Quantum Increasing Sequences generate Quantum Permutation Groups

Paweł Józiak

Faculty of Mathematics and Information Technology, Warsaw University of Technology

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Universitetet i Oslo
Quantum groups and their analysis

Plan of the talk

1. Quantum permutation groups and quantum increasing sequences
2. Motivations and the problem
3. The solution: $\langle I_{k,n}^{+} \rangle = S_n^{+}$

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Quantum permutation groups

A *bistochastic matrix* over A is a square matrix $u \in M_n \otimes A$ such that in each row and column the entries add up to $\mathbb{1}$.

$$\sum_{i=1}^{n} u_{i\,j} = \mathbb{1} = \sum_{j=1}^{n} u_{i\,j}$$

Definition

The *quantum permutation group* over n-letter alphabet is a quantum group S_n^+ such that $C^u(S_n^+)$ is the universal C^* of a $n \times n$ bistochastic matrix consisting of projections. This bistochastic matrix is a fundamental corepresentation:

$$\Delta(u_{i\,k}) = \sum_{j=1}^{n} u_{i\,j} \otimes u_{j\,k}$$

Increasing sequences

Fix $k < n \in \mathbb{Z}_+$. The set of length-k, $[n] = \{1, \ldots, n\}$-valued increasing sequences is:

$$I_{k,n} = \left\{ f : [k] \to [n] : f(i) < f(j) \text{ whenever } i < j \right\}$$

Example

Consider the sequence $(2 < 3 < 5 < 6 < 8) \in I_{5,8}$.

$$(1, 2, 3, 5, 8, 7, 4, 6)$$

Folklore/Example/Exercise

Let $b_{k,n} : I_{k,n} \to S_n$ be the above map. Then $\langle b_{k,n}(I_{k,n}) \rangle = S_n$.

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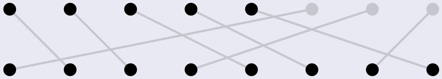

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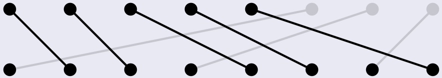

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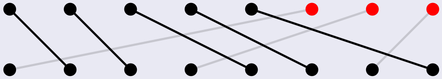

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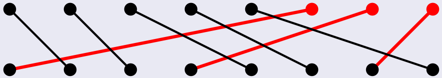

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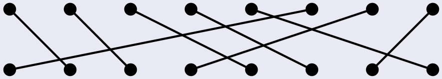

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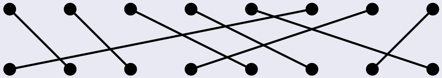

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Increasing sequences – matricial representation

$$I_{k,n} \ni \underline{i} = (i_1 < \ldots < i_k) \mapsto M(\underline{i}) \in M_{n\times k}(\{0,1\})$$

$$M(\underline{i})_{s,t} = \begin{cases} 1 & \text{if } s = i_t \\ 0 & \text{otherwise} \end{cases}$$

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Quantum increasing sequences

Let $k < n \in \mathbb{Z}_+$ and let $C^u(I^+_{k,n})$ be the universal C^*-algebra generated by $p_{i\,j}$, $1 \leq i \leq n$, $1 \leq j \leq k$ subject to the following relations:

1. $p_{i\,j} p^*_{i\,j} = p_{i\,j}$.
2. $\sum_{i=1}^{n} p_{i\,j} = \mathbb{1}$ for each $1 \leq j \leq k$.
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$\beta_{k,n} \colon C^u(S^+_n) \to C^u(I^+_{k,n})$ is given by:

- $u_{i\,j} \mapsto p_{i\,j}$ for $1 \leq i \leq n$, $1 \leq j \leq k$,
- $u_{i\,k+m} \mapsto 0$ for $1 \leq m \leq n-k$ and $i < m$ or $i > m+k$,
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$$u_{m+p\,k+m} \mapsto \sum_{i=0}^{m+p-1} p_{i\,p} - p_{i+1\,p+1},$$

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Motivations

Fix $X_1, \ldots, X_n \in (\mathrm{M}, \tau)$ self-adjoint.

