From invariant integrals to Haar weights

A. Van Daele

Department of Mathematics
KU Leuven (Belgium)

Content of the talk

- Introduction
- Finite quantum groups
- Algebraic quantum groups
- Locally compact quantum groups
- Conclusions
- References

Introduction

For the operator algebraic approach to quantum groups, the existence of the Haar weights, the analogues of the Haar measures on a locally compact group, is crucial.

A first observation is that, up till now, there is no general theory of locally compact quantum groups where the existence of the Haar weights is a theorem, rather than part of the axioms.

On the other hand, existence results are available for finite quantum groups, compact quantum groups, discrete quantum groups and for the dual of a locally compact quantum group. Moreover Haar weights can be constructed following general ideas for most of the, also more complicated, examples.

Therefore, having no general existence theorem is not a big problem. Still, from a theoretical point of view, it would nice to have such a result.

In this talk, I will discuss some aspects of this. The results are available in the literature, but not so easy accessible, and often with more complicated arguments then necessary.

Finite-dimensional Hopf algebras and duality

Let (A, Δ) be a finite-dimensional Hopf algebra. Denote the dual by (B, Δ) and use $(a, b) \mapsto \langle a, b \rangle$ for the pairing. It is non-degenerate.
We consider A as a left A-module and as a left B-module:

$$(a, x) \mapsto ax \quad \text{and} \quad (b, x) \mapsto bx := \textstyle\sum_{(x)} \langle x_{(2)}, b \rangle x_{(1)}$$

where $a, x \in A$ and $b \in B$. Define V in $B \otimes A$ by $\langle V, a \otimes b \rangle = \langle a, b \rangle$.

Proposition

For all x, x' we have $V(x \otimes x') = \Delta(x)(1 \otimes x')$.

Proof.

Let $V = \sum_i b_i \otimes a_i$. Then $V(x \otimes x') = \sum_i b_i x \otimes a_i x$ and so

$$V(x \otimes x') = \textstyle\sum_{i,(x)} \langle x_{(2)}, b_i \rangle x_{(1)} \otimes a_i x' = \sum_{(x)} x_{(1)} \otimes x_{(2)} x'.$$

□

We have $\Delta(a) = V(a \otimes 1)V^{-1}$ and $(S \otimes \iota)V = V^{-1}$.

Existence of a right integral

Proposition

Let $h : x \mapsto hx$ be a linear map from A to itself satisfying $hb = S^2(b)h$. Then $\psi : a \mapsto tr(ah)$ is a right invariant functional.

Proof.

Write $V = \sum b_i \otimes a_i$. Then for all $a \in A$ we have

$$
\begin{aligned}
(\psi \otimes \iota)(V(a \otimes 1)V^{-1}) &= (\mathrm{tr} \otimes \iota)(V(a \otimes 1)V^{-1}(h \otimes 1)) \\
&= \sum (\mathrm{tr} \otimes \iota)((a \otimes 1)(1 \otimes a_i)V^{-1}(hb_i \otimes 1) \\
&= \sum (\mathrm{tr} \otimes \iota)((a \otimes 1)(1 \otimes a_i)V^{-1}(S^2(b_i)h \otimes 1))
\end{aligned}
$$

Now we have

$$(1 \otimes a_i)V^{-1}(S^2(b_i) \otimes 1) = (S \otimes \iota)(S(b_i) \otimes a_i)V).$$

We get 1 and this proves that $(\psi \otimes \iota)\Delta(a) = \psi(a)1$. □

We need enough such operators h to have that at least one such functional is non-zero.

Proposition

For all linear maps y on A that commute with B we have that $h := x \mapsto yS^{-2}(x)$ will satisfy $hb = S^2(b)h$.

