SKEIN MODULES

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Introduction

Suppose $\mathcal{C}$ is a k-linear ribbon category and M an oriented 3-manifold. The $\mathcal{C}$-skein module $\mathrm{Sk}_{\mathcal{C}}(M)$ of M is a k-vector space which we will introduce in these notes. It gives a way to "spread out" or "integrate" $\mathcal{C}$ over the 3-manifold M.

If G is an algebraic group and we denote by $\mathrm{Loc}_G(M) = \mathrm{Hom}(\pi_1(M), G)/G$ the character variety of the 3-manifold, $\mathrm{Sk}_{\mathrm{Rep}(G)}(M) \cong \mathcal{O}(\mathrm{Loc}_G(M))$, the algebra of polynomial functions on the character variety. We can also consider the category $\mathrm{Rep}_q(G)$ of representations of the quantum group. So, $\mathrm{Sk}_{\mathrm{Rep}_q(G)}(M)$ can be viewed as the space of "quantum" functions on $\mathrm{Loc}_G(M)$.

The goal of these lectures is to introduce skein modules and explain how they are related to many interesting algebras appearing in the theory of quantum groups. Towards the end we will also mention a relationship between the theory of skein modules and 3-dimensional TQFTs introduced by Reshetikhin and Turaev (formalising and generalizing the Chern–Simons TQFT).

1. Ribbon categories

1.1. **Definition.** Throughout these notes we fix a ground field k. All categories and all functors will be k-linear. Let us first recall the notion of a dual object in a monoidal category.

Definition 1.1. Let $\mathcal{C}$ be a monoidal category and $x \in \mathcal{C}$. We say $x^\vee \in \mathcal{C}$ is the ***left dual*** to x if we are given morphisms $\mathrm{ev}\colon x \otimes x^\vee \to \mathbf{1}$ and $\mathrm{coev}\colon \mathbf{1} \to x^\vee \otimes x$ such that the composites

$$x \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} x \otimes x^\vee \otimes x \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} x$$

and

$$x^\vee \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} x^\vee \otimes x \otimes x^\vee \otimes x \to \mathrm{id} \otimes \mathrm{ev} x^\vee$$

are equal to the identity. Similarly one can define right duals. A monoidal category $\mathcal{C}$ is ***rigid*** if every object admits a right and a left dual.

Remark 1.2. In the axioms of duality we have omitted some obvious unitors and associators. We will not mention them in the notes to simplify the presentation.

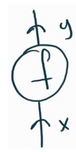
Definition 1.3. A ***ribbon category*** is a category $\mathcal{C}$ together with the following structure:

- Monoidal structure specified by a tensor functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and an associator $\Phi_{x,y,z}\colon x\otimes(y\otimes z) \xrightarrow{(} x \otimes y) \otimes z$ for $x, y, z \in \mathcal{C}$.
- Braided structure specified by $\sigma_{x,y}\colon x \otimes y \xrightarrow{\sim} y \otimes x$ for $x, y \in \mathcal{C}$.
- Ribbon element specified by $\theta\colon x \xrightarrow{\sim} x$ for $x \in \mathcal{C}$ satisfying $\theta_{x\otimes y} = (\theta_x \otimes \theta_y) \circ \sigma_{y,x} \circ \sigma_{x,y}$ and $(\theta_x)^\vee = \theta_{x^\vee}$

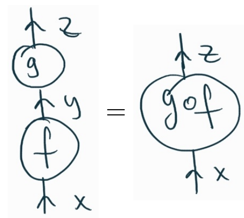
We will also assume that $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$.

The main example of ribbon categories is provided by categories of locally finite representations of quantum groups associated to reductive groups. We will discuss these examples in section 1.3.

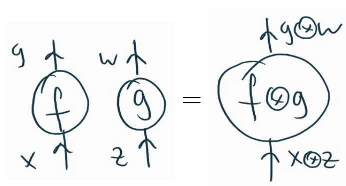
1.2. **Graphical calculus.** An important way to think about ribbon categories is in terms of their graphical calculus. Let $\mathcal{C}$ be a category. A morphism $f\colon x \to y$ is called a ***coupon*** and will be drawn as follows.



Composition of morphisms $f\colon x \to y$ and $g\colon y \to z$ is given by vertical stacking.

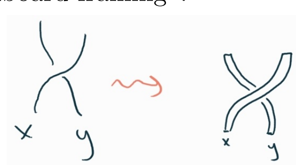


If $\mathcal{C}$ is a monoidal category, the tensor product of morphisms $f\colon x \to y$ and $g\colon z \to w$ is drawn by horizontal stacking.



If $\mathcal{C}$ is a braided monoidal category, the braiding $\sigma_{x,y}\colon x \otimes y \to y \otimes x$ is drawn as a braid.

