Higher dimensional Rokhlin properties for group actions on C*-algebras

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Introduction

The classical Rokhlin Lemma is an approximation Lemma for free measure preserving automorphisms of probability spaces.

The C^*-Rokhlin property of an automorphism is a C^*-analogue of this approximation and can be regarded as an approximation property for the automorphism.

This direct topological analogue however is quite restrictive, requirering existence of many projections in the coefficient algebra, leading to zero dimensional spaces in the commutative case.

This is one of the main motivations for introducing a higher dimensional Rokhlin property. The resulting notion allows to introduce a dimension concept well adapted to nuclear dimension.

Moreover there is a sort of topological version of the Rokhlin Lemma: free $\mathbb{Z}$-actions on finite dimensional compact spaces always have finite Rokhlin dimension.

More recently the definition of Rokhlin dimension has been extended to a larger class of groups (polynominal growth) with analogous results and connections to coarse geometry. A new dimension invariant, the dimension of the box space of the group seems to play an important role.

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There are extensions to larger classes of amenable groups. For monotilable amenable groups $\{0, \ldots, n-1\}$ has to be replaced by Følner tiles. For general amenable groups we have ‘multi-tower’ approximations.

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Main Drawback: Requires existence of lots of projections.

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The same definition is possible for actions of finite groups.

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This can be regarded as a topological version of the Rokhlin Lemma.

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There have been several preliminary definitions of this due to Winter and Kirchberg/Winter.

This one is the most flexible rendering many nuclear C^*-algebras finite dimensional.

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1. Group C*-algebras of f.g. polynominal growth groups have finite nuclear dimension [Eckhard-McKenny].

2. The Roe algebra $C_u^*(X)$ of a discrete metric space X of bounded geometry satisfies $\dim_{nuc}(C_u^*(X)) \leq \operatorname{asdim}(X)$ [Winter-Z].

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Theorem (Szabo)

If α is a $\mathbb{Z}^m$-action then

$$\dim_{nuc}^{+1}(A \rtimes_\alpha \mathbb{Z}^m) \leq 2^m\dim_{Rok}^{+1}(\alpha)\dim_{nuc}^{+1}(A).$$

G discrete group is called residually finite if G has a separating family of homomorphisms into finite groups i.e. for all $g \neq e$ there is a finite index normal subgroup N such that $\bar{g} = gN \neq N = \bar{e}$ in G/N.

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In the sequel we'll consider countable residually finite groups with a fixed countable residually finite approximation (G_n) of G.

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Definition (positive RP for actions of RF groups, Szabo, Wu, Z)

$\alpha : G \curvearrowright A$ has the positive Rokhlin property (with Rokhlin dimension $\leq d$) if for any G_n, $\mathcal{F} \subset A$ finite, $\epsilon > 0$ there are positive elements $f_{\overline{g}}^{(l)} \in A$, $l = 0, \ldots, d$, $\overline{g} \in G/G_n$ with

- $\|f_{\overline{g}}^{(l)} f_{\overline{h}}^{(l)}\| < \epsilon$ ($\overline{g} \neq \overline{h}$, all l)
- $\|\sum_{l,\overline{g}} f_{\overline{g}}^{(l)} - 1\| < \epsilon$
- $\|[f_{\overline{g}}^{(l)}, a]\| < \epsilon$ ($a \in \mathcal{F}$, all $\overline{g}$)
- $\|\alpha_h(f_{\overline{g}}^{(l)}) - f_{\overline{hg}}^{(l)}\| < \epsilon$, for all l.

The smallest such d is called the Rokhlin dimension of α.

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Note that any action $\alpha : G \curvearrowright A$ extends to $\alpha^\infty : G \curvearrowright A_\infty$ and to the central sequence algebra $F(A)$.

Then $\alpha : G \curvearrowright A$ has the positive Rokhlin property with Rokhlin dimension $\leq d$ if for every $n \in \mathbb{N}$, $l = 0, \ldots, d$ there exist equivariant order zero maps

$$\varphi_l : (C(G/G_n), \text{shift}) \to (F(A), \alpha^\infty)$$

with $\varphi_0(1) + \ldots + \varphi_d(1) = 1$.

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Definition (Roe)

Let G, (G_n) be as before, the box space $\square_{(G_n)} G$ is the coarse disjoint union $\bigcup_{n\in\mathbb{N}} G/G_n$, i. e. this disjoint union endowed with a metric such that each subset G/G_n inherits its metric from the right-invariant metric of G, and

$$\operatorname{dist}(G/G_n, G/G_m) \geq \max(\operatorname{diam}(G/G_n), \operatorname{diam}(G/G_m))$$

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Theorem (Szabo, Wu, Z)

With G, (G_n), α as before we have

$$\dim_{nuc}^{+1}(A \rtimes_\alpha G) \leq \operatorname{asdim}^{+1}(\square_{(G_n)}G)\dim_{Rok}^{+1}(\alpha)\dim_{nuc}^{+1}(A)$$

Question: for which RF groups do we have $\operatorname{asdim}(\square_{(G_n)} G) < \infty$?

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This can be generalised to elementary amenable polycyclic groups (Finn-Sell - Wu work in progress).

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G RS with approximation (G_n), $\alpha : G \curvearrowright X$ free, then

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So for groups with finite dimensional box space both dimension invariants are closely related.

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In this case similar bounds for nuclear dimension of crossed products can be obtained.

Many thanks for your attention!