Weak amenability and $\tilde{A}_2$-geometry

Online workshop: C^*-algebras and geometry of semigroups and groups

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Approximation properties

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A fascinating open question

Find an **explicit and natural** separable Banach space without AP.

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Haagerup's work on approximation properties

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Proof relies heavily on the geometry of the tree (Cayley graph). It proves more : it has OAP and even CBAP with constant 1.

A C^*-algebra A has the Operator Space AP (OAP) if there is a net of finite rank operators such that $T_i \otimes \mathrm{id}$ converges pointwise (iff unif. on compacta) to id on $A \otimes \mathcal{K}(\ell_2)$.

Def : A has CBAP if moreover $\sup_i \|T_i \otimes \mathrm{id}\| \leq C < \infty$. And $\inf C = \Lambda_{cb}(A)$.

Haagerup's diagonal averaging trick

A notation : if $\varphi\colon G \to \mathbf{C}$ is a function

$$\|\varphi\|_{cp} := \inf\{\sup_{g,h} \|\xi(g)\|_{\ell_2}\|\eta(h)\|_{\ell_2}) \mid \varphi(g^{-1}h) = \langle \xi(h), \eta(g)\rangle\}.$$

Theorem (Haagerup + many other)

For a discrete group G, $C^*_\lambda(G)$ has CBAP with constant $\leq C$ if and only if G is weakly amenable with constant $\leq C$: there is a net $\varphi_i\colon G \to \mathbf{C}$ of finite support such that $\lim_i \varphi_i(g) = 1$ for all g and $\sup_i \|\varphi_i\|_{cb} \leq C$.

Used a lot, to show :

- (Haagerup, de Cannière, Cowling) A simple connected Lie group is weakly amenable iff $\mathrm{rk}_{\mathbf{R}}(G) \leq 1$.
- (Ozawa) Gromov-hyperbolic groups are weakly amenable.
- (Mizuta + Guentner-Higson) Groups acting on finite dim CAT(0) cube complex.

$\tilde{A}_2$ buildings

$\tilde{A}_2$ buildings, 1

Definition (formal, see later for better)

A (locally finite) $\tilde{A}_2$ building X is a 2-dimensionnal simply connected simplicial complex whose link is the incidence graph of a (finite) projective space.

From now on, X will always denote a locally finite $\tilde{A}_2$ building.

(Here) finite projective space. Take k a (finite) field, $P^2k := k^3 \setminus \{0\}/k^*$. It contains points and lines. Incidence relation $p \sim \ell$ if p belongs to ℓ.

Example : if $F = \mathbf{Q}_p$ and $\mathcal{O} = \mathbf{Z}_p$ or $\mathbf{F}_p((t))$ and $\mathcal{O} = \mathbf{F}_p[[T]]$ then $G/K := \mathrm{PGL}_3(F)/\mathrm{PGL}_3(\mathcal{O})$ with edges $gK \sim hK$ if $\|g^{-1}h\|\|h^{-1}g\| = p$ (+ fill triangles) is an $\tilde{A}_2$ building.

There are many other “exotic” examples, whose automorphism group is in general countable (Radu), sometimes trivial but sometimes quite large (Ronan, Kantor, Radu...).

Approximation properties for $\tilde{A}_2$-lattices.

Today's Theorem (Lécureux-dlS-Witzel 20+)

Let G be discrete group acting by isometries on a locally finite $\tilde{A}_2$ building. Assume that the action is cocompact and with finite stabilizers. Then G is not weakly amenable.

A few remarks :

- More generally, $C^*_\lambda(G)$ does not have *OAP*; neither does $L_p(\mathcal{L}G)$ for $p \notin [\frac{4}{3}, 4]$.
- This generalizes previous results for lattices in $\mathrm{SL}_3(F)$ by Haagerup 86 and Lafforgue-dlS 11, and gives a geometric proof of these results.
- This is one of the outcomes of a broader project where we try to develop harmonic analysis on $\tilde{A}_2$ buildings. Other outcomes include strong property (T) or vanishing of ℓ_p-cohomology.

