Classification problems on the compact quantum groups of Lie type

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Compact quantum groups

Woronowicz: a *compact quantum group* G is given by

- unital C*-algebra $A = C(G)$
- coproduct $\Delta\colon A \to A \otimes A$ which is
  - coassociative $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$
  - cancellative $[(A \otimes 1)\Delta(A)] = A \otimes A = [(1 \otimes A)\Delta(A)]$

Unitary representation of G on H_U is

- unitary element $U \in B(H_U) \otimes C(G)$ s.t. $(\iota \otimes \Delta)(U) = U_{12}U_{13}$

Example

$C(\mathrm{SU}_q(2))$: generated by α and γ such that

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(C(\mathrm{SU}_q(2))) = B(\mathbb{C}^2) \otimes C(\mathrm{SU}_q(2))$$

is a unitary representation.

Compact quantum groups of Lie type

Tensor product rep.: $U \mathbin{ⓣ} V = U_{13}V_{23} \in B(H_U \otimes H_V) \otimes C(G)$

Combinatorial part of the representation theory (fusion ring):

- representation ring $R(G) = \oplus_{U:\ \mathrm{Irr}\, G}\mathbb{Z}[U]$ from irreducible decomposition of tensor product reps
- (classical) dimension function $d_{\mathrm{cl}}\colon R(G) \to \mathbb{Z}, [U] \mapsto \dim H_U$

G is of *Lie type*: $(R(G), d_{\mathrm{cl}}) \simeq (R(G_1), d_{\mathrm{cl}})$ for a compact Lie group G_1

Example

- $\mathrm{SU}_q(n)$ by Woronowicz, Faddeev–Reshetikhin–Takhtadzhyan
- G_q for simple cpt Lie group G from Drinfeld–Jimbo quantization

Problem (Woronowicz)

Classify the compact quantum groups of $\mathrm{SU}(n)$ *type.*

Classification for the $\mathrm{SU}(n)$-type

Theorem (Neshveyev–Y., cf. Ohn for $n = 3$)

The non-Kac cpt quantum groups of $\mathrm{SU}(n)$ type are parametrized by:

- $0 < q = e^{-h} < 1$: *deformation quantization parameter,*
- $\mathbb{T}$-*valued alternating bicharacter on* $\mathbb{Z}^{n-1}$: *Poisson–Lie group structure on* $\mathrm{SU}(n)$*,*
- $\Phi \in H^3(\mathbb{Z}/n; \mathbb{T})$: *associativity data on* $\operatorname{Rep} \mathrm{SU}_q(n)$ *(*$\mathbb{Z}/n$ *is the Pontrjagin dual of* $Z(\mathrm{SU}_q(n))$*).*

Isomorphic quantum groups appear iff these are related by the automorphism group of the root data ($\simeq \mathbb{Z}/2$*).*

- Non-Kac: $S^2 \neq \iota \Leftrightarrow h$ (the Haar state) is not a trace
- Kac case would include the classification of central type factor groups in $\mathrm{SU}(n)$

Twisted $\mathrm{SU}_q(n)$ group

Parameter: $\tau \in \mu_n(\mathbb{C})^{n-1}$, ω alternating bicharacter on $\mathbb{Z}^{n-1}$

$\mathbb{C}[\mathrm{SU}_q^{\tau,\omega}]$: the universal algebra generated by $(v_{ij})_{1\leq i,j\leq n}$ subject to

$$v_{ij}v_{il} = \Big(\prod_{j\leq p<l} \tau_p^{-1}\Big) q\bar{\omega}_{jl}^2 v_{il}v_{ij} \quad (j<l),$$

$$v_{ij}v_{kj} = \Big(\prod_{i\leq p<k} \tau_p\Big) q\omega_{ik}^2 v_{kj}v_{ij} \quad (i<k),$$

$$v_{ij}v_{kl} = \Big(\prod_{i<p\leq k} \tau_p^{-1}\Big)\Big(\prod_{j\leq p<l} \tau_p^{-1}\Big) \omega_{ik}^2\bar{\omega}_{jl}^2 v_{kl}v_{ij} \quad (i>k, j<l),$$

$$\Big(\prod_{j\leq p<l} \tau_p\Big)\omega_{jl}^2 v_{ij}v_{kl} - \Big(\prod_{i\leq p<k} \tau_p\Big)\bar{\omega}_{ki}^2 v_{kl}v_{ij} = (q-q^{-1})v_{il}v_{kj} \quad (i<k, j<l),$$

$$\sum_{\sigma\in S_n} \tau^{m(\sigma)}(-q)^{|\sigma|}\bar{\omega}(1,\dots,n)\omega(\sigma(1),\dots,\sigma(n)) v_{1\sigma(1)}\cdots v_{n\sigma(n)} = 1,$$

$m(\sigma) \in \{\pm 1, 0\}^{n-1}$ (with some rule), $\omega(i_1,\dots,i_n) = \prod_{k<l} \omega_{i_k,i_l}$.

Tannaka–Krein duality

Unitary representations of $G \rightsquigarrow$ rigid C*-tensor category $\operatorname{Rep} G$

Theorem (Woronowicz's Tannaka–Krein duality)

A compact quantum group $(C(G), \Delta)$ can be recovered from:

1. *a rigid C*-tensor category* $\mathcal{C} = \operatorname{Rep} G$
2. *tensor functor (fiber functor)* $\mathcal{C} \to \mathrm{Hilb}_f$, $U \mapsto H_U$.

This can be generalized to the actions of G on C*-algebras

- G-algebras $\leftrightarrow$ $(\operatorname{Rep} G)$-module categories (De Commer–Y., Neshveyev)
- braided commutative Yetter–Drinfeld G-algebras
  $\leftrightarrow$ tensor functors from $\operatorname{Rep} G$ (Neshveyev–Y.)

