Boundary maps, germs and quasi-regular representations

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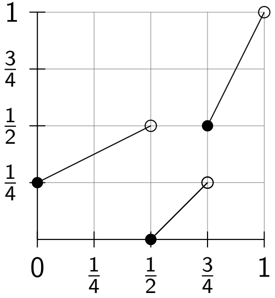
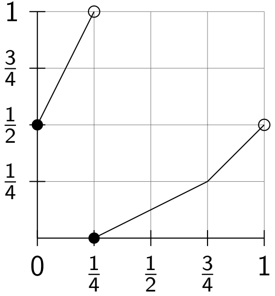
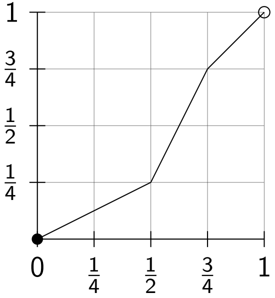
The set of *dyadic rationals* is $\mathbb{Z}[1/2] := \{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$.
Thompson's group V is the group of bijections g on $[0,1)$ for which there exist $t_0, \ldots, t_k \in \mathbb{Z}[1/2]$ and $n_0, \ldots, n_k \in \mathbb{Z}$ such that $0 = t_0 < \cdots < t_k = 1$ and $g|_{[t_i, t_{i+1})}$ is linear with derivative 2^{n_i}, for $i = 0, \ldots, k-1$.
The following is an example of an element of V:



Thompson's group T is the subgroup of V consisting of bijections which, identifying $[0, 1)$ with S^1, are homeomorphisms on S^1.
Example:



Thompson's group F is the subgroup of T consisting of bijections which are homeomorphisms on $[0, 1)$. Example:



Note that $F = \{g \in T : g(0) = 0\}$.
Famous open problem: Is F amenable?

Given a group Γ and $\pi\colon \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation, let $C^*_\pi(\Gamma) = \overline{\mathrm{span}}\{\pi(g) : g \in \Gamma\}$.
If σ is another unitary representation of Γ, we say that π is *weakly contained* in σ ($\pi \prec \sigma$) if $\|\pi(a)\| \leq \|\sigma(a)\|$ for every $a \in \mathbb{C}\Gamma$. This is equivalent to the existence of a $*$-homomorphism $C^*_\sigma(\Gamma) \to C^*_\pi(\Gamma)$ such that $\sigma(g) \mapsto \pi(g)$ for every g.
We say that π is *weakly equivalent to* σ if $\pi \prec \sigma$ and $\sigma \prec \pi$ ($\pi \sim \sigma$).
Given a unitary representation π, we have that $C^*_\pi(\Gamma)$ is simple iff for every σ such that $\sigma \prec \pi$, it holds that $\sigma \sim \pi$.
That is, $C^*_\pi(\Gamma)$ is simple iff π is "weakly irreducible".

Example (Quasi-regular representation)

Let Λ be a subgroup of a group Γ. Denote by $\lambda_{\Gamma/\Lambda}$ the unitary representation of Γ defined by

$$\lambda_{\Gamma/\Lambda}(g)\colon \ell^2(\Gamma/\Lambda) \to \ell^2(\Gamma/\Lambda)$$
$$\delta_{h\Lambda} \mapsto \delta_{gh\Lambda}$$

If $\Lambda = \{e\}$, this is the usual left regular representation λ_Γ. Given groups $\Lambda_1 \leq \Lambda_2 \leq \Gamma$, we have that $\lambda_{\Gamma/\Lambda_2} \prec \lambda_{\Gamma/\Lambda_1}$ iff Λ_1 is co-amenable in Λ_2, in the sense that the action $\Lambda_2 \curvearrowright \Lambda_2/\Lambda_1$ admits invariant mean. If we take $\Lambda_1 = \{e\}$, we conclude that Λ_2 is amenable iff $\lambda_{\Gamma/\Lambda_2} \prec \lambda_\Gamma$.