If $\mathcal{C}$ is a ribbon category, we draw strands as ribbons. Given a picture before, we obtain a ribbon picture by endowing the strands with the "blackboard framing".



The ribbon element $\theta_x\colon x \to x$ is drawn as a $360°$ twist in the ribbon.

The duality of objects is given by reversing the orientation of the arrow. The coevaluation and evaluation maps for the duality are drawn as follows.

We can summarize this discussion as follows. Given a ribbon category $\mathcal{C}$ we can draw a $\mathcal{C}$-labeled ribbon graph Γ in the 3-ball B^3 with objects $x_1, \ldots, x_n$ ending on the lower hemisphere and $y_1, \ldots, y_m$ ending on the upper hemisphere. Then we can ***evaluate*** the graph Γ in $\mathcal{C}$ to obtain a morphism $x_1\otimes\cdots\otimes x_n \to y_1\otimes\cdots\otimes y_m$. This assignment behaves well with respect to isotopies and compositions of ribbon graphs.

1.3. **Categories of representations of quantum groups.** Let G be a split connected reductive algebraic group over the field k. Let $H \subset G$ be a maximal torus. Let $\Lambda = \mathrm{Hom}(H, \mathbf{G}_\mathrm{m})$ be the weight lattice and $\Lambda^\vee = \mathrm{Hom}(\mathbf{G}_\mathrm{m}, H)$ the coweight lattice.

Example 1.4. Consider $G = \mathrm{SL}_n$. The subgroup $H \subset \mathrm{SL}_n$ of diagonal matrices is a maximal torus. The weight lattice is $\Lambda \cong \mathbf{Z}^{n-1}$.

Choose a nonzero element $q \in k$. We denote by $\mathrm{U}_q(\mathfrak{g})$ the Lusztig form of the quantum group. It has the following generators:

- Cartan generators K_μ for $\mu \in \Lambda^\vee$.
- Divided power generators $E_i^{(r)}$, $F_i^{(r)}$ for each simple root α_i and an integer $r \geq 1$.

One should think that $E_i^{(r)} = \frac{E_i^r}{[r]!}$, where the quantum integer is defined to be

$$[r] = \frac{q^r - q^{-r}}{q - q^{-1}}$$

and the quantum factorial is

$$[r]! = \prod_{i=1}^{r} [i].$$

If q is not a root of unity, then $[r] \neq 0$ and so it is enough to consider the generators E_i as $E_i^{(r)}$ can be recovered from the above formula. However, if $q^{2r} = 1$, then $E_i^{(r)}$ is not obtained from E_i and instead we obtain a new relation $E_i^r = 0$. We refer to [Lus10, Chapter 3] for more details on $\mathrm{U}_q(\mathfrak{g})$.

Definition 1.5. $\mathrm{Rep}_q(G)$ is the category of Λ-graded vector spaces

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

equipped with a $\mathrm{U}_q(\mathfrak{g})$-module structure satisfying the following properties:

- For $v \in V_\lambda$ we have $K_\mu v = q^{\langle \lambda, \mu \rangle} v$ for $\langle -, - \rangle$ the canonical pairing between Λ and $\Lambda^\vee$.
- $$E_i^{(r)} \colon V_\lambda \to V_{\lambda + r\alpha_i}, \qquad F_i^{(r)} \colon V_\lambda \to V_{\lambda - r\alpha_i}.$$
- For fixed $v \in V$ and i there is an r such that $E_i^{(r)} v = F_i^{(r)} v = 0$.

Let $\mathrm{Rep}_q^{\mathrm{fd}}(G) \subset \mathrm{Rep}_q(G)$ be the subcategory of finite-dimensional representations.

Remark 1.6. If q is not a root of unity, finite-dimensional representations V of $\mathrm{U}_q(\mathfrak{g})$ carry a grading by Λ such that K_μ acts on the weight λ space by $\pm q^{\langle \lambda, \mu \rangle}$. If V is such that the K_μ-eigenvalues are $q^{\langle \lambda, \mu \rangle}$, one says that V is a *type I* representation. So, $\mathrm{Rep}_q(G)$ is the category of locally finite (union of finite-dimensional) type I representations. Note that if q is a root of unity, a Λ-grading on a finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-module might not exist.

The category $\mathrm{Rep}_q(G)$ has simple objects whose isomorphism classes are labeled by *dominant* weights $\lambda \in \Lambda$. We denote the correspond representations by $L(\lambda) \in \mathrm{Rep}_q(G)$.