Distributional invariance

Let $\rho \colon \mathbb{C}\langle x_1, \ldots, x_n \rangle \to * - alg(X_1, \ldots, X_n) \subset \mathrm{M}$ be the canonical map $x_i \overset{\rho}{\mapsto} X_i$, let $\tau_\rho = \tau \circ \rho$.

Let $\alpha_n \colon \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}\langle x_1, \ldots, x_n \rangle \otimes \mathcal{O}(S_n^+)$ be the defining action of quantum permutation group: $\alpha_n(x_j) = \sum_{i=1}^{n} x_i \otimes u_{ij}$.

We say that *the distribution of $X_1, \ldots, X_n$ is invariant under quantum permutations* or *is quantum exchangeable* if $(\tau_\rho \otimes \mathrm{id}) \circ \alpha_n = \tau_\rho(\,\cdot\,)\mathbb{1}$.

Example

$X_1, \ldots, X_n \in (\mathrm{M}, \tau)$ *exchangeable* is nothing but:

$$(X_1, \ldots, X_n) \overset{d}{=} (X_{\sigma(1)}, \ldots, X_{\sigma(n)})$$

for every $\sigma \in S_n$. $\underline{A} \overset{d}{=} \underline{B} \iff \forall_{p\colon \text{polynomial}} \tau(p(\underline{A})) = \tau(p(\underline{B}))$

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Let $\rho\colon \mathbb{C}\langle x_1, \ldots, x_n\rangle \to * - alg(X_1, \ldots, X_n) \subset \mathrm{M}$ be the canonical map $x_i \overset{\rho}{\mapsto} X_i$, let $\tau_\rho = \tau \circ \rho$.

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Example

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for every $\sigma \in S_n$. $\underline{A} \overset{d}{=} \underline{B} \iff \forall_{p\colon \text{polynomial}} \tau(p(\underline{A})) = \tau(p(\underline{B}))$

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Theorem (Köstler, Speicher)

Let $X_1, X_2, \dots \in (\mathrm{M}, \tau)$ be an infinite sequence of self-adjoint elements. TFAE

- each initial sub-segment $(X_1, \dots, X_n)$ is quantum exchangeable
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for every $\sigma \in S_n$. $X_1, \dots, X_n \in (\mathrm{M}, \tau)$ is *spreadable* if:

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Let $X_1, X_2, \dots$ be an infinite sequence of real random variables. TFAE

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Motivations III

Definition

$X_1, \dots, X_n \in (\mathrm{M}, \tau)$ is *quantum exchangeable* if:

$$(\tau_\rho \otimes \mathrm{id}) \circ \alpha_n = \tau_\rho(\,\cdot\,)\mathbb{1}$$

Unpacking this amounts to a condition: $\forall_{P \in \mathbb{C}\langle x_1, \dots, x_n \rangle}$

$$(\tau \otimes \mathrm{id})(P(\sum_{i=1}^{n} X_i \otimes u_{i\,1}, \dots, \sum_{i=1}^{n} X_i \otimes u_{i\,n})) = \tau(P(X_1, \dots, X_n))\mathbb{1}$$

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Theorem (S. Curran)

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Question (Skalski, Sołtan)

What is the quantum subgroup of S_n^+ generated by $I_{k,n}^+$?

Hopf image in quantum groups

$I_{k,n} \subset I_{k,n}^+$, thus $S_n = \langle I_{k,n}\rangle \subset \langle I_{k,n}^+\rangle \subset S_n^+$. If $k = 1, n-1$, then $I_{k,n}^+ = I_{k,n}$!