A *trace* on a unital C^*-algebra A is a unital positive linear functional $\phi\colon A \to \mathbb{C}$ such that $\phi(ab) = \phi(ba)$ for every $a, b \in A$. We have that $C^*_{\lambda_\Gamma}(\Gamma)$ always admits a trace given by $\tau(a) = \langle a\delta_e, \delta_e\rangle$. Moreover, this trace is *faithful* (in the sense that $\tau(a^*a) = 0 \implies a = 0$). This is called the *canonical trace*.

For quasi-regular representations, the situation is much different:

Example (Haagerup, Olesen)

Let $X := \mathbb{Z}[1/2] \cap [0,1)$. Then X is V-invariant. Let $\pi\colon V \to B(\ell^2(X))$ be given by $\pi(g)\delta_x = \delta_{gx}$. Then $C^*_{\pi|_F}(F) \subsetneq C^*_{\pi|_T}(T) \subsetneq C^*_\pi(V) = \mathcal{O}_2$ (in particular, $C^*_\pi(V)$ is simple and admits no traces).
Since the action of T on X is transitive, we have that $\pi|_T = \lambda_{T/F}$ (and likewise π is a quasi-regular representation of V).

Recently, there has been a lot of progress in understanding when $C^*_{\lambda_\Gamma}(\Gamma)$ is simple and when it admits a unique trace, although it can be difficult to determine these properties for individual examples. For example, it was shown by (Haagerup, Olesen) and (Le Boudec, Matte Bon) that Thompson's group F is non-amenable iff $C^*_{\lambda_T}(T)$ is simple.
Our initial goal was to give a conceptual explanation for why $C^*_\pi(V)$ is simple and has no traces, and then decide these properties for $C^*_{\lambda_{T/F}}(T)$. We were able to do this for certain unitary rerpesentations coming from dynamical systems.

Boundary actions

Let X be a compact Hausdorff space and $\Gamma \curvearrowright X$.

1. Given $g \in \Gamma$, let $\mathrm{Fix}_g = \{x \in X : gx = x\}$;
2. Given $x \in X$, let $\Gamma_x = \{g \in \Gamma : gx = x\}$ and

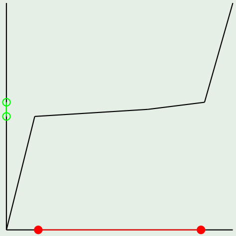
$$\Gamma_x^0 := \{g \in \Gamma : \ g \text{ fixes an open neighborhood of } x\}.$$

One can show that $\Gamma_x^0 \trianglelefteq \Gamma_x$. The action is:

1. minimal if, $\forall x \in X$, $\overline{\Gamma x} = X$;
2. faithful if, $\forall g \in \Gamma \setminus \{e\}$, $\mathrm{Fix}_g \subsetneq X$;
3. topologically free if, $\forall g \in \Gamma \setminus \{e\}$, $\operatorname{int} \mathrm{Fix}_g = \emptyset$;
4. strongly proximal if, given $\mu, \nu \in \mathrm{Prob}(X)$, there is a net $(g_i) \subset \Gamma$ such that $\lim g_i\mu = \lim g_i\nu$.
5. a boundary if it is minimal and strongly proximal (in this case, we say that X is a Γ-boundary);
6. an extreme boundary if, given $C \subsetneq X$ closed and $U \subset X$ non-empty and open, there is $g \in \Gamma$ such that $g(C) \subset U$. It was shown by Glasner that extreme boundary implies boundary.

Example (Le Boudec, Matte Bon)

The action of T on S^1 is an extreme boundary action.



Given a group Γ, there is a Γ-boundary, denoted by $\partial_F \Gamma$ and called the *Furstenberg boundary* of Γ, which is "universal", in the sense that, given another Γ-boundary X, there is a Γ-equivariant continuous map $\partial_F \Gamma \to X$.