To introduce a braiding on $\mathrm{Rep}_q(G)$ we need to choose an extra data. Choose an integer d, a d-th root $q^{1/d} \in \mathbf{C}$ of q and a symmetric bilinear form $B \colon \Lambda \times \Lambda \to \frac{1}{d}\mathbf{Z}$ such that

$$B(\alpha_i, \alpha_j) = \alpha_i \cdot \alpha_j.$$

Remark 1.7. If G is simple, the minimal such choice of d is given by the determinant of the Cartan matrix. For instance, for $G = \mathrm{SL}_n$ it is $d = n$.

Remark 1.8. If G is semisimple, B is unique if it exists, since the simple roots span $\mathfrak{h}^* = \Lambda \otimes_{\mathbf{Z}} \mathbf{C}$, the dual of the Cartan subalgebra.

Lusztig [Lus10, Chapter 32] introduces a braided monoidal structure on $\mathrm{Rep}_q(G)$ as follows. Consider $V, W \in \mathrm{Rep}_q(G)$. The braiding consists of the following three pieces:

- Define
$$\Pi\colon V \otimes W \longrightarrow V \otimes W$$
so that for $v \in V_\lambda$ and $w \in W_\mu$ we have
$$\Pi(v \otimes w) = q^{-B(\lambda,\mu)} v \otimes w.$$
- Let $\tau\colon V \otimes W \to W \otimes V$ be the map $v \otimes w \mapsto w \otimes v$ given by the flip of tensor factors.
- $\Theta \in \mathrm{U}_q(\mathfrak{g}) \widehat{\otimes} \mathrm{U}_q(\mathfrak{g})$ is the ***quasi R-matrix***.

Example 1.9. For $G = \mathrm{SL}_2$ we have
$$\Theta = \sum_{n=0}^{\infty} (-1)^n q^{-n(n-1)/2} (q - q^{-1})^n [n]! F^{(n)} \otimes E^{(n)}.$$
Note that it depends on $q^{1/2}$.

The braiding is
$$\sigma_{V,W} = \Theta \circ \Pi \circ \tau \colon V \otimes W \longrightarrow W \otimes V.$$

Next, to introduce a ribbon element we choose a homomorphism $\phi\colon \Lambda \to \mathbf{Z}/2$ such that $\phi(\alpha_i)$. Consider $V \in \mathrm{Rep}_q(G)$. The ribbon structure introduced in [ST09] consists of the following three pieces:

- Define
$$J\colon V \longrightarrow V$$
so that for $v \in V_\lambda$ we have
$$J(v) = q^{B(\lambda,\lambda)/2 + (\lambda,\rho)} v,$$
where $\rho \in \Lambda$ is the half-sum of positive roots (the Weyl vector).
- Define
$$\Phi \longrightarrow V$$
so that for $v \in V_\lambda$ we have
$$\Phi(v) = (-1)^{\phi(\lambda)} v.$$
- $T_{w_0} \in \widehat{\mathrm{U}_q(\mathfrak{g})}$ is the quantum Weyl group action by the longest element $w_0 \in W$ of the Weyl group.

Example 1.10. For $G = \mathrm{SL}_2$ we have
$$T_{w_0}(v) = \sum_{a,b,c \geq 0,\ a-b+c = B(\lambda,\alpha_i)} (-1)^b q^{ac-b} E^{(a)} F^{(b)} E^{(c)} v,$$
where $v \in V_\lambda$.

The ribbon element is
$$\theta_V = \Phi \circ (J \circ T_{w_0})^{-2} \colon V \longrightarrow V.$$

Proposition 1.11. *For $L(\lambda) \in \mathrm{Rep}_q(G)$ the simple representation with highest weight $\lambda \in \Lambda$ the ribbon element $\theta_{L(\lambda)}$ is given by multiplication by the scalar $(-1)^{\langle 2\lambda, \rho^\vee \rangle + \alpha(\lambda)} q^{B(\lambda,\lambda) + 2(\lambda,\rho)}$.*

So, in this way we obtain a ribbon structure on $\mathrm{Rep}_q^{\mathrm{fd}}(G)$.

1.4. **Semisimplified representation category.** For q a root of unity the category $\mathrm{Rep}_q(G)$ is not semisimple. One can obtain a semisimple category in the following way. Suppose q^2 is a primitive ℓ'th root of unity. Let $\mathrm{Rep}_q^{\mathrm{tilt}}(G) \subset \mathrm{Rep}_q^{\mathrm{fd}}(G)$ be the subcategory of *tilting modules*. It turns out the ribbon structure on $\mathrm{Rep}_q^{\mathrm{fd}}(G)$ restricts to one on $\mathrm{Rep}_q^{\mathrm{tilt}}(G)$.

Example 1.12. A tilting module for the quantum SL_2 is a direct summand of $L(1)^{\otimes n}$, where $L(1)$ is the irreducible two-dimensional representation.

Definition 1.13. let $\mathcal{C}$ be a ribbon category. A morphism $f\colon x \to y$ in $\mathcal{C}$ is ***negligible*** if for every $g\colon y \to x$ one has $\mathrm{tr}(f \circ g) = 0$.

