Purely infinite inverse semigroup crossed products and Cartan pairs

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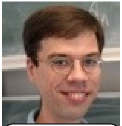
joint work with  Ralf Meyer in  PROGRES

1) Inverse semigroup crossed products

Let $A \subseteq B$ be a non-degenerate C^*-subalgebra of a C^*-algebra B.

$$N(A) := \{b \in B : \ bAb^* \subseteq A, \ b^*Ab \subseteq A\} \qquad (\textit{normalizers})$$

We say that $A \subseteq B$ is a **regular C^*-inclusion** if $B = \overline{\operatorname{span}}\, N(A)$.

Lem. (Exel 2011)

The family of "noncommutative bisections":

$\mathrm{Bis}(B, A) := \{M \subseteq N(A) : M$ is a closed linear space $AM \subseteq M, MA \subseteq M\}$

with operations inherited from B is an inverse semigroup with unit A:

$$M, N \in \mathrm{Bis}(B, A) \implies MN \in \mathrm{Bis}(B, A)$$

$$M \in \mathrm{Bis}(B, A) \implies M^* \in \mathrm{Bis}(B, A)$$

$$M \in \mathrm{Bis}(B, A) \implies MM^*M = M$$

$$M \in \mathrm{Bis}(B, A) \implies M^*M \lhd A, \quad MM^* \lhd A$$

1) Each $M \in \mathrm{Bis}(B, A)$ is naturally a Hilbert A-bimodule.

2) $A \subseteq B$ is regular if and only if $\sum_{M \in \mathrm{Bis}(B,A)} M$ is dense in B.

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Let S be an inverse semigroup with unit 1. Let A be a C^*-algebra.

Def. (Buss, Meyer 2017) $\equiv$ (saturated Fell bundles over S - Sieben, Exel)

An **action of** S **on** A **by Hilbert bimodules** is a semigroup $\bigsqcup_{t \in S} \mathcal{E}_t$ where

1) $\mathcal{E}_t$ is a Hilbert A-bimodule, for every $t \in S$, and $\mathcal{E}_1 = A$,

2) semigroup multiplication induces isomorph. $\mathcal{E}_t \otimes_A \mathcal{E}_s \cong \mathcal{E}_{ts}$ for $t, s \in S$.

Let $\mathcal{G}$ an étale groupoid with locally compact and Hausdorff unit space X.

Ex.1 (Saturated Fell bundles over $\mathcal{G}$ - Kumjian 1998)

Let $\mathcal{A} = (A_\gamma)_{\gamma\in\mathcal{G}}$ be a saturated Fell bundle over $\mathcal{G}$. Put

$S := \mathrm{Bis}(\mathcal{G})$ - the inverse semigroup of open bisections of $\mathcal{G}$

A_U - the space of continuous sections of $\mathcal{A}$ vanishing outside U.

Then $\bigsqcup_{U\in S} A_U$ with operations coming from $\mathcal{A}$ is an action of S on A_X.

Ex.2 (Actions on commutative algebras - Buss, Exel 2012)

Actions $\mathcal{E} = \bigsqcup_{t\in S} \mathcal{E}_t$ on $A = C_0(X) \iff$ line bundles over $\mathcal{G}$

$\iff$ twisted étale groupoids $(\mathcal{G}, \Sigma)$

Def. (Groupoid dual to an action $\mathcal{E}$)

Let $\mathcal{E} = \bigsqcup_{t\in S} \mathcal{E}_t$ be an action on A. It induces an action $\widehat{\mathcal{E}} = (\widehat{\mathcal{E}}_t)_{t\in S}$ of the inverse semigroup S by partial homeomorphisms on the spectrum $\widehat{A}$ (for $t \in S$ we have $\widehat{\mathcal{E}}_t : \widehat{D}_t \xrightarrow{\sim} \widehat{D}_{t*}$ where $D_t := \langle \mathcal{E}_t, \mathcal{E}_t \rangle_A \lhd A$).

$\widehat{A} \rtimes S$ - **the transformation groupoid for** $\widehat{\mathcal{E}}$.