Given C^*-algebras A and B, we say a linear map $\phi\colon A \to B$ is completely positive if for every $n \in \mathbb{N}$ the extension $\phi^{(n)}\colon M_n(A) \to M_n(B)$ is positive.

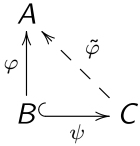
Example:

1. Any $*$-homomorphism is completely positive.
2. If A is unital and $\phi\colon A \to B$ is linear positve, and either A or B are abelian, then ϕ is completely positive.

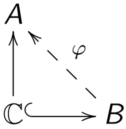
Given a group Γ, a Γ-*C^*-algebra* A is a unital C^*-algebra endowed with an action $\Gamma \curvearrowright A$ by automorphisms.

A Γ-*map* between Γ-C^*-algebras A and B is a Γ-equivariant unital and completely positive map $\phi\colon A \to B$.

We say A is Γ-*injective* if, given a Γ-equivariant injective unital $*$-homomorphism $\psi\colon B \to C$ and a Γ-map $\varphi\colon B \to A$, there is a Γ-map $\tilde{\varphi}\colon C \to A$ such that $\tilde{\varphi} \circ \psi = \varphi$:



Notice that, if A is Γ-injective, given another Γ-C^*-algebra B, there is at least one Γ-map $\varphi\colon B \to A$:



Theorem (Kalantar, Kennedy)

Given a group Γ,

1. $C(\partial_F\Gamma)$ *is* Γ*-injective;*
2. *The only* Γ*-map* $\psi\colon C(\partial_F\Gamma) \to C(\partial_F\Gamma)$ *is* $\mathrm{Id}_{C(\partial_F\Gamma)}$ *(rigidity).*

Corollary (Kalantar, Kennedy)

Let X *be a* Γ*-boundary. There is a unique* Γ*-equivariant map* $\mathfrak{b}_X\colon \partial_F\Gamma \to X$ *and a unique* Γ*-map* $\psi\colon C(X) \to C(\partial_F\Gamma)$. *Moreover,* ψ *is given by* $\psi(f) = f \circ \mathfrak{b}_X$, *for* $f \in C(X)$.

Applying $\partial_F\Gamma$ for investigating traces

Given a unitary representation $\pi\colon \Gamma \to B(\mathcal{H}_\pi)$, we have that both $C^*_\pi(\Gamma)$ and $B(\mathcal{H}_\pi)$ can be seen as Γ-C^*-algebras with action given by $g.a := \pi(g)a\pi(g^{-1})$.
A trace $\tau\colon C^*_\pi(\Gamma) \to \mathbb{C}$ is a Γ-map, since $\tau(\pi(g)a\pi(g^{-1})) = \tau(a)$.
Furthermore, τ can be seen as taking values into $C(\partial_F\Gamma)$:

$$\begin{aligned} C^*_\pi(\Gamma) &\to C(\partial_F\Gamma) \\ a &\mapsto \tau(a)1_{C(\partial_F\Gamma)} \end{aligned}$$

Conversely, any Γ-map $\psi\colon C^*_\pi(\Gamma) \to C(\partial_F\Gamma)$ whose image consists of constant functions arises from a trace.

Applying $\partial_F\Gamma$ for simplicity

Fix π a unitary representation of Γ.
Suppose there is a unique Γ-map $\psi\colon C^*_\pi(\Gamma) \to C(\partial_F\Gamma)$ which is, moreover, faithful.
Given a surjective $*$-homomorphism $\rho\colon C^*_\pi(\Gamma) \to B$, we have that B can also be seen as a Γ-C^*-algebra, via $g.b = \rho(\pi(g))b\rho(\pi(g^{-1}))$ and, in this way, ρ is a Γ-map.
By Γ-injectivity, there is a Γ-map $\phi\colon B \to C(\partial_F\Gamma)$:

$$C^*_\pi(\Gamma) \xrightarrow[\rho]{} B \xrightarrow[\phi]{} C(\partial_F\Gamma)$$

Then $\phi \circ \rho = \psi$. Since ψ is faithful, we conclude that ρ is injective.
Therefore, $C^*_\pi(\Gamma)$ is simple.
Given a Γ-C^*-algebra A, a *boundary map* is a Γ-map $\psi\colon A \to C(\partial_F\Gamma)$.