For $x, y \in \mathcal{C}$ we denote by $\mathcal{N}(x, y) \subset \mathrm{Hom}_{\mathrm{Rep}_q^{\mathrm{tilt}}(G)}(x, y)$ the subspace of negligible morphisms.

Definition 1.14. The ***semisimplification*** $\mathrm{Rep}_q^{\mathrm{ss}}(G)$ ***of*** $\mathrm{Rep}_q(G)$ is the category with the same objects as $\mathrm{Rep}_q^{\mathrm{tilt}}(G)$ and with morphisms given by

$$\mathrm{Hom}_{\mathrm{Rep}_q^{\mathrm{ss}}(G)}(x, y) = \mathrm{Hom}_{\mathrm{Rep}_q^{\mathrm{tilt}}(G)}(x, y)/\mathcal{N}(x, y).$$

Proposition 1.15. $\mathrm{Rep}_q^{\mathrm{ss}}(G)$ *is a semisimple ribbon category such that the projection functor* $\mathrm{Rep}_q^{\mathrm{tilt}}(G) \to \mathrm{Rep}_q^{\mathrm{ss}}(G)$ *is ribbon.*

It turns out that the semisimplification for the categories of representations of quantum groups has a certain nondegeneracy property.

Definition 1.16. A ribbon category $\mathcal{C}$ is ***modular*** if it is semisimple, has finitely many isomorphism classes of simple objects and the following condition holds:

- (***modularity***) If $x \in \mathcal{C}$ is such that $\sigma_{y,x} \circ \sigma_{x,y} = \mathrm{id}_{x \otimes y}$ for every $y \in \mathcal{C}$, then x is a direct sum of a number of copies of the unit $\mathbf{1} \in \mathcal{C}$.

It turns out that $\mathrm{Rep}_q^{\mathrm{ss}}(G)$ is modular for good roots of unity.

Example 1.17. It is shown in [Bru00] that the category $\mathrm{Rep}_q^{\mathrm{ss}}(\mathrm{SL}_n)$ for $q = \exp(z\pi i/\ell)$ is modular if, and only if, $\gcd(z, n\ell) = 1$.

2. Skein modules

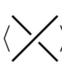
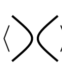
2.1. Skein modules.

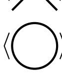
Definition 2.1. Suppose $\mathcal{C}$ is a ribbon category and M an oriented 3-manifold. The ***skein module*** $\mathrm{Sk}_{\mathcal{C}}(M)$ is the k-vector space spanned by isotopy classes of $\mathcal{C}$-ribbon graphs in M modulo the following relation:

- (***skein relation***) Given two $\mathcal{C}$-labeled ribbon graphs Γ_1, Γ_2 in M which coincide outside of a 3-ball $B^3 \subset M$ and which evaluate to the same morphism in $\mathcal{C}$ are declared to be equivalent.

There is a ***distinguished element*** $\mathrm{Dist}_M \in \mathrm{Sk}_{\mathcal{C}}(M)$ corresponding to the empty ribbon graph.

Example 2.2. Consider $G = \mathrm{SL}_2$ with the ribbon structure on $\mathrm{Rep}_q^{\mathrm{fd}}(\mathrm{SL}_2)$ determined by $\phi = 0$. Then $\mathrm{Sk}_{\mathrm{SL}_2}(M)$ is the k-vector space spanned by isotopy classes of framed unoriented links in M modulo the following local relations:

$\langle$  $\rangle = q^{1/2} \langle$  $\rangle + q^{-1/2} \langle$  $\rangle$

$\langle$  $\rangle = -(q + q^{-1}) \langle \varnothing \rangle.$

In this way we obtain the ***Kauffman bracket skein algebra*** introduced in [Prz91; Tur91].

Remark 2.3. Above we have described in simpler terms the skein module. Such a description was possible due to a simple "generators-and-relations" description of $\mathrm{Rep}_q(\mathrm{SL}_2)$ in terms of graphical calculus. An analogous graphical calculus is known in the following cases:

- For $G = \mathrm{SL}_3, \mathrm{Sp}_4, \mathrm{G}_2$ [Kup96].
- For $G = \mathrm{SL}_n$ [Sik05; CKM14].
- For $G = \mathrm{Sp}_{2n}$ [Bod+21].

Proposition 2.4. *Suppose $\mathcal{C}$ is a ribbon category. Then the map $k \to \mathrm{Sk}_{S^3}(\mathcal{C})$ given by the inclusion of the distinguished element is an isomorphism.*

The main example we will consider in these notes is $\mathcal{C} = \mathrm{Rep}_q^{\mathrm{fd}}(G)$, so we denote

$$\mathrm{Sk}_G(M) = \mathrm{Sk}_{\mathrm{Rep}_q^{\mathrm{fd}}(G)}(M).$$

In this notation we suppress q, the choice of the bilinear form B and the choice of the homomorphism ϕ.

