Continuous orbit equivalence rigidity

Xin Li

Dynamical systems and operator algebras

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  A.: Crossed product construction.

- Q.: Is there a way back???
- More precisely: Given $G \curvearrowright X$, $H \curvearrowright Y$, do we have $C_0(X) \rtimes_r G \cong C_0(Y) \rtimes_r H \Rightarrow G \curvearrowright X \ \sim \ H \curvearrowright Y$???

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$G \curvearrowright X$ and $H \curvearrowright Y$ are conjugate if there exist a homeomorphism $\varphi : X \xrightarrow{\cong} Y$ and an isomorphism $\rho : G \xrightarrow{\cong} H$ with $\varphi(g.x) = \rho(g).\varphi(x)$ for all $g \in G$, $x \in X$.

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Topological dynamics and C*-algebras

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$G \curvearrowright X$, $H \curvearrowright Y$: topologically free topological dynamical systems.
$G \curvearrowright X \ \sim_{coe}\ H \curvearrowright Y$ if and only if there is a C-isomorphism*
$\Phi:\ C_0(X) \rtimes_r G \xrightarrow{\cong} C_0(Y) \rtimes_r H$ with $\Phi(C_0(X)) = C_0(Y)$.

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- Can these arrows be reversed?

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Assume $G \curvearrowright X \sim_{coe} H \curvearrowright Y$.

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If G is fin. gen., then so is H, and G and H are quasi-isometric.

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- A. Thom and R. Sauer have shown that for two groups G and H, there exist top. free systems $G \curvearrowright X$, $H \curvearrowright Y$ on compact spaces X and Y with $G \curvearrowright X \ \sim_{\text{coe}} \ H \curvearrowright Y$ if and only if G and H are bi-Lipschitz equivalent.

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Assume:

- *X compact, $C(X, \mathbb{Z}) \cong \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules with* $\mathrm{pd}_{\mathbb{Z}G}(N) < \mathrm{cd}(G) - 1$

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- *G: duality group, H: solvable group.*

Then $G \curvearrowright X \sim_{coe} H \curvearrowright Y \Rightarrow G \curvearrowright X \sim_{conj} H \curvearrowright Y$.

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- $G \curvearrowright X_0^G$, X_0 *compact,* $|X_0| > 1$, G*: solvable duality group;*
- *top. free subshift of* $G \curvearrowright \{0, \ldots, N\}^G$ *whose forbidden words avoid the letter* 0, G*: solvable duality group;*

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- *top. free subshift of* $G \curvearrowright \{0, \ldots, N\}^G$ *whose forbidden words avoid the letter* 0, G*: solvable duality group;*
- *chessboards* $\mathbb{Z}^2 \curvearrowright X^{(n)}$ *with* $n \geq 4$ *colours.*

The End

Thank you!