Conjugacy and dynamics in tree almost automorphism groups

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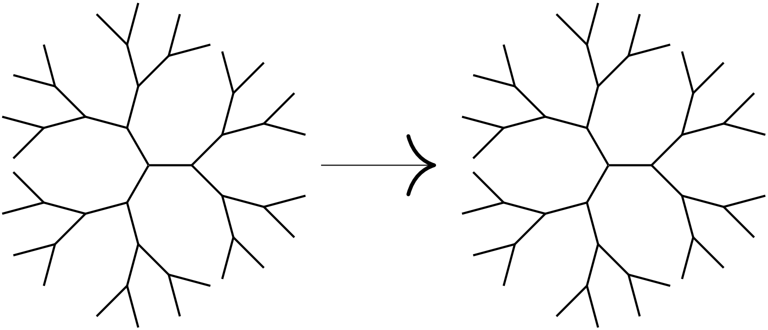
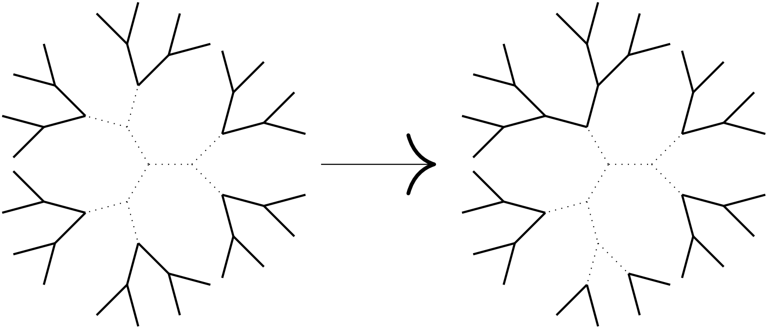
Tree almost automorphisms

Tree automorphism: bijection between vertices and edges preserving adjacency



Tree almost automorphism: (Equivalence class of) bijection between trees, *outside a finite set*



aAut($\mathcal{T}$): the group of almost automorphisms of the tree $\mathcal{T}$
t.d.l.c. group with a natural topology, Aut($\mathcal{T}$) $\leq$ aAut($\mathcal{T}$) open

There is an obvious action $\mathrm{aAut}(\mathcal{T}) \curvearrowright \partial\mathcal{T}$, can consider $\mathrm{aAut}(\mathcal{T}) \leq \mathrm{Homeo}(\partial\mathcal{T})$.

If $\mathcal{T}$ is regular, $\mathrm{aAut}(\mathcal{T})$ is called **Neretin's group**.
It was defined by Neretin in the 90s.

Theorem

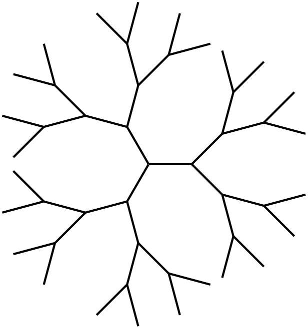
Neretin's group is

1. *simple, [Kapoudjian]*
2. *compactly generated, (Higman–Thompson, Caprace–De Medts)*
3. *has no lattices. (Bader–Caprace–Gelander–Mozes)*

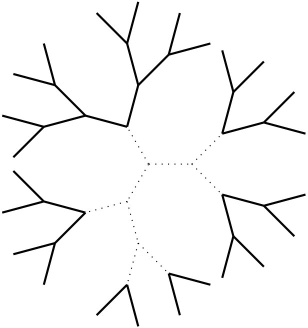
Theorem (Tits)

Tree automorphisms come in three different types:

- **elliptic:** *fixing a vertex,*
- **inversion:** *flipping an edge,*
- **translation:** *translating along a bi-infinite line.*



In aAut($\mathcal{T}$), elements may show "mixed" behaviour - i.e, behave differently over different areas of $\partial\mathcal{T}$



Theorem (Le Boudec – Wesolek)

Tree almost automorphisms come in two different types. The following are equivalent for $g \in \mathrm{aAut}(\mathcal{T})$:

1. $\overline{\langle g \rangle}$ *is compact,*
2. g *can be represented as* $\mathcal{T} \setminus T \to \mathcal{T} \setminus T$,
3. *some power of* g *is an elliptic tree automorphism,*

These are called **elliptic**. *The other elements are called* **translations**.