Example 2.5. Consider $\mathcal{C} = \mathrm{Rep}_q^{\mathrm{fd}}(\mathrm{SL}_2)$ with the ribbon structure determined by $\phi = 0$. Suppose $K \subset S^3$ is a framed oriented knot. Then it defines an element $[K] \in \mathrm{Sk}_{\mathrm{SL}_2}(S^3)$ which corresponds to the ribbon graph where we label K by the two-dimensional irreducible representation $L(1) \in \mathrm{Rep}_q^{\mathrm{fd}}(\mathrm{SL}_2)$. Then the image of $[K]$ under the isomorphism $\mathrm{Sk}_{\mathrm{SL}_2}(S^3) \cong k$ is the ***Kauffman bracket*** invariant of K. Rescaling by a simple factor involving the writhe of the knot we obtain the ***Jones polynomial*** of K.

The following is an important theorem (proved in [Tur91; BFK99]) which gives a way to think about $\mathrm{Sk}_G(M)$.

Theorem 2.6. *Suppose $q = 1$ and M is connected. Then $\mathrm{Sk}_G(M) \cong \mathcal{O}(\mathrm{Loc}_G(M))$, the space of polynomial functions on the character variety $\mathrm{Loc}_G(M) = \mathrm{Hom}(\pi_1(M), G)/G$ of the 3-manifold M.*

So, $\mathrm{Sk}_G(M)$ for a general q is a quantum version of the character variety $\mathrm{Loc}_G(M)$. The following result (proved in [GJS19], previously conjectured by Witten) shows that the skein module for generic q becomes simpler.

Theorem 2.7. *If q is not a root of unity, then $\mathrm{Sk}_G(M)$ is finite-dimensional.*

Example 2.8. Suppose q is not a root of unity and Σ is a Riemann surface of genus g. Then it was shown in [GM18; DW20] that

$$\dim \mathrm{Sk}_{\mathrm{SL}_2}(\Sigma \times S^1) = 2^{2g+1} + 2g - 1.$$

2.2. **Skein categories.** Previously we have introduced skein modules, k-vector spaces, associated to 3-manifolds. Now we will introduce a k-linear category associated to a surface Σ.

Definition 2.9. Let $\mathcal{C}$ be a ribbon category and Σ an oriented surface. The ***skein category*** $\mathrm{SkCat}_{\mathcal{C}}(\Sigma)$ is defined as follows:

- Its objects are collections of points of Σ equipped with a nonzero tangent vector labeled by objects of $\mathcal{C}$.
- The space of morphisms from one labeling of Σ to another labeling of Σ is the space spanned by $\mathcal{C}$-labeled ribbon graphs in $\Sigma \times [0,1]$ intersecting the boundary transversely, whose endpoints at $\Sigma \times \{0\}$ and at $\Sigma \times \{1\}$ give the source and target labelings of the morphisms, modulo the skein relations in $\Sigma \times (0,1)$.
- Composition of morphisms is given by stacking along the $[0,1]$ direction.

As for skein modules, there is a ***distinguished object*** $\mathrm{Dist}_\Sigma \in \mathrm{SkCat}_{\mathcal{C}}(\Sigma)$ corresponding to the empty labeling of Σ. We have the following statement which implies that $\mathrm{SkCat}_G(\Sigma)$ can be thought of as a quantization of the character variety.

Remark 2.10. It is shown in [Coo19] that $\mathrm{SkCat}_{\mathcal{C}}(\Sigma)$ is the factorization homology

$$\int_\Sigma \mathcal{C}$$

of $\mathcal{C}$ over Σ in the sense of [AF15].

Proposition 2.11. *Suppose $q = 1$ and Σ is connected. Then $\mathrm{End}_{\mathrm{SkCat}_G(\Sigma)}(\mathrm{Dist}_\Sigma) \cong \mathcal{O}(\mathrm{Loc}_G(\Sigma))$.*

The following statement follows from the well-definedness of the evaluation map for ribbon categories.

Proposition 2.12. *Let D be the disk. The functor $\mathcal{C} \to \mathrm{SkCat}_{\mathcal{C}}(D)$ given by sending an object x to the origin of D labeled by x and a morphism $f\colon x \to y$ to the vertical skein with a coupon f is an equivalence.*

If the surface Σ is connected equipped with a basepoint $p \in \Sigma$, the natural functor $\mathcal{C} \to \mathrm{SkCat}_{\mathcal{C}}(\Sigma)$ given by sending $x \in \mathcal{C}$ to the point $p \in \Sigma$ labeled by x is essentially surjective.

