On the existence of bases in the orbit of unitary group representations

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Online workshop: C*-algebras and geometry of groups and semigroups

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- Intuition: For $\pi(\Gamma)\eta$ to "span" $\mathcal{H}_\pi$, Γ must be "sufficiently dense", and for $\pi(\Gamma)\eta$ to be "linearly independent", Γ must be "sufficiently sparse".

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- $(e_j)_j$ is a *Riesz basis* if it is both a frame and a Riesz sequence.

Motivating example

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1. Let $G = \mathbb{R}^{2d} \cong \mathbb{R}^d \times \mathbb{R}^d$ and let $\pi\colon \mathbb{R}^{2d} \to \mathcal{U}(L^2(\mathbb{R}^d))$ be the Heisenberg representation given by

   $$\pi(x,\omega)\xi(t) = e^{2\pi i \omega \cdot t}\xi(t-x) \quad \text{for } (x,\omega) \in \mathbb{R}^{2d} \text{ and } \xi \in L^2(\mathbb{R}^d).$$

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2. π is a σ-projective representation:

   $$\pi(x,\omega)\pi(x',\omega') = e^{-2\pi i x\cdot\omega'}\pi(x+x',\omega+\omega'),$$

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3. In this context, families $(\pi(\gamma)\eta)_{\gamma \in \Gamma}$ for $\eta \in L^2(\mathbb{R}^d)$ and a discrete subset $\Gamma \subseteq \mathbb{R}^{2d}$ are known as Gabor systems, and have been extensively studied in Gabor analysis.

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1. Let $G = A \times \widehat{A}$ for a locally compact abelian group A and let $\pi\colon G \to \mathcal{U}(L^2(A))$ be the Heisenberg representation given by

$$\pi(x,\omega)\xi(t) = \omega(t)\xi(x^{-1}t) \quad \text{for } (x,\omega) \in G \times \widehat{G} \text{ and } \xi \in L^2(A).$$

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Square-integrable representations

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- For the rest of the talk, we assume that G is a unimodular, second countable, locally compact group, and that π is a σ-projective, irreducible, unitary representation of G which is square-integrable, i.e., there exist nonzero $\xi, \eta \in \mathcal{H}_\pi$ such that

$$\int_G |\langle \xi, \pi(x)\eta \rangle|^2 \, \mathrm{d}x < \infty.$$

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$$\int_G |\langle \xi, \pi(x)\eta \rangle|^2 \, dx < \infty.$$

- Under these assumptions, the following orthogonality relations hold for all $\xi, \xi', \eta, \eta' \in \mathcal{H}_\pi$:

$$\int_G \langle \xi, \pi(x)\eta \rangle \overline{\langle \xi', \pi(x)\eta' \rangle} \, dx = d_\pi^{-1} \langle \xi, \xi' \rangle \overline{\langle \eta, \eta' \rangle}.$$

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- The number d_π is called the formal dimension of π and depends on the choice of Haar measure on G.

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Theorem (Romero–Van Velthoven 2020)

With π a σ-projective unitary representation of G as before, let Γ be a lattice in G. Then the following hold for $\eta \in \mathcal{H}_\pi$:

1. If $\pi(\Gamma)\eta$ is a frame for $\mathcal{H}_\pi$, then $d_\pi \, \mathrm{vol}(G/\Gamma) \leq 1$.
2. If $\pi(\Gamma)\eta$ is a Riesz sequence for $\mathcal{H}_\pi$, then $d_\pi \, \mathrm{vol}(G/\Gamma) \geq 1$.
3. If $\pi(\Gamma)\eta$ is a Riesz basis for $\mathcal{H}_\pi$, then $d_\pi \, \mathrm{vol}(G/\Gamma) = 1$.

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- The density theorem has a long history for the Heisenberg representation of $\mathbb{R}^{2d}$.

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3. An element $\gamma \in \Gamma$ is called σ-regular if $\sigma(\gamma, \gamma') = \sigma(\gamma', \gamma)$ whenever $\gamma\gamma' = \gamma'\gamma$. We say that (Γ, σ) satisfies Kleppner's condition if every nontrivial σ-regular conjugacy class is infinite.

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Romero–Van Velthoven (2020)

If (Γ, σ) satisfies Kleppner's condition, there exists $\eta \in \mathcal{H}_\pi$ such that $\pi(\Gamma)\eta$ is a frame if and only if $d_\pi \operatorname{vol}(G/\Gamma) \leq 1$. Analagous statements hold for Riesz sequences and Riesz bases.

Characterization of existence

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Theorem (Bekka, 2004)

Let G be a unimodular, second countable, locally compact group and let π be a square-integrable, irreducible, unitary representation of G. Let Γ be a lattice in G. Let $\eta \in \mathcal{H}_\pi$ be a unit vector. Define a function $\phi \in \ell^\infty(\Gamma)$ by

$$\phi(\gamma) = \frac{d_\pi}{|C_\gamma|} \int_{G/\Gamma_\gamma} \langle \eta, \pi(y^{-1}\gamma y)\eta \rangle \, \mathrm{d}(y\Gamma_\gamma)$$

if the conjugacy class C_γ is finite, and $\phi(\gamma) = 0$ otherwise. Then:

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Theorem (E.)

Let G be a unimodular, second countable, locally compact group with 2-cocycle σ and let π be a σ-projective, square-integrable, irreducible, unitary representation of G. Let Γ be a lattice in G. Let $\eta \in \mathcal{H}_\pi$ be a unit vector. Define a function $\phi \in \ell^\infty(\Gamma)$ by

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3. $\exists \eta \in \mathcal{H}_\pi$: $\pi(\Gamma)\eta$ is a Riesz basis iff $\phi = \delta_e$.