The translations show **"mixed"** behaviour, and therefore we wish to decompose them into **hyperbolic** and **elliptic** components.

Definition

Let $g \in \mathrm{aAut}(\mathcal{T})$. The **stable area** of g is "where g looks elliptic":

$$\mathrm{St}(g) := \{x \in \partial\mathcal{T} \mid \forall \text{ nbhd } U \text{ of } x \exists B \subset U \, \exists n > 0 \colon g^n(B) = B\}.$$

$\mathrm{St}(g)$ is clopen.

Definition

$g \in \mathrm{aAut}(\mathcal{T})$ is **hyperbolic** if $g|_{\mathrm{St}(g)} = id$.

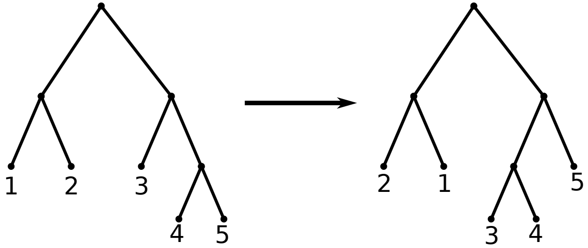
Every $g \in \mathrm{aAut}(\mathcal{T})$ has a unique decomposition $g = g_e g_h = g_h g_e$ with g_e **elliptic**, g_h **hyperbolic** and $\mathrm{supp}(g_e) \cap \mathrm{supp}(g_h) = \emptyset$.

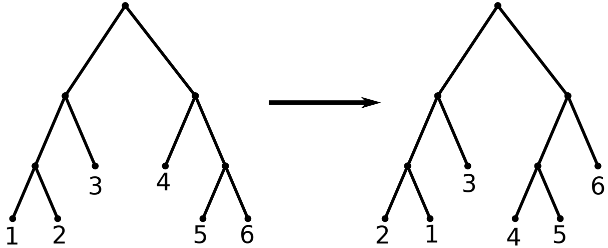
Proposition (Goffer–L.)

Let $g, h \in \mathrm{aAut}(\mathcal{T})$. Then, g is conjugate to h if and only if

1. *g_e is conjugate to h_e,*
2. *g_h is conjugate to h_h,*
3. *either $\mathrm{supp}(g) = \mathrm{supp}(h) = \partial\mathcal{T}$ or $\mathrm{supp}(g), \mathrm{supp}(h) \neq \partial\mathcal{T}$.*

Example





Theorem (Gawron – Nekrashevych – Sushchansky)

In Aut($\mathcal{T}$)*:*

- *Two translations are conjugate if and only if they have the same translation length.*
  In particular, their conjugacy class is open.
- *Two elliptic elements, or two inversions, are conjugate if and only if the labelled trees* $\langle g\rangle\backslash\mathcal{T}$ *with label* $\ell([v]) = |[v]|$ *are isomorphic as labeled trees.*
  This labeled tree is called **orbital type***.*

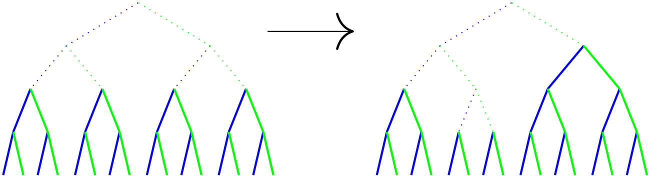
For **elliptic elements**: we generalize conjugacy in $\mathrm{Aut}(\mathcal{T})$

Theorem (Goffer – L.)

Two elliptic elements in $\mathrm{aAut}(\mathcal{T})$ *are conjugate if and only if their* **orbital types** $\langle g\rangle\backslash(\mathcal{T}\setminus T)$ *are isomorphic as labeled forests, outside a finite set.*

As for **hyperbolic elements**: we generalize conjugacy in **Thompson's** V.

Endow $\mathcal{T}$ with a **plane order**: For every vertex, a total order on its children.