$\tilde{A}_2$-building, 2

Recall

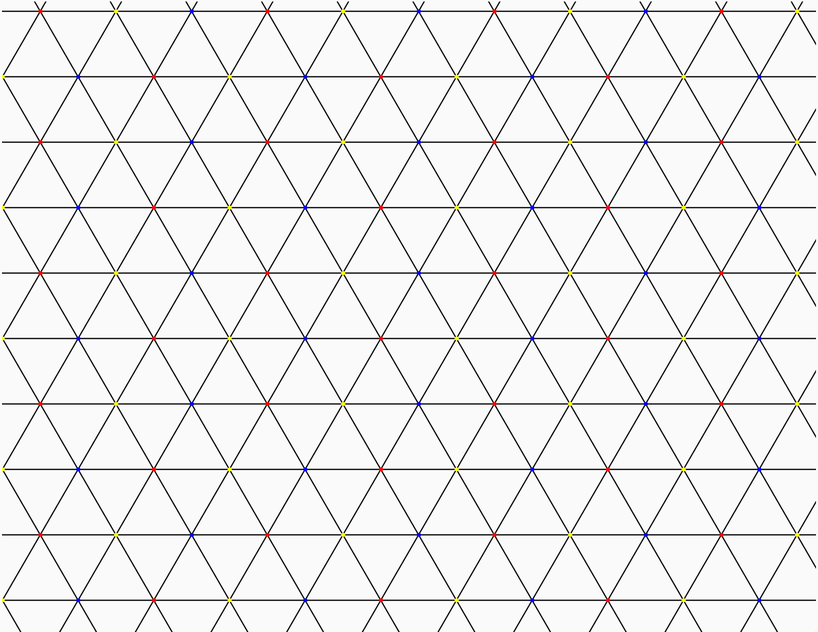
Definition

An $\tilde{A}_2$ building is a 2-dimensionnal simply connected simplicial complex whose link is the incidence graph of a finite projective space.

The way we think of it : a kind of 2-dimensionnal tree, obtained by pasting in a tree-like structure infinitely many copies of a $\mathbf{R}^2$ tesselated by equilateral triangles.

To connect with standard terminology : $\mathbf{R}^2$ = appartments ; equilateral triangles = chambers.

$\mathbb{R}^2$ tesselated by equilateral triangles...



… arranged in a tree-like way

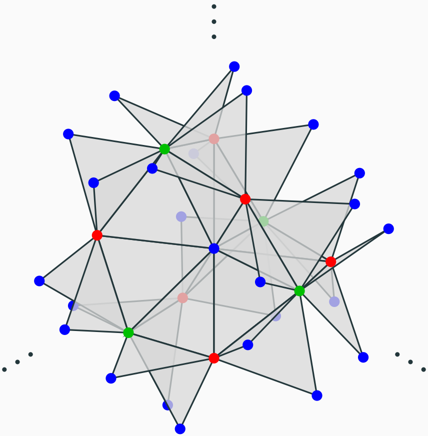


Figure 1 – A fragment of an $\tilde{A}_2$ building for $q = 2$ (picture by Greg Kuperberg)

Parameter q of a building : $q + 1$ is the number of triangles to which every edge belongs.

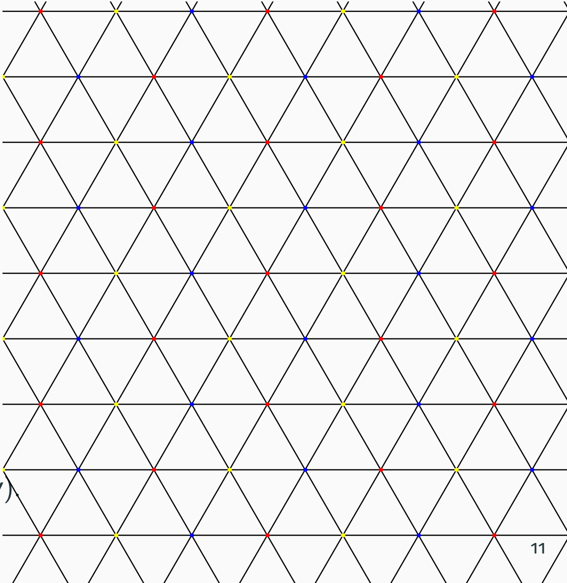
Harmonic analysis on an $\tilde{A}_2$ building, after Cartwright-Młotkowski 94

- Relative position of a pair $(x, y) \in X$ is given by $\sigma(x, y) = \lambda \in \mathbf{N}^2$.
- The distance $d(x, y)$ is $\lambda_1 + \lambda_2$.
- Define the sphere $S_\lambda(x)$

$$\{y \in X \mid \sigma(x, y) = \lambda\}.$$

- $A_\lambda \in B(\ell_2(X))$ the averaging operator

$$A_\lambda f(x) = \frac{1}{|S_\lambda|} \sum_{y \in S_\lambda(x)} f(y).$$



Harmonic analysis on an $\tilde{A}_2$ building 2

Theorem (Cartwright-Młotkowski 94)

The $*$-algebra generated by $\{A_\lambda \mid \lambda \in \mathbf{N}^2\}$ is commutative.