Let $\mathcal{E} = \bigsqcup_{t\in S} \mathcal{E}_t$ be an action of S on A (we have $\mathcal{E}_t^* \cong \mathcal{E}_{t*}$ For $t \in S$)

- $\bigoplus_{t\in S} \mathcal{E}_t$ is a $*$-algebra equipped with operations induced from $\mathcal{E}$
- $A \rtimes_{\text{alg}} S$ is a quotient of $\bigoplus_{t\in S} \mathcal{E}_t$ by an ideal generated by "inclusions", e.g. if for each $t \in S$ we put

$$D_t := \langle \mathcal{E}_t, \mathcal{E}_t \rangle_A \lhd A, \qquad I_t := \overline{\bigcup_{v \leqslant t,1} D_v} \lhd A,$$

there is a natural isomorphism $\theta_t \colon \mathcal{E}_t \cdot I_t \xrightarrow{\sim} I_t \subseteq A$.

Def. (Buss, Exel, Meyer) $\equiv$ (Sieben, Exel)

The **crossed product** $A \rtimes_{\mathcal{E}} S$ is the maximal C^*-completion of $A \rtimes_{\text{alg}} S$

The **reduced crossed product** $A \rtimes_{\mathcal{E}}^r S$ is the C^*-completion of $A \rtimes_{\text{alg}} S$ admitting a faithful completely positive map $E : A \rtimes_{\mathcal{E}}^r S \to A''$ such that

$$E(\xi_t) = weak\text{-}\lim_{\lambda} \theta_t(\xi_t \cdot \mu_\lambda) \qquad \text{for } \xi_t \in \mathcal{E}_t$$

where $\{\mu_\lambda\}$ is an approximate unit for I_t (*E-weak conditional expectation*).

The **reduced crossed product** $A \rtimes^r_{\mathcal{E}} S$ is the C^*-completion of $A \rtimes_{\mathrm{alg}} S$ admitting a faithful completely positive map $E : A \rtimes^r_{\mathcal{E}} S \to A''$ such that

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where $\{\mu_\lambda\}$ is an approximate unit for I_t (*E-weak conditional expectation*).

Thm. (Buss, Exel, Meyer) The following conditions are equivalent:

(1) The weak conditional expectation $E\colon A \rtimes^r_{\mathcal{E}} S \to A''$ is A-valued

(2) The ideal I_t is complemented in the larger ideal D_t for all $t \in S$.
Then $\mathcal{E}_t = \mathcal{E}_t D_t = \mathcal{E}_t I_t \oplus \mathcal{E}_t I_t^{\perp}$ is a direct sum of Hilbert bimodules:

$$E(\xi_t) = \theta_t(\xi_{t,1}) \qquad \xi_t = \xi_{t,1} + \xi_{t,1}^{\perp} \in \mathcal{E}_t I_t \oplus \mathcal{E}_t I_t^{\perp}.$$

(3) The subset of units $\widehat{A}$ is closed in the dual groupoid $\widehat{A} \rtimes S$.

Def.

The action $\mathcal{E} = \bigsqcup_{t\in S} \mathcal{E}_t$ is **closed** if the above equivalent conditions hold.

2) Aperiodic actions

Let $\mathcal{E} = \bigsqcup_{t\in S} \mathcal{E}_t$ be a **closed** action, i.e. $\mathcal{E}_t = \mathcal{E}_t I_t \oplus (\mathcal{E}_t I_t)^\perp$ for $t \in S$.

Def.

$\mathcal{E} = \bigsqcup_{t\in S} \mathcal{E}_t$ is **aperiodic** if for every $t \in S$ the Hilbert bimodule $(\mathcal{E}_t I_t)^\perp$ satisfies *Kishimoto's condition*:

$$\forall_{x\in(\mathcal{E}_t I_t)^\perp}\ \forall_{0\neq D\subseteq A \text{ hereditary}} \quad \inf\{\|a\cdot x\cdot a\| : a \in D^+,\ \|a\| = 1\} = 0$$

Thm. Assume A contains an essential ideal which is separable or of Type I

$$\mathcal{E} \text{ is aperiodic} \iff \text{the dual groupoid } \widehat{A} \rtimes S \text{ is effective.}$$

If S is countable, then $\widehat{A} \rtimes S$ is effective $\iff$ $\widehat{A} \rtimes S$ is topologically principal

Thm. If $\mathcal{E}$ is aperiodic then

(1) A detects ideals in $A \rtimes^r_{\mathcal{E}} S$ $\qquad 0 \neq J \lhd A \rtimes^r_{\mathcal{E}} S \Longrightarrow J \cap A \neq 0$

