The boundary path space of graphs, labelled spaces and topological graphs

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Directed graphs and their paths

- By a *(directed) graph* we mean a quadruple $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$, where:
  - $\mathcal{E}^0$ is the set of vertices,
  - $\mathcal{E}^1$ is the set of edges,
  - $s : \mathcal{E}^1 \to \mathcal{E}^0$ is the source map,
  - $r : \mathcal{E}^1 \to \mathcal{E}^0$ is the range map.
- A *path* α on $\mathcal{E}$ is a sequence (finite or infinite) of edges $\alpha = \alpha_1 \dots \alpha_n (\dots)$ such that $r(\alpha_i) = s(\alpha_{i+1})\ \forall i$.
- A vertex $v \in \mathcal{E}^0$ is called a *sink* if $|s^{-1}(v)| = 0$ and an *infinite emitter* if $|s^{-1}(v)| = \infty$.
- The set of all finite paths is denoted by $\mathcal{E}^*$ and the set of all infinite paths by $\mathcal{E}^\infty$.

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Path spaces

- Graphs with no sinks and no infinite emitters - the space of *all infinite paths* with the product topology. (Kumjian, Pask, Raeburn, Renault - 1997)
- Arbitrary graphs - the space of *all finite and infinite paths*, with a certain topology. (Paterson - 2002 / Farthing, Muhly, Yeend - 2005)
- Arbitrary graphs - *the boundary path space*, which consists of all infinite paths together with finite paths that ends in sinks and infinite emitters. (Paterson - 2002 / Farthing, Muhly, Yeend - 2005)

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An inverse semigroup point of view

- One can define an inverse semigroup with zero from a graph.
- The set of idempotents E is a semilattice with zero.
- There are bijections between the following sets:
  - The set of *all finite and infinite paths*.
  - The set of *semicharacters preserving zero* of E.
  - The set of *filters* in E.
- The topology is the one coming from $\{0, 1\}^E$.

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The tight spectrum

- The set of *tight filters* is the closure of the set of ultrafilters and is called the *tight spectrum*. (Exel - 2008)
- Ultrafilters correspond to infinite paths and finite paths that end in a sink.
- Tight filters which are not ultrafilters correspond to paths that end in an infinite emitter.

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Labelled spaces

- A *labelled space* is a triple $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ where $\mathcal{E}$ is a graph, $\mathcal{L} : \mathcal{E}^1 \to \mathcal{A}$ is a labelling map and $\mathcal{B}$ is a family of subsets of $\mathcal{E}^0$ satisfying certain conditions.
- The C*-algebra associated to a labelled space contains projections p_A for each $A \in \mathcal{B}$ subject to the relations $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{B}$. (Bates, Pask - 2007)
- One can define an inverse semigroup in such a way that the tight spectrum is homeomorphic to the spectrum of the diagonal C*-subalgebra. (Boava, de C., Mortari - 2017)

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The boundary path space of a labelled space

- To describe the tight spectrum, we need a *labelled path* together with a *family of ultrafilters*, each in a different boolean algebra, but satisfying certain compatibility conditions.
- Ultrafilters consists of the generalization of the infinite paths and paths ending in a sink.
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Topological graphs

- *Topological graphs* are directed graphs $\mathcal{E}$ in which $\mathcal{E}^0$ and $\mathcal{E}^1$ are locally compact Hausdorff spaces, s is a continuous map and *r is a local homeomorphism*. (Katsura - 2004)
- We change the boundary space in order to accommodate the topology. For that, we need the following subspaces of $\mathcal{E}^0$.
  - $\mathcal{E}^0_{sink} = \{v \in \mathcal{E}^0 \mid \text{there is a neighbourhood } U \text{ of } v \text{ such that } s^{-1}(U) = \emptyset\}$.
  - $\mathcal{E}^0_{inf} = \{v \in \overline{\mathcal{E}^0 \mid} s^{-1}(U) \text{ is not compact for all neighbourhoods } U \text{ of } v\}$.
  - $\mathcal{E}^0_{sg} = \overline{\mathcal{E}^0_{sink}} \cup \mathcal{E}^0_{inf}$.

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The boundary space of a topological graph

- *The boundary space* of a topological graph $\partial\mathcal{E}$ is the set of all infinite paths and finite paths that ends in a element of $\mathcal{E}^0_{sg}$. (Yeend - 2007)
- There is a topology on $\partial\mathcal{E}$ given by a basis of the form $Z(U) \cap Z(K)^c$, for $U \subseteq \mathcal{E}^*$ open and $K \subseteq \mathcal{E}^*$ compact, where $Z(A)$ is called a cylinder set and represents the set of all paths with beginning in $A \subseteq \mathcal{E}^*$. (Yeend - 2007)
- Can we recover the boundary path space using inverse semigroups? More specifically, can we recover it from a certain semilattice?