Proposition 2.13. *Let S^2 be the 2-sphere. The functor $\mathcal{C} \to \mathrm{SkCat}_{\mathcal{C}}(S^2)$ given by sending an object x to the origin of S^2 labeled by x is a quotient where we set $\sigma_{y,x} \circ \sigma_{x,y} = \mathrm{id}_{x\otimes y}\colon x \otimes y \to x \otimes y$ for every $x, y \in \mathcal{C}$.*

Definition 2.14. The endomorphism algebra $\mathrm{End}_{\mathrm{SkCat}_{\mathcal{C}}(\Sigma)}(\mathrm{Dist}_\Sigma)$ is the ***skein algebra*** of Σ.

By definition the skein algebra is $\mathrm{Sk}(\Sigma \times (0, 1))$ as a vector space with the multiplication given by stacking in the $(0, 1)$ direction.

2.3. Modules over categories.

Definition 2.15. Let $\mathcal{C}$ be a category. A ***(left)*** ***$\mathcal{C}$-module*** is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Vect}$ to the category of vector spaces. The category of all $\mathcal{C}$-modules is denoted by $\widehat{\mathcal{C}}$.

Example 2.16. Let A be an algebra. Let $\mathcal{C}$ be the category with a unique object $*$ and $\mathrm{End}_{\mathcal{C}}(*) = A$. Then a $\mathcal{C}$-module is the same as an A-module.

Example 2.17. Suppose q is not a root of unity. Then $\widehat{\mathrm{Rep}_q^{\mathrm{fd}}(G)} \cong \mathrm{Rep}_q(G)$.

There is a natural fully faithful Yoneda functor $\mathcal{C} \to \widehat{\mathcal{C}}$ which sends $x \in \mathcal{C}$ to the functor $\mathrm{Hom}_{\mathcal{C}}(-, x)$. It is an analog of the regular representation of an algebra. If $\mathcal{C}$ is a ribbon category, $\widehat{\mathcal{C}}$ inherits a natural braided monoidal (in fact, balanced) structure such that $\mathcal{C} \to \widehat{\mathcal{C}}$ is braided monoidal.

We can combine skein modules and skein categories in the following way.

Definition 2.18. Let M be an oriented 3-manifold with boundary Σ. The ***skein module*** $\mathrm{Sk}_{\mathcal{C}}(M)$ is the functor $\mathrm{SkCat}_{\mathcal{C}}(\Sigma)^{\mathrm{op}} \to \mathrm{Vect}$ which sends a labeling of Σ to the k-vector space spanned by $\mathcal{C}$-labeled ribbon graphs in M ending on the given labeling modulo skein relations in the interior of M.

Remark 2.19. $\mathrm{SkCat}_{\mathcal{C}}(\varnothing) = *$ is the one-object category and a functor $* \to \mathrm{Vect}$ is the same as a vector space. So, if M has no boundary we recover the previous definition of the skein module.

2.4. Reflection equation algebra.

Definition 2.20. Suppose $\mathcal{C}$ is a ribbon category. The ***reflection equation algebra*** $\mathcal{F} \in \widehat{\mathcal{C}}$ is the object defined by the following universal property. For $V \in \widehat{\mathcal{C}}$ the space $\mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathcal{F}, V)$ consists of maps $K_x\colon x \to x \otimes V$ for every $x \in \mathcal{C}$ compatible with morphisms $f\colon x \to y$.

In particular, the universal property gives us maps $K_x\colon x \to x \otimes \mathcal{F}$. The object $\mathcal{F}$ was introduced in [LM94; Lyu95] where it was equipped with the structure of a braided Hopf algebra in $\widehat{\mathcal{C}}$:

- The unit $\mathbf{1} \to \mathcal{F}$ is given by $K_{\mathbf{1}}$.
- The product $m\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ is defined so that the composite
$$x \otimes y \xrightarrow{K_x \otimes K_y} x \otimes \mathcal{F} \otimes y \otimes \mathcal{F} \xrightarrow{\mathrm{id}\otimes\sigma_{\mathcal{F},y}\otimes\mathrm{id}} x \otimes y \otimes \mathcal{F} \otimes \mathcal{F} \xrightarrow{\mathrm{id}\otimes m} x \otimes y \otimes \mathcal{F}$$
is $K_{x\otimes y}$.
- The counit $\epsilon\colon \mathcal{F} \to \mathbf{1}$ is defined so that $x \xrightarrow{K_x} x \otimes \mathcal{F} \xrightarrow{\mathrm{id}\otimes\epsilon} x$ is the identity.
- The coproduct is $K_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$.