It is a generalization of the (Hecke) algebra of K-biinvariant functions on $G = \mathrm{SL}_3(F)$.

Philosophy : in our generality, there is no G and no K (so no Gelfand pair !), but the spectrum of the Gelfand pair is there somewhere hidden.

Later : Cartwright-Mantero-Steger-Zappa compute explicitely the spectrum of the universal representation of the above commutative $*$-algebra. Consequence : $\tilde{A}_2$ groups have property (T).

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Our contribution : developp **local** tools to further perform harmonic analysis on X.

Local harmonic analysis in $\tilde{A}_2$-buildings

Two main ingredients

- Local harmonic analysis : finite volume, fine analysis.
- Global analysis : exploring the whole building at large scales using the local analysis.

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Strongly inspired by Vincent Lafforgue, who introduced these two ingredients for $\mathrm{SL}_3(F)$. In that case, the ingredients become

- Harmonic analysis in the maximal compact subgroup K.
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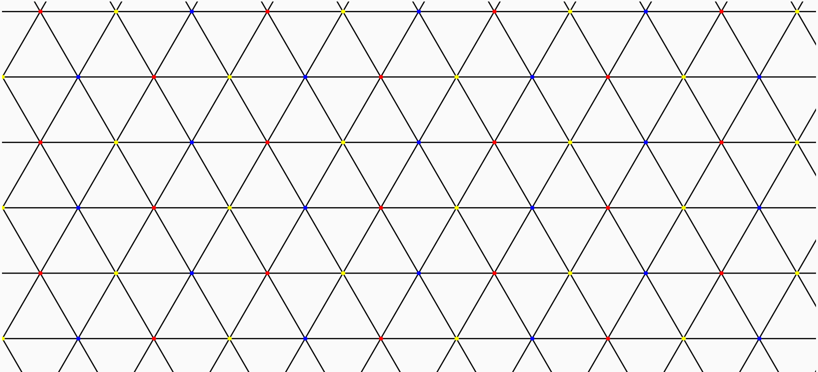
Difficulty : there is no locally compact group, typically vertex stabilizers in $\mathrm{Aut}(X)$ are trivial !

A finite-volume geometric object : biaffine Hemlslev planes

Classical object : an affine plane = a projective plane without a line and all points incident to it.

We introduce : a biaffine place = a projective plane without an incident pair (p, ℓ) and all lines/points adjacent to them.

A biaffine Hemlslev plane = a similar object but on the ℓ_1 spheres in X.



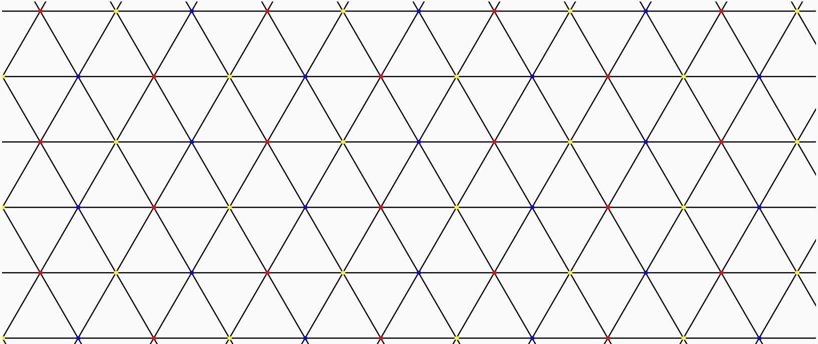
Averaging operators on biaffine Hemlslev planes

If p, ℓ is a point at distance n from the origin in a biaffine Hemslev plane, we define

$$(p, \ell)_o = \max\{s \mid (op_s\ell_s) \text{ is a regular triangle.}$$

Averaging operator : $T_s f(p) = \mathbb{E}[f(\ell) \mid (p, \ell)_o = s]$.

Main result $\|T_s - T_{s+1}\|_{L^2 \to L^2} \leq Cq^{-\frac{s}{2}}$.



The main technical result

Assume G acts nicely on X. Let $\psi\colon X \times X \to \mathbf{C}$ be a G-invariant Schur multiplier (that is $\psi(x,y) = \langle \xi_x, \eta_y \rangle$ for bounded functions $\xi, \eta\colon X \to \ell_2$).

Then for every $x \in X$, there is $\psi_\infty(x) \in \mathbf{C}$ such that

$$\frac{1}{|\Gamma_x|} \sum_{x \in X/\Gamma} \left| \psi_\infty(x) - \frac{1}{|S_\lambda(x)|} \sum_{y \in S_\lambda(x)} \psi(x,y) \right| \leq C q^{-|\lambda|/2} \|\psi\|_{cb}.$$