Proof.

$$
\begin{aligned}
S^{-2}(bx) &= \textstyle\sum_{(x)} \langle x_{(2)}, b\rangle S^{-2}(x_{(1)}) \\
&= \textstyle\sum_{(S^{-2}(x))} \langle S^2(S^{-2}(x))_{(2)}, b\rangle (S^{-2}x)_{(1)}) \\
&= \textstyle\sum_{(S^{-2}(x))} \langle (S^{-2}(x))_{(2)}, S^2(b)\rangle (S^{-2}x)_{(1)})
\end{aligned}
$$

and we see that

$$hbx = yS^{-2}(bx) = yS^2(b)S^{-2}(x) = S^2(b)yS^{-2}(x) = S^2(b)hx$$

□

One can now prove that there are elements y so that the corresponding ψ is non-zero. It is also true that a non-zero right invariant functional is automatically faithful.

Uniqueness of integrals

Proposition

Assume that ψ' is a right invariant functional and that ψ is a faithful right invariant functional. Then ψ' is a scalar multiple of ψ.

Proof.

For all $a, a' \in A$ we have

$$\Delta(a')(a \otimes 1) = \sum_{(a)} (\Delta(a' a_{(1)})(1 \otimes S(a_{(2)})).$$

If we apply ψ on the first factor, we obtain

$$(\psi \otimes \iota)(\Delta(a')(a \otimes 1)) = S((\psi \otimes \iota)((a' \otimes 1)\Delta(a)))$$

Now choose a so that $\psi(\,\cdot\, a) = \varepsilon$, the counit. Then we get for the left hand side a'.
Apply ψ' and use that $\psi' \circ S$ is left invariant. Then we get $\psi'(a') = \psi'(S(a))\psi(a')$. □

Existence for other cases

There are various other cases where existence theorems can be proven:

- for compact quantum groups,
- for discrete quantum groups,
- for the dual quantum groups.

The techniques vary and seem to be far away from the ones used to prove the existence of the Haar measure on a locally compact group.

As far as I know, there is no general result proving the existence of Haar weights on a general locally compact quantum group.

The proof that I gave for finite quantum groups might be a basis for a general existence result in the case where a multiplicative unitary (with the right properties) is already available. In fact, we have the candidate for the integral. The missing point is to show that it is non-trivial and densely defined.

The method has been used to find the Haar weights on the examples of locally compact quantum groups, with the muliplicative unitaries constructed by Woronowicz.

Invariance of the Haar weights - Problem

There is a major difficulty proving invariance of the Haar weight on locally compact quantum groups.

In many cases, the invariant integral is first constructed on a dense subalgebra. Next it has to be extended to a weight on the operator algebra (C^*-algebra or von Neumann algebra). Finally one has to show that invariance still holds for this extension.

That this is not obvious is illustrated by the following result.

Proposition

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and M the von Neumann algebra of all bounded linear operators on $\mathcal{H}$. Then there exists normal, semi-finite weights φ_1 and φ_2 on M, with orthogonal support, but equal on a weakly dense *-subalgebra.

I will illustrate the solution to this problem for the extension of the integral on an algebraic quantum group to the generated locally compact quantum group. Observe that it is a problem that has to be solved in other situations as well.

Algebraic quantum groups

We use the following terminology.

Definition

A multiplier Hopf *-algebra with a positive right integral is called an algebraic quantum group.

Recall that a multiplier Hopf algebra is like a Hopf algebra but for algebras without identity. This means e.g. that the coproduct Δ does not map into the algebraic tensor product but has values in the multiplier algebra $M(A \otimes A)$. It is assumed that elements of the form

$$\Delta(a)(1 \otimes a') \quad \text{and} \quad (1 \otimes a')\Delta(a)$$

belong to $A \otimes A$. Here we also assume that A is a *-algebra and that Δ is a *-homomorphism.

A right integral is a linear functional ψ on A satisfying $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ in the multiplier algebra $M(A)$.

Algebraic quantum groups - Duality

Denote by B the set of linear functionals on A of the form $\psi(\,\cdot\, a)$ where $a \in A$.

Theorem

The space B is an algebraic quantum group where the product and the coproduct are adjoint to the coproduct and the product on A. The involution on B is defined by $\langle a, b^* \rangle = \langle S(a)^*, b \rangle^-$.

We use $(a, b) \mapsto \langle a, b \rangle$ on $A \times B$.

Proposition

There is an element V in the multiplier algebra $M(B \otimes A)$ satisfying $\langle V, a \otimes b \rangle = \langle a, b \rangle$ for all a, b.