Example (quantum homogeneous space)

Q. subgrp. $H < G \rightsquigarrow G \curvearrowright C(G/H)$ corresponds to $\operatorname{Rep} G \to \operatorname{Rep} H$

Kazhdan–Wenzl deformation scheme

G semisimple compact Lie group

- $\operatorname{Rep} G_q$ is graded over $\widehat{Z(G_q)} = \widehat{Z(G)}$ (take central characters)
- $\mathbb{T}$-valued 3-cocycle Φ on $\widehat{Z(G_q)}$ gives a new associativity morphisms: for irreducible U, V, W,

$$(U \mathbin{\bigcirc\!\!\!\!\top} V) \mathbin{\bigcirc\!\!\!\!\top} W \to U \mathbin{\bigcirc\!\!\!\!\top} (V \mathbin{\bigcirc\!\!\!\!\top} W) \text{ by } \Phi(\chi_U, \chi_V, \chi_W)\iota_{H_U \otimes H_V \otimes H_W}$$

$\rightsquigarrow$ new C*-tensor category $(\operatorname{Rep} G_q; \Phi)$

Theorem (Kazhdan–Wenzl, Jordans)

Any semisimple C-tensor category with the fusion rule of* $\mathrm{SU}(n)$ *is of the form* $(\operatorname{Rep} \mathrm{SU}_q(n); \Phi)$.

Neshveyev–Y.: $\exists$ CQG realization for $(\operatorname{Rep} G_q; \Phi)$ when the image of Φ is trivial in $H^3(\hat{T}; \mathbb{T})$ for the maximal torus $T < G$

Classifying fiber functors

Maximal Kac quantum subgroup

- $\exists$ maximal quantum subgroup of Kac type $K < G$ (Vaes)
- T is the maximal Kac quantum subgroup in G_q (Tomatsu)

Theorem (Neshveyev–Y.)

Suppose G is coamenable ($d_{\mathrm{cl}}(U) =$ fusion norm of $[U] \in R(G)$). Then any fiber functor $F\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$ with $\dim F(U) = \dim H_U$ factors through $\operatorname{Rep} K$ in an essentially unique way.

- G_q (and the CQG realization of $(\operatorname{Rep} G_q; \Phi)$) is coamenable
- fiber functors on $\operatorname{Rep} T$: classified by alt. bichars. on $\hat{T} \simeq \mathbb{Z}^{\operatorname{rk} G}$

Categorical Poisson boundary

- (Longo–Roberts) "intrinsic dimension" $d(X)$ on rigid semisimple C*-tensor category $\mathcal{C}$
- (N.–Y.) prob. measure μ on $\operatorname{Irr}\mathcal{C} \rightsquigarrow$ new C*-$\otimes$ category $\mathcal{P}$, $\otimes$-functor $\Pi \colon \mathcal{C} \to \mathcal{P}$ ("Poisson boundary")
- when $\mathcal{C} = \operatorname{Rep} G$, $(\mathcal{P}, \Pi)$ corresponds to Izumi's noncommutative Poisson boundary $H^\infty(\hat{G}; \mu)$

Theorem (Neshveyev–Y.)

Suppose μ defines an ergodic random walk on $\operatorname{Irr}\mathcal{C}$. Then Π is the universal $\otimes$-functor $F \colon \mathcal{C} \to \mathcal{C}'$ such that $d(F(X)) = \|[X]\|_{B(\ell_2(R(\mathcal{C})))}$.

- to show $d(\Pi(X)) = \|[X]\|$: construct type III subfactor $N \subset N_X$ s.t. $d(\Pi(X))^2 =$ statistical dimension, use the relative entropy
- for universality: do "Poisson integral" $\Theta \colon \mathcal{P} \to \mathcal{C}'$, use operator theory to show the multiplicativity

Isomorphism problems

$G \simeq G'$ (that is, $(C(G), \Delta) \simeq (C(G'), \Delta)$) means

- $\exists$ C*-$\otimes$-equivalence $E\colon \operatorname{Rep} G \to \operatorname{Rep} G'$
- the fiber functors $F\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$, $F'\colon \operatorname{Rep} G' \to \mathrm{Hilb}_f$ are related by a natural isomorphism $F'E \simeq F$

Theorem (Neshveyev–Y.)

If G is simple, $(\operatorname{Rep} G_q; \Phi)$ are mutually nonequivalent for different $\Phi \in H^3(\widehat{Z(G)}, \mathbb{T})$.

Theorem (Neshveyev–Tuset, Neshveyev–Y.)

If G is simple not of type D_{2m} ($Z(G)$ is cyclic), the group of autoequivalences of $(\operatorname{Rep} G_q; \Phi)$ is isomorphic to that of the root data.

Tidbits

- quasi-triangular quasi-Hopf algebra argument shows that Φ^2 is the obstruction for the existence of braiding on $(\operatorname{Rep} G_q; \Phi)$
- (X, c) is a *unitary* half-braiding on $X \in \mathcal{C} \Rightarrow X$ generates amenable subcategory (G is coamenable and Kac)
- $\exists \mu$ s.t. $H^\infty(\hat{G}; \mu) = \mathbb{C} \Leftrightarrow G$ is coamenable Kac
- G semisimple cpt Lie grp, $\Phi \in H^3(\widehat{G}; \mathbb{T}) \rightsquigarrow \exists$ fiber functor F on $(\operatorname{Rep} G_q; \Phi)$ with $\dim F(U) = \dim H_U \Leftrightarrow \Phi$ is trivial in $H^3(\hat{T}; \mathbb{T})$ (Bichon–Neshveyev–Y.)
- 3-cocycle deformation scheme works more generally (B.–N.–Y.)