cong \ell^2R$ and the endomorphisms induce a family of isometries s_α of L^2H.

The C*-algebra generated by $C(H)$ together with all the s_α was studied under the name 'ring C*-algebra' by Cuntz-Li and denoted by $\mathcal{A}[R]$ (it is related to Bost-Connes systems).

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Theorem (Cuntz-Li) $\mathcal{A}[R]$ is simple and purely infinite. It can be described as a universal C*-algebra with a natural set of generators and relations.

K-theory In order to compute the K-theory of $\mathcal{A}[R]$ we use a duality result. Assume for simplicity that $R = \mathbb{Z}$, $K = \mathbb{Q}$. Then we show that

$$\mathcal{K} \otimes \mathcal{A}[\mathbb{Z}] \cong C_0(\mathbb{R}) \rtimes \mathbb{Q} \rtimes \mathbb{Q}^\times$$

From this the K-theory can be computed with the result that $K_*(\mathcal{A}[\mathbb{Z}])$ is a free exterior algebra with one generator for each prime number p.

Structure of the left regular C*-algebra $C^*_\lambda(R \rtimes R^\times)$

The C*-algebra $\mathcal{A}[R]$ is generated by the natural representation of the semidirect product semigroup $R \rtimes R^\times$ on $\ell^2 R$. However it is a natural question to also consider the regular C*-algebra $C^*_\lambda(R \rtimes R^\times)$ generated by the left regular representation of $R \rtimes R^\times$ on $\ell^2(R \rtimes R^\times)$.

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It also carries a natural one-parameter action with an interesting *KMS*-structure including a symmetry breaking over the class group $Cl_R = \{\text{ideals of } R\}/\{\text{principal ideals}\}$ of R for large inverse temperatures (Cuntz-Deninger-Laca).

K-theory

Theorem (Cuntz-Echterhoff-Li) Let R^* be the group of units in R and Cl_R the class group. Choose for every ideal class $\gamma \in Cl_R$ an ideal I_γ of R which represents γ. The K-theory of the left regular C*-algebra $C^*_\lambda(R \rtimes R^\times)$ is given by the formula

$$K_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{\gamma \in Cl_R} K_*(C^*_\lambda(I_\gamma \rtimes R^*)).$$

This is a special case of the following general theorem. We consider a semigroup P which is a subsemigroup of a group G.

Theorem (Cuntz-Echterhoff-Li) Assume that the following conditions are satisfied:

1. $P \subseteq G$ satisfies the K-theoretic Toeplitz condition;
2. The set $\mathcal{J}_{P \subseteq G}$ of constructible right P-ideals in G is independent;
3. G satisfies the Baum-Connes conjecture with coefficients.

Then there is a canonical isomorphism

$$K_*(C^*_\lambda P) \cong \bigoplus_{[X]} K_*(C^*(G_X)).$$

The sum is over all $[X]$ in the set of G-orbits in $\mathcal{I}_{P \subseteq G} \setminus \emptyset$ and G_X denotes the stabilizer group of $[X]$.

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This theorem can be used to compute the K-theory of $C^*_\lambda P$ for many more semigroups P.

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