We need the extension of the pairing between $B \otimes A$ and $A \otimes B$ to $M(B \otimes A) \times A \otimes B$.

As in the finite-dimensional case, V is unitary and $V^* = (S \otimes \iota)V$.

Algebraic quantum groups - Actions

Fix a positive right integral ψ. Denote by $\mathcal{H}$ the Hilbert space completion of A with respect to the scalar product $(x, x') \mapsto \psi(x'^* x)$. We use $x \mapsto \Lambda(x)$ for the canonical embedding of A in $\mathcal{H}$.

Proposition

There is a unitary operator on $\mathcal{H} \otimes \mathcal{H}$:

$$\Lambda(x) \otimes \Lambda(x') \mapsto \sum_{(x)} \Lambda(x_{(1)}) \otimes \Lambda(x_{(2)} x').$$

One has the obvious left action of A on $\Lambda(A)$ given by multiplication. We use $a\Lambda(x) = \Lambda(ax)$. We also have the left action of B on $\Lambda(A)$ written as

$$b\Lambda(x) = \sum_{(x)} \langle x_{(2)}, b \rangle \Lambda x_{(1)}.$$

We combine this to an action of $B \otimes A$ and extend it to the multiplier algebra $M(B \otimes A)$. Then we have that the unitary operator coincides with V on $\Lambda(A) \otimes \Lambda(A)$.

It is not entirely obvious, but as a consequence we get:

Proposition

i) *The algebra A acts with bounded operators on $\mathcal{H}$.*

ii) *The algebra B acts with bounded operators on $\mathcal{H}$.*

Moreover we have *-representations and non-degenerate actions. In particular we have:

Proposition

For all $b \in B$ we have $\langle b\xi, \xi' \rangle = \langle \xi, b^* \xi' \rangle$.

Proof.

Take $x, x' \in A$ and $b \in B$. Then $\langle b\Lambda(x), \Lambda(x') \rangle$ is equal to

$$\begin{aligned} \sum_{(x)} \langle x_{(2)}, b \rangle \psi(x'^* x_{(1)}) &= \sum_{(x')} \langle S^{-1}(x'^*_{(2)}), b \rangle \psi(x'^*_{(1)} x) \\ &= \sum_{(x')} \langle x'_{(2)}, b^* \rangle^- \psi(x'^*_{(1)} x) \\ &= \langle \Lambda(x), b^* \Lambda(x') \rangle. \end{aligned}$$

□

The associated locally compact quantum group

Define M as the von Neumann algebra generated by A on $\mathcal{H}$ and a coproduct Δ on M by $\Delta(x) = V^*(x \otimes 1)V$.

Theorem

The pair (M, Δ) is a locally compact quantum group containing A. The right Haar weight on M is the unique extension of the right integral ψ on A.

In fact, the right Haar weight is easy to define with the following notion from the theory of left Hilbert algebras:

Definition

An element $\eta \in \mathcal{H}$ is called *right bounded* if there is a bounded linear operator $\pi_r(\eta)$ on $\mathcal{H}$ satisfying $\pi_r(\eta)\Lambda(a) = a\eta$ for all $a \in A$.

The right Haar weight

Definition

Define, for $x \in M$ and $x \geq 0$,

$$\overline{\psi}(x) = \sup\{\langle x\eta, \eta\rangle \mid \eta \text{ is right bounded and } \|\pi_r(\eta)\| \leq 1\}.$$

We will give some indications how it is shown that $\overline{\psi}$ is a faithful normal semi-finite weight.

Proposition

The map $\Lambda(a) \mapsto \Lambda(a^*)$ is pre-closed.

We have

$$\langle \Lambda(a^*), \Lambda(a')\rangle = \psi(a'^* a^*) = \psi(a^* \sigma(a'^*)) = \langle \Lambda(\sigma(a'^*)), \Lambda(a)\rangle$$

This is used to prove (using left Hilbert algebra techniques) the following:

Lemma

The space of right bounded elements is dense.

