Exotic crossed products and K-theory

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Motivation

Exotic group algebras

There has been extensive recent work on exotic group algebras

$$C^*_{max}(G) \to C^*_{\mu}(G) \to C^*_r(G)$$

for locally compact groups G by Brown-Guentner, Wirsma and others. They allow the construction of corresponding exotic crossed-products $A \rtimes_{\alpha,\mu} G$ due to Brown-Guenter and Kaliszewski-Landstad-Quigg.

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Baum-Connes conjecture (Baum-Guentner-Willett 2014)

$$\mu : K^{\mathrm{top}}_*(G;A) \xrightarrow{\cong} K_*(A \rtimes_{\alpha,r} G)$$

Fails to be true for the "Gromov Monster group" due to failure of "exactness" of $(A,\alpha) \mapsto A \rtimes_{\alpha,r} G$.

Idea: Replace $A \rtimes_{\alpha,r} G$ by the smallest exact Morita compatible crossed product functor $A \rtimes_{\alpha,\epsilon} G$.

Exotic group algebras

Let $C_c(G)$ be equipped with the usual involution and convolution and recall that $C_u^*(G)$ and $C_r^*(G)$ are completions of $C_c(G)$ with respect to

$$\|f\|_u = \sup\{\|U(f)\| : U \text{ a unit. rep. of } G\} \quad \text{and} \quad \|f\|_r = \|\lambda(f)\|$$

with

$$U(f) = \int_G f(t) U_t \, dt, \quad f \in C_c(G).$$

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Hence, the identity on $C_c(G)$ induces a sequence of surjective $*$-homom.

$$C^*_u(G) \twoheadrightarrow C^*_\mu(G) \twoheadrightarrow C^*_r(G).$$

Let $B(G) = \{s \mapsto \langle U_s\xi, \eta\rangle : U$ unitary rep. of G on H_U, $\xi, \eta \in H_U\}$ the Fourier-Stieltjes algebra of G. Note that

$$B(G) = C^*_u(G)' \quad \text{via} \quad C^*_u(G) \ni x \mapsto \langle U(x)\xi, \eta\rangle.$$

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Proposition (Kaliszeswki-Landstad-Quigg)

There is a one-to-one correspondence between

1. Exotic group algebras $C_\mu^*(G)$ of G.
2. weak*-closed left- and right translation invariant subspaces E of $B(G)$ which contain $B_c(G) = C_c(G) \cap B(G)$.

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If $C_\mu^*(G)$ is given, let $E = C_\mu^*(G)' \subseteq C_u^*(G)' = B(G)$.

Conversely: If $E \subseteq B(G)$, then $C_E^*(G) = C_u^*(G)/I_E$ with $I_E = {}^\perp E$.

Examples

For $p \in [2, \infty]$ let $E_p = \overline{L^p(G) \cap B(G)}^{w*}$ and $E_0 = \overline{C_0(G) \cap B(G)}^{w*}$.

We write $C^*_p(G) := C^*_{E_p}(G)$.

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Theorem (Brown-Guentner '13, Okayasu '14)

1. G is amenable if and only if $\exists p < \infty$ with $C^*_p(G) = C^*_u(G)$.
2. G has the Haagerup property if and only if $C^*_0(G) = C^*_u(G)$.
3. (Okayasu) If G is discrete and $F_2 \subseteq G$, then $C^*_p(G) \neq C^*_q(G)$ for all $p, q \in [2,\infty]$ with $p < q$.

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Note also: All $E_p \subseteq B(G)$ as above are ideals in $B(G)$. In general:

$E \subseteq B(G)$ is an ideal $\Leftrightarrow$ $s \mapsto s \otimes s : G \to UM(C_E(G) \otimes C^*_u(G))$ integrates to a coaction $\delta_E : C^*_E(G) \to M(C^*_E(G) \otimes C^*_u(G))$.

Exotic crossed products

Definition (Baum-Guentner-Willett)

An exotic crossed-product functor is a functor $(A, G, \alpha) \to A \rtimes_{\alpha,\mu} G$ from the category of G-C*-algebras into the category of C*-algebras with (G-) *-homomorphisms, such that the identity on $C_c(G, A)$ induces morphisms

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Examples: Let $C^*_E(G)$ be given with quot. map $U : C^*_u(G) \to C^*_E(G)$.

BG-crossed products (Brown-Guentner): For $f \in C_c(G, A)$ let

$$\|f\|_{E_{BG}} = \sup\{\|\rho \rtimes V(f)\| : (\rho, V) \text{ covariant } V \prec U\}.$$

Then

$$(A, \alpha) \mapsto A \rtimes_{\alpha,E_{BG}} G := \overline{C_c(G, A)}^{\|\cdot\|_{E_{BG}}}$$

is functor with $\mathbb{C} \rtimes_{E_{BG}} G = C^*_E(G)$.

Examples of crossed-product functors

KLQ-crossed products (Kaliszewski-Landstad-Quigg):

Let $0 \neq E \subseteq B(G)$ be a G-invariant ideal and $U : C^*_u(G) \twoheadrightarrow C^*_E(G)$ the quotient map.

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Consider the "coaction"

$$\widehat{\alpha}_E : A \rtimes_u G \to M(A \rtimes_u G \otimes C_E^*(G))$$

$$a \mapsto i_A(a) \otimes 1, \qquad s \mapsto i_G(s) \otimes U_s.$$

Define $A \rtimes_{E_{KLQ}} G := \overline{C_c(G, A)}^{\|\cdot\|_{E_{KLQ}}}$ with $\|f\|_{E_{KLQ}} = \|\widehat{\alpha}_E(f)\|$.