Theorem (Kennedy)

$C^*_{\lambda_\Gamma}(\Gamma)$ is simple iff $C^*_{\lambda_\Gamma}(\Gamma)$ admits a unique boundary map.

Definition

Let $\Gamma \curvearrowright X$, where X is a compact space. A *groupoid representation* of (Γ, X) is a pair (π, ρ) such that:

1. $\pi\colon \Gamma \to B(\mathcal{H}_\pi)$ is a unitary representation;
2. $\rho\colon C(X) \to B(\mathcal{H}_\pi)$ is a Γ-equivariant unital $*$-homomorphism;
3. $\pi(g)\rho(f) = \rho(f), \quad \forall g \in \Gamma, f \in C(X)$ with $\operatorname{supp} f \subset \operatorname{int} \operatorname{Fix}_g$.

In this case, we also say that π is a groupoid representation of Γ (relative to X).

Given an action $\Gamma \curvearrowright X$, one can associate to it an étale (not necessarily Hausdorff) groupoid G (*groupoid of germs*). The term "groupoid representation" from the definition above comes from the fact that these representations are in one-to-one correspondence with representations of $C^*(G)$.

Proposition

Let $\Gamma \curvearrowright X$, where X is a compact space. The following are groupoid representations:

(i) *Fix $x \in X$ and $H \leq \Gamma$ is such that $\Gamma_x^0 \leq H \leq \Gamma_x$. The pair $(\lambda_{\Gamma/H}, \mathcal{P}_x)$ is a groupoid representation, where $\mathcal{P}_x\colon C(X) \to B(\ell^2(\Gamma/H))$ (Poisson map) is given by:*

$$\begin{aligned} \mathcal{P}_x(f)\colon \ell^2(\Gamma/H) &\to \ell^2(\Gamma/H) \\ \delta_{gH} &\mapsto f(gx)\delta_{gH}. \end{aligned}$$

(ii) *The pair (κ_ν, ρ), where ν is a σ-finite quasi-invariant measure on X, κ_ν is the Koopman representation of Γ on $L^2(X, \nu)$ and $\rho\colon C(X) \to B(L^2(X, \nu))$ is the representation by multiplication operators.*

Theorem

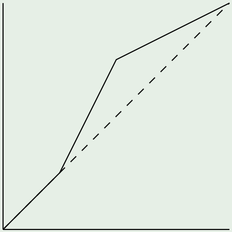
Let X be a Γ-boundary and π a groupoid representation of Γ. Then there is a unique Γ-map $\psi\colon C^*_\pi(\Gamma) \to C(\partial_F \Gamma)$. Furthermore, $\psi(\pi(g)) = 1_{\overline{\mathfrak{b}_X^{-1}(\operatorname{int}\operatorname{Fix}_g)}}$ for all $g \in \Gamma$.

Corollary

Let X be a faithful Γ-boundary and π a groupoid representation of Γ. Then $C^*_\pi(\Gamma)$ admits a trace if and only if X is topologically free.

Example

Since we defined T as a group of homeomorphisms on S^1, it is clear that $T \curvearrowright S^1$ is fatifhul.
On the other hand, the action is not topologically free:



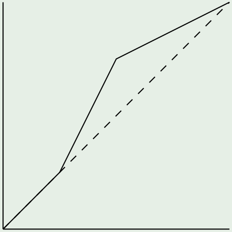
Recall that $F = \{g \in T : g(1) = 1\}$ (i.e., $F = T_1$). Therefore, $C^*_{\lambda_{T/F}}(T)$ does not admit traces.