Again we need a property of the theory of left Hilbert algebras:

Lemma

The set of right bounded elements η with $\pi_r(\eta)^* = \pi_r(\eta')$ for some right bounded element η' gives a non-degenerate *-algebra of operators on $\mathcal{H}$.

Proposition

If $a \in A$ then $\overline{\psi}(a^*a) = \psi(a^*a)$.

Proof.

For η right bounded we have

$$\langle a^*a\eta, \eta\rangle = \langle a\eta, a\eta\rangle = \langle \pi_r(\eta)\Lambda(a), \pi_r(\eta)\Lambda(a)\rangle = \langle \pi_r(\eta)^*\pi_r(\eta)\Lambda(a), \Lambda(a)\rangle.$$

□

One needs to prove additivity and then it follows that $\overline{\psi}$ is a faithful normal semi-finite weight on M extending ψ.

Right invariance of ψ

We now prove that this extension is still right invariant.

Theorem

The weight $\overline{\psi}$ is right invariant.

Proof.

For all $a \in A$ and η right bounded and any vector ξ we have

$$\Delta(a)(\eta \otimes \xi) = \sum_{(a)} (\pi_r(\eta) \otimes 1)(\Lambda(a_{(1)}) \otimes a_{(2)}\xi) = (\pi_r(\eta) \otimes 1)V(\Lambda(a) \otimes \xi).$$

By a result from the theory of left Hilbert algebras, this equation will also hold for $x \in M$ satisfying $\overline{\psi}(x^*x) < \infty$. Then, when $\overline{\psi}(x^*x) < \infty$, we have

$$\overline{\psi}((\iota \otimes \langle \cdot \xi, \xi)(\Delta(x^*x)) = \overline{\psi}(x^*x)\langle \xi, \xi \rangle$$

and $\overline{\psi}$ is right invariant. □

Conclusions

- We see that for the passage of algebraic quantum groups to locally compact quantum groups, some of the techniques used in the theory of left Hilbert algebras are needed, but not the full strength of this theory.
- On the other hand, for the development of the general theory of locally compact quantum group, you do need the whole theory.
- To understand locally compact quantum groups, a good start are the algebraic quantum groups. After all these contain already the compact quantum groups and their duals, the discrete quantum groups.
- We see that the proof of the existence of the Haar weights in the different situations uses different techniques. Understanding these different cases may help in the future to find a notion of locally compact quantum groups where the existence of the Haar weights is no longer part of the axioms, but a result, just like in the classical theory.

Some references

- S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987).
- S.L. Woronowicz, *Compact quantum groups.* Quantum symmetries/Symmétries quantiques. Proceedings of the Les Houches summer school 1995, North-Holland, Amsterdam (1998).
- A. Van Daele, *The Haar Measure on a finite quantum group*, Proc. Amer. Math. Soc. 123 (1995).
- A. Van Daele, *Discrete quantum groups*, J. of Alg. 180 (1996).
- J. Kustermans & A. Van Daele, *C^*-algebraic quantum groups arising from algebraic quantum groups*, Int. J. Math. 8 (1997).
- A. Van Daele, *An algebraic framework for group duality*, Adv. in Math. 140 (1998).
- A. Maes & A. Van Daele, *Notes on compact quantum groups*, Nieuw Archief voor Wiskunde, Vierde serie 16 (1998).
- J. Kustermans & S. Vaes, *Locally compact quantum groups* Ann. Sci. Éc. Norm. Sup. 33 (2000).

- A. Van Daele, *The Haar measure on some locally compact quantum groups*. Preprint University of Leuven (2001). Arxiv math.OA/0109004.
- K. De Commer & A. Van Daele, *Multiplier Hopf algebras embedded in C^*-algebraic quantum groups*, Rocky Mountain Journal of Mathematics 40 (2010).
- M. Takesaki *Theory of operator algebras II*, Encyclopedia of Mathematical Sciences 125 (2001), Springer, Heidelberg.
- A. Van Daele, *Locally compact quantum groups. A von Neumann algebra approach*. Preprint University of Leuven (2006), 49p. Arxiv math.OA/0602212.
- A. Van Daele: *Algebraic quantum groups and duality*. Preprint KU Leuven (In preparation).