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Note: If $F_2 \subseteq G$, there are infinitely many different BG- and KLQ-functors!

The smallest exact Morita compatible functor

Let $\{\rtimes_\mu : \mu \in M\}$ be a family of crossed-product functors with ideals

$$I_\mu := \ker(A \rtimes_u G \to A \rtimes_\mu G) \subseteq A \rtimes_u G.$$

Put $I_{\operatorname{Inf}\mu} := \overline{\sum_\mu I_\mu} \subseteq A \rtimes_u G, \quad A \rtimes_{\operatorname{Inf}\mu} G := (A \rtimes_\mu G)/I_{\operatorname{Inf}\mu}.$

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Theorem (Baum-Guentner-Willett)

- $(A, \alpha) \to A \rtimes_{\mathrm{Inf}\,\mu} G$ is a crossed-product functor.
- If all functors $\rtimes_\mu$ are exact, then so is $\rtimes_{\mathrm{Inf}\,\mu}$.
- If all functors $\rtimes_\mu$ are Morita compatible, then so is $\rtimes_{\mathrm{Inf}\,\mu}$.

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Corollary (B-G-W)

There exists a smallest exact, Morita compatible crossed-product functor

$$(A, \alpha) \mapsto A \rtimes_{\mathcal{E}} G.$$

Some questions

Question 1: Suppose G is K-amenable in the sense of Cuntz (like F_2). For which crossed product functors are the canonical quotient maps

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Notice: This is not always the case! Let G be a non-amenable group and

$$C^*_\mu(G) := \lambda \oplus 1_G(C^*_u(G)) \cong C^*_r(G) \oplus \mathbb{C}.$$

The corresponding BG-functor does not have the above property!

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Question 2: Which crossed product functors allow a descent in KK-theory

$$J_\mu : KK^G(A,B) \to KK(A \rtimes_\mu G, B \rtimes_\mu G)?$$

Does it exist for the minimal exact Morita compatible functor $\rtimes_{\mathcal{E}}$ in the reformulation of the Baum-Connes conjecture by Baum-Guentner-Willett?

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Notice: The descent was an important tool for proving that for K-amenable G the quotient map $A \rtimes_u G \twoheadrightarrow A \rtimes_r G$ is a KK-equivalence!

Correspondence functors

Recall that a cycle in $KK^G(A, B)$ consists of a triple $(\mathcal{E}_B, \Phi, T)$ in which $(\mathcal{E}_B, \Phi)$ is a G-equivariant correspondence between (A, α) and (B, β).

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which is functorial for compositions of Kasparov cycles.

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In particular, we need an assignment $(\mathcal{E}_B, \Phi) \mapsto (E_B \rtimes_\mu G, \Phi \rtimes_\mu G)$ s.t.

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Notation: If $(A,\alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is functorial for correspondences, we call it a correspondence functor.

A characterisation of correspondence functors

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Application: Every KLQ-functor is a correspondence functor:

$$\begin{array}{ccc} pAp \rtimes_{E_{KLQ}} G & \xrightarrow{\iota \rtimes_E G} & A \rtimes_{E_{KLQ}} G \\ \widehat{\alpha}_E \big\downarrow & & \big\downarrow \widehat{\alpha}_E \\ M(pAp \rtimes_u G \otimes C^*_E(G)) & \xrightarrow{(\iota \rtimes G)\otimes \mathrm{id}} & M(A \rtimes_u G \otimes C^*_E(G)). \end{array}$$

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Note: BG-funtors are (almost) never correspondence functors!

Some further results

Theorem (Buss-E-Willett)

- Every correspondence functor admits a KK-descent.
- If $\rtimes_\mu$ is a correspondence functor and G is K-amenable, then

$$A \rtimes_u G \twoheadrightarrow A \rtimes_\mu G \twoheadrightarrow A \rtimes_r G \qquad \text{are } KK\text{-equivalences.}$$

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This applies in particular for all KLQ-functors for K-amenable groups, hence to $C_u^*(G) \twoheadrightarrow C_E^*(G) \twoheadrightarrow C_r^*(G)$ for G-invariant ideals $E \subseteq G$!

Corollary: If $G = F_2$, then $C_u^*(G) \sim_{KK} C_p^*(G) \sim_{KK} C_r^*(G)$ are KK-equivalences for all $p \in [2, \infty]$.

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$$A \rtimes_u G \twoheadrightarrow A \rtimes_\mu G \twoheadrightarrow A \rtimes_r G \qquad \text{are } KK\text{-equivalences.}$$

This applies in particular for all KLQ-functors for K-amenable groups, hence to $C_u^*(G) \twoheadrightarrow C_E^*(G) \twoheadrightarrow C_r^*(G)$ for G-invariant ideals $E \subseteq G$!

Corollary: If $G = F_2$, then $C_u^*(G) \sim_{KK} C_p^*(G) \sim_{KK} C_r^*(G)$ are KK-equivalences for all $p \in [2, \infty]$.

Theorem (Buss-E-Willett)

The minimal exact Morita compatible crossed-product functor $\rtimes_{\mathcal{E}}$ of Baum-Guentner-Willett is a correspondence functor (at least if restricted to separable systems). Hence it has a KK-descent!

Thank you!