Given a group Γ, let $\mathrm{Sub}(\Gamma)$ be the set of subgroups of Γ endowed with the Chabauty topology (coming from $\mathrm{Sub}(\Gamma) \subset \{0,1\}^{\Gamma}$). Given a group Γ acting on a compact space X, let $\mathrm{Stab}^0 \colon X \to \mathrm{Sub}(\Gamma)$ be the map $x \mapsto \Gamma_x^0$. Denote by X_0^0 the points on which Stab^0 is continuous. Then:

1. $X_0^0 = \left(\bigcup_{g \in \Gamma} \partial \,\mathrm{int}\, \mathrm{Fix}_g \right)^{c}$;
2. if Γ is countable, then X_0^0 is dense in X;
3. $X_0^0 = X \iff$ the groupoids of germs of the action is Hausdorff.

Example

Consider $T \curvearrowright S^1$ and let $g \in T$ be as follows:



Note that $1 \in \partial \operatorname{int} \operatorname{Fix}_g$. It is easy to see that $(S^1)^0_0 = S^1 \setminus \{e^{2\pi i\theta} : \theta \in \mathbb{Z}[1/2]\}$.

Corollary

Let X be a Γ-boundary and $x \in X$. If $x \in X^0_0$, then $C^*_{\lambda_{\Gamma/\Gamma^0_x}}(\Gamma)$ is simple.

Our next goal is to show that this result may fail if $x \notin X^0_0$.

Let Γ be a group acting on a set X. Given $g \in \Gamma$, let $\operatorname{supp} g := \{x \in X : gx \neq x\}$.
The following fact is behind the proof by Haagerup and Olesen that $\lambda_T \not\prec \lambda_{T/F}$.

Proposition

Let Γ be a group acting on a set X and $\pi_X \colon \Gamma \to B(\ell^2(X))$ the associated unitary representation. Given $g, h \in \Gamma$, we have that

$$\operatorname{supp} g \cap \operatorname{supp} h = \emptyset \iff (1 - \pi_X(g))(1 - \pi_X(h)) = 0.$$

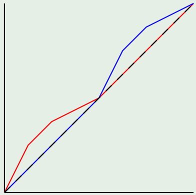
This is a useful tool for studying weak containment in the following way: Suppose we have two actions on sets $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$. If there are $g, h \in \Gamma$ such that
$\operatorname{supp}_X g \cap \operatorname{supp}_X h = \emptyset \neq \operatorname{supp}_Y g \cap \operatorname{supp}_Y h$, then $\pi_Y \not\prec \pi_X$.

Example

Consider again $T \curvearrowright S^1$. One can show that $T_1^0 = [F, F]$. By amenability, $\lambda_{T/F} \prec \lambda_{T/[F,F]}$. Let $X := \mathbb{Z}[1/2] \cap [0, 1)$. Recall $T \curvearrowright X$ and $T/F = X$. Also T acts on $X \times \mathbb{Z} \times \mathbb{Z}$ by

$$g(x, m, n) = (g(x), m + \log_2 g'_-(x), n + \log_2 g'_+(x)).$$

Moreover, $T/[F, F] = X \times \mathbb{Z} \times \mathbb{Z}$.
Consider the following elements g,$h$$\in T$:



Then g and h have disjoint supports with respect to $T \curvearrowright X$, but $(1/2, 0, 0) \in \operatorname{supp} g \cap \operatorname{supp} h$ (with respect to $T \curvearrowright X \times \mathbb{Z} \times \mathbb{Z}$). Therefore, $\lambda_{T/[F,F]} \not\prec \lambda_{T/F}$ and $C^*_{\lambda_{T/[F,F]}}(T)$ is not simple, even though it admits a unique boundary map.

