Almost commuting matrices, cohomology, and dimension

Joint work with Dominic Enders

March 30, 2021

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$\| \ \|$ is the operator norm, that is $\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$.

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(Sizes of matrices can grow!)

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YES. Lin, 1995

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1-dimensional CW-complexes	Yes (Loring 89)

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for all $n \in \mathbb{N}$.

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E.g. in $K_0(\prod M_n(\mathbb{C}))$ there are no infinitesimals.

Work of Eilers, Loring, Pedersen

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If X is a 2-dimensional CW-complex, then this is the only obstruction!

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All infinitesimals in $K_0(C(X))$ are torsion $\Rightarrow$

all infinitesimals are killed $\Rightarrow$ $C(X)$ is matricially stable

From K-theory to cohomology

Corollary (Eilers-Loring-Pedersen '89)

Let X be a 2-dimensional CW-complex. If all infinitesimals in $K_0(C(X))$ are torsion, then $C(X)$ is matricially stable.

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Chern character:
$K_0(C(X)) \to H^0(X, \mathbb{Q}) \oplus H^2(X, \mathbb{Q}) \oplus H^4(X, \mathbb{Q}) \oplus \ldots$.

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Proposition

Let X be a 2-dimensional CW-complex. If $H^2(X; \mathbb{Q}) = 0$, then $C(X)$ is matricially stable.

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2) $\dim X \geq 3 \Rightarrow C(X)$ is not matricially stable.

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Guess: iff $\dim X \leq 2$ and $H^2(X, \mathbb{Q}) = 0$.

2 things to prove:

1) For X of dim ≤ 2, $C(X)$ is matricially stable $\Leftrightarrow H^2(X; \mathbb{Q}) = 0$,

2) $\dim X \geq 3 \Rightarrow C(X)$ is not matricially stable.

Theorem

For X of dim ≤ 2, $C(X)$ is matricially stable if and only if $H^2(X; \mathbb{Q}) = 0$.

Strategy for proving that $\dim X$ cannot be ≥ 3

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(if X is a CW-complex, just embed S^2)

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Theorem

Suppose $n < \dim X < \infty$. Then there exists a closed subset A of X such that $\dim A = n$ and $H^n(A,\mathbb{Q}) \neq 0$.

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Why the assumption $\dim X < \infty$?

Matricial stability of $C(X)$

Question: For which compact metric X is $C(X)$ matricially stable?

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Suppose $\dim X < \infty$. Then $C(X)$ is matricially stable if and only if $\dim(X) \leq 2$ and $H^2(X;\mathbb{Q}) = 0$.

(In terms of generators and relations, this means that we solve the questions for *finite* families of matrices (almost) satisfying possibly infinitely many relations)

Some applications

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Theorem

Let X be a compact subset of the plane. The following are equivalent:
(i) Any normal element of the Calkin algebra with spectrum contained in X lifts to a normal operator;

(ii) $\dim X \leq 1$ and $H^1(X) = 0$.

Some applications

Theorem

The following are equivalent:

1) Any pointwise limit of liftable $*$-homomorphisms from $C(X)$ to $Q(H)$ is liftable itself;

2) $C(X)$ has the following lifting property:

$$\begin{tikzcd} & & B(H) \arrow[d, "\pi"] \\ C(X) \arrow[rru, dotted] \arrow[r, "\phi"'] & \prod M_{d_n} / \bigoplus M_{d_n} \arrow[r, "j"'] & Q(H) \end{tikzcd}$$

Some applications

2) **Lifting homomorphisms from the Calkin algebra**

$$
\begin{tikzcd}
 & B(H) \arrow[d] \\
C(X) \arrow[r] \arrow[ur, dashed] & B(H)/K(H)
\end{tikzcd}
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Some applications

2) **Lifting homomorphisms from the Calkin algebra**

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For what X are all $*$-homomorphisms from $C(X)$ to the Calkin algebra liftable?

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BDF-theory deals with (lifting of) **injective** $*$-homomorphisms from $C(X)$ to the Calkin algebra.

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Conjecture: The following are equivalent:
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Missing ingredient (Question): Does $\infty > \dim X > n$ imply that there exists a closed subset $Y \subseteq X$ with $\dim Y = n$ and $Hom(H^n(Y), \mathbb{Z}) \neq 0$?

Some applications

3) **Blackadar's l-closedness**

Definition (Blackadar) A C^*-algebra A is *l-closed* (*l-open*) if for any C^*-algebra B and any ideal I in B, the set of liftable $*$-homomorphisms from A to B/I is closed (open) w.r.t. the topology of pointwise convergence in the set $Hom(A, B/I)$.

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B. Blackadar: "It seems reasonable that if X is any absolute neighborhood retract, then $C(X)$ is l-closed".

Theorem

Let X be a CW-complex. If $C(X)$ is l-closed, then $\dim X \leq 3$.

Some applications

4) **Matricial stability for CW-complexes**

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Corollary (Eilers-Loring-Pedersen '89)

Let X be a 2-dimensional CW-complex. If all infinitesimals in $K_0(C(X))$ are torsion, then $C(X)$ is matricially stable.

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Thank you!