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Revisiting the case of discrete graphs

- The *semilattice of idempotents* in a graph inverse semigroup can be seen as *cylinder sets* in the space of all paths *ordered by inclusion*.
- In general, the topology on $\mathcal{E}^* \cup \mathcal{E}^\infty$ given by the cylinder sets is not the same as considering them as semicharacters.
- To correct it, we take the *patch topology*, which is the coarsest topology that contains the original one and the co-compact topology coming from it.
- We then consider only the *tight filters* to arrive at the *boundary path space*. (More about this in a moment.)
- This *does not work for* an arbitrary *topological graphs* because the original topology is lost in the process.

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Some techniques in pointless topology

- We work with *frames* (locales) instead of topological spaces. There is a duality if we restrict to certain subcategories.
- Frames can be *presented used generators and relations*. If one starts with a semilattice, we only need join relations.
- We can *impose new relations* in a frame F to arrive at a new frame G. If the F is the topology on a set X, then G *defines a subspace* Y (although G may not be isomorphic to the induced topology on Y).

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Revisiting the case of discrete graphs II

- The relation

$$p_v = \sum_{s(e)=v} s_e s_e^*$$

if $v \in \mathcal{E}^0$ is not a sink nor a infinite emitter defining the graph C*-algebra can be seen as imposing that

$$Z(v) = \bigcup_{s(e)=v} Z(e)$$

in the set of all paths.

- Notice that for v as above, $v \in Z(v)$, *but* $v \notin Z(e)$ for all $e \in \mathcal{E}^1$ if we consider *all paths*, however $v \notin \partial\mathcal{E}$ so that the *above equality* of cylinder sets *is true in the boundary path space*.
- This is the same relation one arrives when considering the tight spectrum.

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Back to topological graphs

- Start with the *semilattice of cylinders $Z(U)$*, for $U \subseteq \mathcal{E}^*$ open, ordered by inclusion.
- Use the *topologies on $\mathcal{E}^0$ and $\mathcal{E}^1$* to define *relations* and arrive at a frame.
- This frame corresponds to the set of *all paths* with a *basis* for topology given by $\{Z(U)\}$.
- Take the *patch topology*.
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- For $A \subseteq s(\mathcal{E}^1)$ open, we define a "covering" of A as a finite family of *pairwise disjoint open subsets* of $\mathcal{E}^1$, $\{B_1, \ldots, B_k\}$, such that for each $i = 1, \ldots, k$, $r|_{B_i}$ *is as homeomorphism*, B_i *is relatively compact*, and

$$s^{-1}(A) = \bigcup_{i=1}^{k} B_i.$$

- Let $U \subseteq \mathcal{E}^n$ be such that $r(U) \subseteq s(\mathcal{E}^1)$, we define the new relations as

$$Z(U) \cap Z(K)^c = \bigcup_{i=1}^{k} Z(U \times B_i) \cap Z(K)^c$$

whenever $\{B_1, \ldots, B_k\}$ is a covering of $r(U)$.

- At the end, one arrives at the boundary path space with the correct topology.

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An inverse semigroup from a topological graph

Definition

For a topological graph $\mathcal{E}$, we define the inverse semigroup $S(\mathcal{E})$ to be the inverse semigroup with zero generated by elements e_A for $A \in \Omega(\mathcal{E}^0)$ and elements x_M for $M \in \Omega(\mathcal{E}^1)$ such that $r|_M$ is a homeomorphism, satisfying the relations:

1. $e_\emptyset = x_\emptyset = 0$,
2. $e_A e_B = e_{A \cap B}$,
3. $e_A x_M = x_{M \cap s^{-1}(A)}$,
4. $x_M e_A = x_{M \cap r^{-1}(A)}$,
5. $x_M^* x_N = e_{r(M \cap N)}$.

An inverse semigroup from a topological graph

Definition

For a topological graph $\mathcal{E}$, we define the inverse semigroup $S(\mathcal{E})$ to be the inverse semigroup with zero generated by elements e_A for $A \in \Omega(\mathcal{E}^0)$ and elements x_M for $M \in \Omega(\mathcal{E}^1)$ such that $r|_M$ is a homeomorphism, satisfying the relations:

1. $e_\emptyset = x_\emptyset = 0$,
2. $e_A e_B = e_{A\cap B}$,
3. $e_A x_M = x_{M\cap s^{-1}(A)}$,
4. $x_M e_A = x_{M\cap r^{-1}(A)}$,
5. $x_M^* x_N = e_{r(M\cap N)}$.

To be done

- Show that the idempotents of $S(\mathcal{E})$ are related to cylinder sets.
- Find a concrete model for $S(\mathcal{E})$, or prove, for example, that for $A, B \in \Omega(\mathcal{E}^0)$, if $A \neq B$ then $e_A \neq e_B$.
- Find an action of $S(\mathcal{E})$ on $\partial\mathcal{E}$.
- Prove that Yeend's groupoid is isomorphic to the groupoid of germs of the above action.

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Thank you!