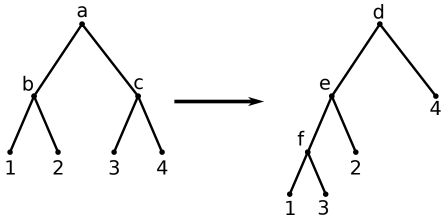


Definition

The **Higman–Thompson group** V is

$$V := \{g \in \mathrm{aAut}(\mathcal{T}) \mid \text{ a representative of } g \text{ preserves the plane order}\}.$$

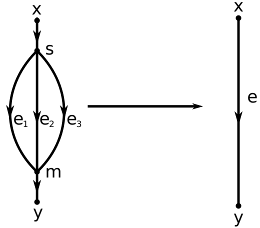
Example



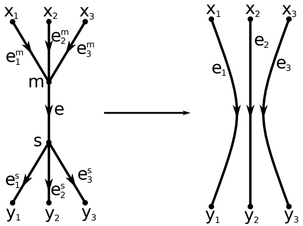
Belk and Matucci give an algorithm to determine conjugacy in V. Given a tree pair φ representing an element $g \in V$, they successively perform on φ three types of reductions, to obtain a finite diagram called the **reduced BM-diagram**.

Theorem (Belk–Matucci)

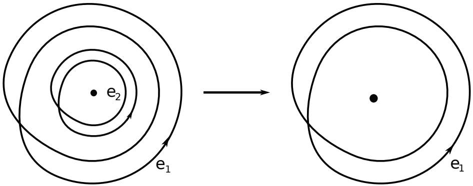
The BM-diagram fully determines the conjugacy class of g.



(a) Type I



(b) Type II



(c) Type III

We denote leaves of a finite tree T by $\mathcal{L}T$.

Theorem (Goffer–L.)

Let $g, h \in \mathrm{aAut}(\mathcal{T})$ be hyperbolic elements such that there exist revealing pairs $\varphi, \psi \colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2$ with $g|_{\mathcal{L}T_1} = h|_{\mathcal{L}T_1}$. Then, g and h are conjugate.

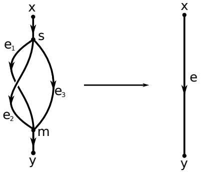
Namely, a hyperbolic $\mathrm{aAut}(\mathcal{T})$ element is conjugate to every sufficiently close Higman-Thompson element.

Corollary (Goffer–L.)

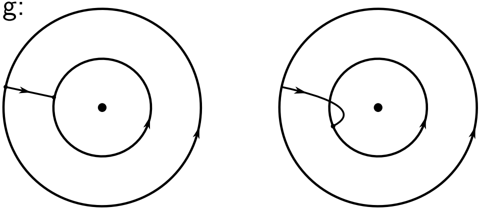
The conjugacy class of a hyperbolic element g is closed. It is open if and only if $\mathrm{supp}(g) = \partial\mathcal{T}$.

Difference between conjugacy in V and hyperbolics in $\mathrm{aAut}(\mathcal{T})$:

- In $\mathrm{aAut}(\mathcal{T})$, we start with a revealing pair.
- If no Type II can be done anymore, we can also do Type I*:



- Forget about the plane order, there is no difference between the following:



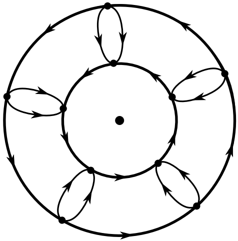
Call the result the ***-reduced BM-diagram** for g.

Theorem (Goffer–L.)

*Two hyperbolic tree almost automorphisms are conjugate if and only if they have isomorphic *-reduced BM-diagrams.*

Corollary

A hyperbolic element is conjugate to a translation in $\mathrm{Aut}(\mathcal{T})$ *if and only if its *-reduced BM-diagram looks as follows.*



Conclusion

"Algorithm" to determine the conjugacy class:

1. Check whether $\text{supp}(g) = \partial\mathcal{T}$
2. Calculate the decomposition $g = g_e g_h$
3. Form the (almost) orbital type of g_e
4. Find a revealing pair for g_h
5. Determine the *-reduced BM-diagram of this revealing pair

Questions?