(2) A supports $A \rtimes^r_{\mathcal{E}} S$ $\qquad b \in (A \rtimes^r_{\mathcal{E}} S)^+ \setminus \{0\} \Longrightarrow \exists_{a\in A^+\setminus\{0\}}\ a \precsim b$

Thm. (Characterisation of Cartan pairs)

Let $A \subseteq B$ be a commutative C^*-subalgebra. The following are equivalent:

(1) A is a **Cartan subalgebra** of B:

- $A \subseteq B$ is regular,
- A is maximal abelian in B,
- there is a faithful conditional expectation $E : B \to A \subseteq B$.

(2) $A \subseteq B$ is **regular** and there is a **unique faithful** conditional expectation $E\colon B \to A \subseteq B$,

(3) $B \cong A \rtimes_{\mathrm{r}} S$ for a **closed and aperiodic action** $\mathcal{E}$ of an inverse semigroup S on A by Hilbert bimodules,

(4) $B \cong \mathrm{C}^*_{\mathrm{r}}(\mathcal{G}, \Sigma)$ for a **twist** $(\mathcal{G}, \Sigma)$ of a **locally compact Hausdorff, effective étale groupoid** $\mathcal{G}$.

Jean Renault

(1) $\iff$ (4)

3) Ideal structure and pure infiniteness

Let $\mathcal{E} = \bigsqcup_{t\in S} \mathcal{E}_t$ be a closed action. We put

$$\mathbb{I}^{\mathcal{E}}(A) := \{I \lhd A : \mathcal{E}_t I = I\mathcal{E}_t \text{ for all } t \in S\} \qquad \mathcal{E}\text{-invariant ideals}$$

If $I \in \mathbb{I}^{\mathcal{E}}(A)$ then $\begin{cases} \mathcal{E}|_I := \bigsqcup_{t\in S} \mathcal{E}_t I \text{ is a closed action on } I \\ \mathcal{E}|_{A/I} := \bigsqcup_{t\in S} \mathcal{E}_t/(\mathcal{E}_t I) \text{ is a closed action on } A/I \end{cases}$

For every $I \in \mathbb{I}^{\mathcal{E}}(A)$ we have an exact sequence

$$0 \to I \rtimes S \xrightarrow{\iota \rtimes S} A \rtimes S \xrightarrow{q \rtimes S} A/I \rtimes S \to 0$$

Def.

$\mathcal{E}$ is **exact** if for every $I \in \mathbb{I}^{\mathcal{E}}(A)$ the following sequence is exact:

$$0 \to I \rtimes_{\mathrm{r}} S \xrightarrow{\iota \rtimes_{\mathrm{r}} S} A \rtimes_{\mathrm{r}} S \xrightarrow{q \rtimes_{\mathrm{r}} S} A/I \rtimes_{\mathrm{r}} S \to 0$$

Rem. If $\mathcal{E}$ is **amenable**, i.e. $A \rtimes S \cong A \rtimes_{\mathrm{r}} S$, then $\mathcal{E}$ is exact.

Thm. (Ideal structure)

If $\mathcal{E}$ is exact and **residually aperiodic**, i.e. $\mathcal{E}|_{A/I}$ is aperiodic for every $I \in \mathbb{I}^{\mathcal{E}}(A)$, then $A \rtimes_{\mathrm{r}} S \rhd J \longmapsto J \cap A \in \mathbb{I}^{\mathcal{E}}(A)$ is a lattice isomorphism

Thm. (Pure infiniteness)

Let $\mathcal{E}$ be exact and residually aperiodic, and let $\mathcal{F} \subseteq A^+$ residually support A. Put $B := A \rtimes_{\mathrm{r}} S$. Assume one of the following conditions:

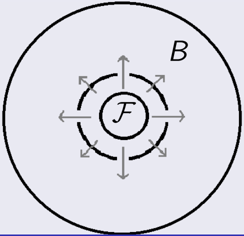
(i) $\mathbb{I}^{\mathcal{E}}(A)$ is finite;

(ii) $\mathcal{F}$ consists of projections;

(iii) $\mathcal{F} = A^+$ and the projections in A separate the ideals in $\mathbb{I}^{\mathcal{E}}(A)$.