The name "reflection equation algebra" has the following origin. Suppose $F\colon \widehat{\mathcal{C}} \to \mathrm{Vect}$ is a monoidal functor. Then we have ***universal K-matrices*** $K_x \in F(\mathcal{F}) \otimes \mathrm{End}(F(x))$ for every $x \in \mathcal{C}$. For $x, y \in \mathcal{C}$ let $R \in \mathrm{End}(F(x) \otimes F(y))$ be the image of the braiding in $\mathcal{C}$ under F. Then for $x, y \in \mathcal{C}$ we obtain the ***reflection equation*** [DKM03]
$$R_{21}K_1R_{12}K_2 = K_2R_{21}K_1R_{12}$$
in $F(\mathcal{F}) \otimes \mathrm{End}(F(x) \otimes F(y))$.

So, $\mathcal{F}$ is the universal receptacle for solutions of the reflection equation.

Example 2.21. Suppose $\mathcal{C} = \mathrm{Rep}_q^{\mathrm{fd}}(G)$ and q is not a root of unity. Then $\widehat{\mathcal{C}} = \mathrm{Rep}_q(G)$. The reflection equation algebra $\mathcal{F} \in \mathrm{Rep}_q(G)$ is usually denoted by $\mathcal{O}_q(G) \in \mathrm{Rep}_q(G)$. It is a quantization of G equipped with the adjoint action of G on itself.

Recall the notion of the Drinfeld center.

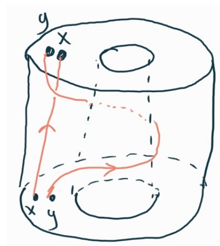
Definition 2.22. Let $\mathcal{C}$ be a monoidal category. The ***Drinfeld center*** $\mathrm{Z}_{\mathrm{Dr}}(\mathcal{C})$ is the category of pairs (z, τ), where $z \in \mathcal{C}$ and $\tau_x : x \otimes z \to z \otimes x$ is a natural isomorphism satisfying standard compatibilities.

The following was shown in [BBJ18].

Proposition 2.23. *Let* Ann *be the annulus. There is an equivalence of categories*

$$\widehat{\mathrm{SkCat}_{\mathcal{C}}(\mathrm{Ann})} \cong \mathrm{LMod}_{\mathcal{F}}(\widehat{\mathcal{C}}) \cong \mathrm{Z}_{\mathrm{Dr}}(\widehat{\mathcal{C}}).$$

Explicitly, a functor $F : \mathrm{SkCat}_{\mathcal{C}}(\mathrm{Ann}) \to \mathrm{Vect}$ is the same as a functor $F : \mathcal{C} \to \mathrm{Vect}$ equipped with a compatible family of isomorphisms $F(x \otimes y) \cong F(y \otimes x)$ for $x, y \in \mathcal{C}$ coming from wrapping y around the center of the annulus.



2.5. **Alekseev–Grosse–Schomerus algebras.** We now want to describe skein categories for arbitrary surfaces. We will use the following general formalism. Suppose $\mathcal{C}$ is a monoidal category and $A, B \in \mathcal{C}$ are two algebras. To specify an algebra structure on $A \otimes B$ we need to define an isomorphism $B \otimes A \to A \otimes B$ (the *cross relation*) satisfying some obvious compatibilities.

Definition 2.24. The ***algebra*** $\mathcal{H}$ ***of quantum differential operators*** is $\mathcal{H} = \mathcal{F} \otimes \mathcal{F} \in \widehat{\mathcal{C}}$ with the cross relation $f : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ defined by the commutative diagram

$$\begin{array}{ccccc} x \otimes \mathcal{F} & \xrightarrow{K \otimes \mathrm{id}} & x \otimes \mathcal{F} \otimes \mathcal{F} & \xrightarrow{\mathrm{id} \otimes f} & x \otimes \mathcal{F} \otimes \mathcal{F} \\ \downarrow{\scriptstyle \sigma^{-1}} & & & & \downarrow{\scriptstyle \sigma \otimes \mathrm{id}} \\ \mathcal{F} \otimes x & & \xrightarrow{\mathrm{id} \otimes K} & & \mathcal{F} \otimes x \otimes \mathcal{F} \end{array}$$

Example 2.25. Suppose $\mathcal{C} = \mathrm{Rep}_q^{\mathrm{fd}}(G)$ and q is not a root of unity. Then $\mathcal{H} \in \mathrm{Rep}_q(G)$ is usually denoted by $D_q(G) \in \mathrm{Rep}_q(G)$. It is the algebra of quantum differential operators considered in [BK06; VV10]. It is a quantization of $G \times G$ equipped with its action of G by simultaneous conjugation.

The following algebras were introduced by Alekseev, Grosse and Schomerus [AGS96].

Definition 2.26. The ***genus*** g ***AGS algebra*** is $\mathcal{H}_g = \mathcal{H}^{\otimes g} \in \widehat{\mathcal{C}}$, the braided tensor product of the algebras $\mathcal{H}$.