Corollary

Let X be a Γ-boundary. Given $x \in X$, we have that $C^*_{\lambda_{\Gamma/\Gamma_x}}(\Gamma)$ is simple iff $\frac{\Gamma_x}{\Gamma_x^0}$ is amenable.

Example

Consider again $T \curvearrowright S^1$. Then $\frac{T_1}{T_1^0} = \frac{F}{[F,F]} = \mathbb{Z}^2$. Hence, $C^*_{\lambda_{T/F}}(T)$ is simple.

Example

Let $\mathbb{P}^2(\mathbb{R}) := \frac{\mathbb{R}^3 \setminus \{0\}}{\{v \sim \lambda v : v \in \mathbb{R}^3 \setminus \{0\}, \lambda \in \mathbb{R} \setminus \{0\}\}}$. Then $\Gamma := SL_3(\mathbb{Z}) \curvearrowright \mathbb{P}^2(\mathbb{R})$ is a topologically free boundary action (Furstenberg).

Let $x = [(1,0,0)]$. Since the action is topologically free, we have $\Gamma_x^0 = \{e\}$. On the other hand, for any $B \in SL_2(\mathbb{Z})$, we have that

$$\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \in \Gamma_x.$$

Therefore, $SL_2(\mathbb{Z}) \leq \Gamma_x$, hence Γ_x is not amenable. Therefore, $C^*_{\Gamma/\Gamma_x}(\Gamma)$ is not simple, although there is a unique Γ-map $C^*_{\lambda_{\Gamma/\Gamma_x}}(\Gamma) \to C(\partial_F \Gamma)$.

Given a compact Γ-space X, let $\partial(\Gamma, X)$ be the spectrum of the Γ-*injective envelope* of $C(X)$. In particular, there is a Γ-equivariant continuous map $\mathfrak{b}_X \colon \partial(\Gamma, X) \to X$.
For example, $\partial(\Gamma, \{*\}) = \partial_F\Gamma$.

Theorem (Kawabe)

Let X be a minimal Γ-space. TFAE:

(i) *The action $\Gamma \curvearrowright \partial(\Gamma, X)$ is topologically free;*

(ii) *There is a unique Γ-map $\psi \colon C(X) \rtimes_r \Gamma \to C(\partial(\Gamma, X))$ such that $\psi|_{C(X)} = \mathrm{Id}_{C(X)}$;*

(iii) *$C(X) \rtimes_r \Gamma$ is simple;*

(iv) *There is $x \in X$ such that for every amenable $\Lambda \leq \Gamma_x$ there is a net $(g_i) \subset \Gamma$ such that $g_i \Lambda g_i^{-1}$ converges to $\{e\}$.*

Proposition

Let X be a minimal Γ-space. Given a Γ-map $\psi\colon C(X) \rtimes \Gamma \to C(\partial(\Gamma, X))$ such that $\psi(f) = f \circ \mathfrak{b}_X$ for $f \in C(X)$, we have that

$$\operatorname{supp} \psi(\pi(g)) \subset \overline{\mathfrak{b}_X^{-1}(\operatorname{int} \operatorname{Fix}_g)}, \quad \forall g \in \Gamma, f \in C(X)$$

Theorem

Let X be a minimal Γ-space. TFAE:

(i) *The action $\Gamma \curvearrowright \partial(\Gamma, X)$ is faithful;*

(ii) *There is a unique Γ-map $\psi\colon C(X) \rtimes_r \Gamma \to C(X)$ such that $\psi|_{C(X)} = \mathrm{Id}_{C(X)}$ and $\psi|_{C_r^*(\Gamma)}$ is a trace;*

(iii) *Given $x \in X$, the stabilizer Γ_x does not contain any non-trivial amenable normal subgroup of Γ;*

(iv) *There is $x \in X$ such that Γ_x^0 does not contain any non-trivial amenable normal subgroup of Γ.*