Then the following statements are equivalent:

1. $\mathcal{F} \setminus \{0\}$ consists of elements that are properly infinite in B;
2. B is purely infinite;
3. B is purely infinite and Prim B has topological dimension zero;
4. B is purely infinite and has the ideal property;
5. B is strongly purely infinite.



Let $\mathcal{G}$ be an étale locally compact Hausdorff groupoid with unit space X.

Def.

A set $V \subseteq X$ is **$\mathcal{G}$-infinite** if there are open $V_1, \dots, V_n \in \mathrm{Bis}(\mathcal{G})$ such that

$$V = \bigcup_{i=1}^{n} s(V_i), \quad \overline{\bigcup_{i=1}^{n} r(V_i)} \subsetneq V, \quad \text{and} \quad r(V_i) \cap r(V_j) = \emptyset \text{ for all } i \neq j.$$

V is **residually $\mathcal{G}$-infinite** if $V \cap F$ is $\mathcal{G}|_F$-infinite for all $\mathcal{G}$-invariant closed $F \subseteq X$.

Cor. (Anantharaman-Delaroche 1997, Brown, Clark, Sierakowski 2016) - minimal case (Bönicke, Li; Rainone, Sims) - ample case

Assume $\mathcal{G}$ is exact, residually effective. Let $\mathcal{F}$ be a basis of topology on X. If there are finitely many $\mathcal{G}$-invariant open sets in X or $\mathcal{F}$ consists of compact open sets, then

$$\left(\begin{array}{c} \text{every } V \in \mathcal{F} \text{ is} \\ \text{residually } \mathcal{G}\text{-infinite} \end{array} \right) \Longrightarrow \left(\begin{array}{c} C_r^*(\mathcal{G}) \text{ is} \\ \text{strongly purely infinite} \end{array} \right)$$

4) Dichotomy for simple C^*-algebras with Cartan subalgebras

Assume that

- $A \subseteq B$ is a Cartan C*-subalgebra;
- the spectrum of $A = C_0(X)$ is totally disconnected;
- B is simple.

Equivalently, $B = \mathrm{C}^*_{\mathrm{r}}(\mathcal{G}, \Sigma)$ where

- $(\mathcal{G}, \Sigma)$ is a twist of an ample, étale, Hausdorff, effective and minimal groupoid $\mathcal{G}$ with unit space X

Def. (Bönicke, Li; Rainone, Sims)

The **type semigroup** $S(\mathcal{G})$ of $\mathcal{G}$ is the quotient of $C_c(X,\mathbb{Z})^+$ by the equivalence relation where: for $f, g \in C_c(X,\mathbb{Z})^+$ we write $f \sim_{\mathcal{G}} g$ if there are compact open bisections $V_1, \ldots, V_n \in \mathrm{Bis}(\mathcal{G})$ such that

$$f = \sum_{i=1}^{n} 1_{s(V_i)}, \qquad g = \sum_{i=1}^{n} 1_{r(V_i)}.$$

For $f, g \in C_c(X,\mathbb{Z})^+$ we put $[f] + [g] = [f+g]$ and write $[f] \leqslant [g]$ if $[f] + [h] = [g]$ for some $h \in C_c(X,\mathbb{Z})^+$

Thm. (Purely infinite vs stably finite)

Let B be a **SIMPLE** C^*-algebra which contains a Cartan subalgebra A of real rank zero. Let $\mathcal{G}$ be the groupoid dual to inclusion $A \subseteq B$.

Consider the following conditions:

(i) The semigroup $S(\mathcal{G})$ is purely infinite, i.e. $2\theta \leqslant \theta$ for every $\theta \in S(\mathcal{G})$;

(ii) Every non-empty compact open $V \subseteq X$ is $\mathcal{G}$-infinite;

(iii) The C^*-algebra B is **PURELY INFINITE**;

(iv) The C*-algebra B is traceless;

(v) B is not **STABLY FINITE**;

(vi) The semigroup $S(\mathcal{G})$ admits no non-trivial state.

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi). If $S(\mathcal{G})$ is almost unperforated, then (vi)$\Rightarrow$(i) and all conditions are equivalent.

An abelian semigroup $(S, +)$ is **almost unperforated** if
$\forall_{\theta,\eta\in S}\ \forall_{0<m<n}\ n\theta \leqslant m\eta \implies \theta \leqslant \eta$