The following is proven in [BBJ18].

Proposition 2.27. *Let* Σ *be a surface of genus* g *with one boundary component. There is an equivalence of categories*

$$\widehat{\mathrm{SkCat}_{\mathcal{C}}(\Sigma)} \cong \mathrm{LMod}_{\mathcal{H}_g}(\widehat{\mathcal{C}}).$$

Example 2.28. Consider the case $g = 1$, so that $\Sigma = T^2 \setminus D$. The mapping class group of Σ is $\mathrm{SL}_2(\mathbf{Z})$ and therefore it acts on the category $\mathrm{LMod}_{\mathcal{H}}(\widehat{\mathcal{C}})$. As explained in [BJ17], it can be explicitly described in terms of the Fourier transform for quantum differential operators.

3. Topological quantum field theories

In this section we explain a relationship between skein modules and 3-dimensional TQFTs introduced by Reshetikhin and Turaev [RT91].

3.1. **TQFTs.** Let us briefly recall the structure of 3-dimensional TQFTs. Broadly speaking, it is given by the following assignments:

- To a closed oriented surface Σ we assign a vector space $Z(\Sigma)$. For the empty surface we assign the ground field k and for a disjoint union we assign the tensor product.
- To a compact oriented 3-manifold M with boundary decomposed as $\overline{\Sigma}_- \coprod \Sigma_+$ we assign a linear map
$$Z(M)\colon Z(\Sigma_-) \longrightarrow Z(\Sigma_+).$$
Diffeomorphisms of M relative to the boundary give rise to equal maps. In the case M is closed we get a map $k \to k$, i.e. a number.

Given a modular tensor category $\mathcal{C}$ Reshetikhin and Turaev have constructed such invariants Z_{WRT} of surfaces and 3-manifolds. Our goal will be to explain how these invariants are related to skein modules.

A large source of modular tensor categories is given by $\mathrm{Rep}_q^{\mathrm{ss}}(G)$ for appropriate roots of unity.

3.2. **TQFTs from skein modules.** The construction explained in this section is due to Walker [Wal]. Throughout this section we fix a modular tensor category $\mathcal{C}$.

The first fact is that the skein module $\mathrm{Sk}_{\mathcal{C}}(M)$ dramatically simplifies if $\mathcal{C}$ is modular.

Proposition 3.1. *Suppose $\mathcal{C}$ is a modular tensor category and M a closed oriented 3-manifold. Then $\mathrm{Sk}_{\mathcal{C}}(M)$ is one-dimensional.*

This fact has the following implication.

Proposition 3.2. *Suppose Σ is a closed oriented surface and M_1, M_2 are two compact oriented 3-manifolds with boundary Σ. Then*
$$\mathrm{Sk}_{\mathcal{C}}(M_1) \cong \mathrm{Sk}_{\mathcal{C}}(M_2) \otimes \mathrm{Sk}_{\mathcal{C}}(M_1 \coprod_{\Sigma} \overline{M}_2).$$

So, for every oriented surface Σ we may canonical associate a projective space $\mathbb{P}(\mathrm{Sk}_{\mathcal{C}}(M))$, where M is any bounding 3-manifold. These spaces can be easily computed to be
$$\mathrm{Sk}_{\mathcal{C}}(M) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{F}^{\otimes g}),$$
where g is the genus.

Proposition 3.3. *There is an isomorphism $\mathrm{Sk}_{\mathcal{C}}(M) \cong Z_{WRT}(\Sigma)$.*

Remark 3.4. If we fix Σ, the left-hand side is well-defined up to tensoring with a line. Similarly, the space of states $Z_{WRT}(\Sigma)$ is not canonically associated to Σ due to a *framing anomaly* of the corresponding TQFT.

If M is a closed oriented 3-manifold and W is a bounding 4-manifold Walker constructs an isomorphism
$$Z_{CY}(W)\colon \mathrm{Sk}_{\mathcal{C}}(M) \longrightarrow k.$$

Remark 3.5. if M is empty, then $\mathrm{Sk}_{\mathcal{C}}(\varnothing) = k$ and $Z_{CY}(W)$ coincides with the Crane–Yetter state sum of W as defined in [CKY97].

Proposition 3.6. *Let M be a closed oriented 3-manifold and W a bounding 4-manifold. Then the value of $Z_{CY}(W)$ on $\mathrm{Dist}_M \in \mathrm{Sk}_{\mathcal{C}}(M)$ coincides with $Z_{WRT}(M)$.*

Recently a 3-dimensional TQFT was constructed from *non-semisimple* modular tensor categories (for instance, the category of representations of the small quantum group at a good root of unity), see e.g. [De +19]. It would be interesting to relate those invariants to skein modules.

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