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Corollary (E.)

If G is abelian and (G, σ) satisfies Kleppner's condition, then $\phi = d_\pi \operatorname{vol}(G/\Gamma)\delta_e$.

Twisted group von Neumann algebras

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1. Let Γ be a discrete group with 2-cocycle σ. The σ-twisted left regular representation λ_σ of Γ on $\ell^2(\Gamma)$ is given by

   $$\lambda_\sigma(\gamma)f(\gamma') = \sigma(\gamma, \gamma^{-1}\gamma')f(\gamma^{-1}\gamma') \quad \text{for } \gamma, \gamma' \in \Gamma \text{ and } f \in \ell^2(\Gamma).$$

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3. The action of Γ on $\mathcal{H}_\pi$ via $\pi|_\Gamma$ extends to give $\mathcal{H}_\pi$ the structure of a left Hilbert $\mathsf{L}(\Gamma, \sigma)$-module.

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$$C\xi = (\langle \xi, \pi(\gamma)\eta \rangle)_{\gamma \in \Gamma}$$

is bounded and injective, with closed range. It also intertwines $\pi|_\Gamma$ and $\lambda_\sigma|_\Gamma$.

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- Conversely, if $C\colon \mathcal{H}_\pi \to \ell^2(\Gamma)$ is an isometry that intertwines $\pi|_\Gamma$ and $\lambda_\sigma|_\Gamma$, then $\pi(\Gamma)\eta$ is a frame, where $\eta = P\delta_e$ (P projection of $\ell^2(\Gamma)$ onto $\mathcal{H}_\pi$).

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- Conclusion: There exists a frame $\pi(\Gamma)\eta$ for some $\eta \in \mathcal{H}_\pi$ if and only if $\mathcal{H}_\pi$ is a subrepresentation of $\ell^2(\Gamma)$. This extends to an inclusion of Hilbert $\mathsf{L}(\Gamma, \sigma)$-modules: $\mathcal{H}_\pi \leq \ell^2(\Gamma)$.

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- Conclusion: There exists a frame $\pi(\Gamma)\eta$ for some $\eta \in \mathcal{H}_\pi$ if and only if $\mathcal{H}_\pi$ is a subrepresentation of $\ell^2(\Gamma)$. This extends to an inclusion of Hilbert $\mathrm{L}(\Gamma,\sigma)$-modules: $\mathcal{H}_\pi \leq \ell^2(\Gamma)$.
- Hilbert modules over finite von Neumann algebras are entirely determined by their center-valued dimension: $\mathcal{H}_\pi \leq \ell^2(\Gamma)$ if and only if $\operatorname{cdim}\mathcal{H}_\pi \leq \operatorname{cdim}\ell^2(\Gamma) = I$.

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- Set
$$\mathcal{H}_\pi^1 = \Big\{ \xi \in \mathcal{H}_\pi : \int_G |\langle \xi, \pi(x)\xi \rangle| \, \mathrm{d}x < \infty \Big\}.$$

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- Often, one wants additional regularity of η, e.g., through decay and/or smoothness of the matrix coefficients of η.
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$$\mathcal{H}_\pi^1 = \Big\{ \xi \in \mathcal{H}_\pi : \int_G |\langle \xi, \pi(x)\xi \rangle| \, \mathrm{d}x < \infty \Big\}.$$
- What can we say about existence of a frame of the form $\pi(\Gamma)\eta$ for some $\eta \in \mathcal{H}_\pi^1$?

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- Generalizes to locally compact abelian groups A with noncompact identity component (E., Jakobsen, Luef, Omland).

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Let $\Gamma = M\mathbb{Z}^{2d}$ be a lattice in $\mathbb{R}^{2d}$ for $M \in GL_{2d}(\mathbb{R})$ such that the matrix $M^t JM$ contains at least one irrational entry. Here J denotes the standard symplectic $2n \times 2n$ matrix. Then the following hold:

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- Frames $\pi(\Gamma)\eta$ with $\eta \in \mathcal{H}_\pi^1$ can be interpreted as single generators of Rieffel's Heisenberg module over $C^*(\Gamma, \sigma) \cong A_\Theta$, $\Theta = M^t JM$.

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  where the balls $B_r(x)$ are defined using a choice of word metric on G.
- If Γ is a lattice in G, then $D^{-}(\Gamma) = D^{+}(\Gamma) = 1/\operatorname{vol}(G/\Gamma)$.

Density and Balian–Low Theorems for irregular point sets

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Theorem (Führ–Gröchenig–Haimi–Klotz–Romero 2017)

Let G be a compactly generated, locally compact group with polynomial growth, and let π be a square-integrable, irreducible, unitary representation of G. Let $\Gamma \subseteq G$ be discrete and $\eta \in \mathcal{H}_\pi$. Then:

1. If $\pi(\Gamma)\eta$ is a frame for $\mathcal{H}_\pi$, then $D^-(\Gamma) \geq d_\pi$.
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Theorem (Gröchenig–Romero–Van Velthoven 2019)

Let G be a homogeneous Lie group and let π be a square-integrable, irreducible, unitary representation of G. Let $\Gamma \subseteq G$ be discrete and $\eta \in \mathcal{H}_\pi^1$. Then:

1. If $\pi(\Gamma)\eta$ is a frame for $\mathcal{H}_\pi$, then $D^-(\Gamma) > d_\pi$.
2. If $\pi(\Gamma)\eta$ is a Riesz sequence for $\mathcal{H}_\pi$, then $D^+(\Gamma) < d_\pi$.

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