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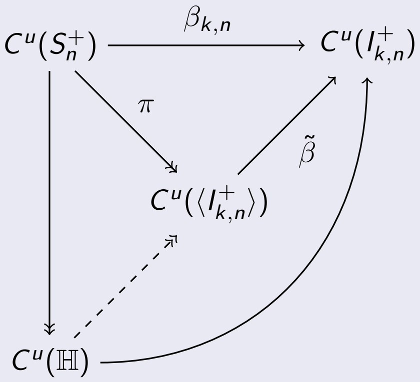

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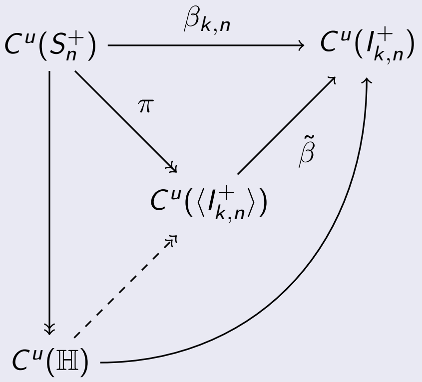

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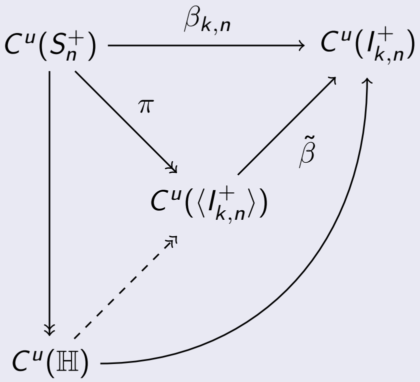

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Plan of the talk

Observation

Let $S_n \subsetneq \mathbb{G}_n \subset S_n^+$. Then $\mathbb{G}_n = S_n^+$, if there is a commuting diagram:

$$\begin{array}{ccc} C^u(S_n^+) & \twoheadrightarrow & C^u(\mathbb{G}_n) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ C^u(S_{n-1}^+) & \twoheadrightarrow & C^u(\mathbb{G}_{n-1}) \end{array}$$

Idea

- Banica-Bichon: if $S_4 \subsetneq \mathbb{G} \subset S_4^+$, then $\mathbb{G} = S_4^+$.
- Brannan-Chirvasitu-Freslon: $\langle S_{n-1}^+, S_n \rangle = S_n^+$ (uses Banica '18 at $n = 5$).
- Induction: at $n = 4$ clear from Banica-Bichon. Commuting diagram $\implies S_{n-1}^+ = \mathbb{G}_{n-1} \subset \mathbb{G}_n$, use B-C-F.

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Corollary

$S_n \subsetneq \langle I_{k,n}^+ \rangle \subset S_n^+$. Then $\langle I_{k,n}^+ \rangle = S_n^+$ if there is a commuting diagram:

$$\begin{array}{ccc} C^u(S_n^+) & \xrightarrow{\beta_{k,n}} & C^u(I_{k,n}^+) \\ \Big\downarrow q_n & & \Big\downarrow \tilde{\eta}_{k,n} \\ C^u(S_{n-1}^+) & \xrightarrow[\beta_{k,n-1}]{} & C^u(I_{k,n-1}^+) \end{array} \qquad \begin{array}{ccc} C^u(S_n^+) & \xrightarrow{\beta_{k,n}} & C^u(I_{k,n}^+) \\ \Big\downarrow \bar{q}_n & & \Big\downarrow \dot{\eta}_{k,n} \\ C^u(S_{n-1}^+) & \xrightarrow[\beta_{k-1,n-1}]{} & C^u(I_{k-1,n-1}^+) \end{array}$$

Lemma

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Idea

- q_n and $\tilde{\eta}_{k,n}$ are induced by the inclusion $[n-1] \cong \{1, \ldots, n-1\} \subset [n]$
- $\bar{q}_n$ and $\dot{\eta}_{k,n}$ is induced by the inclusion $[n-1] \cong \{2, \ldots, n\} \subset [n]$ plus juxtaposition of 1 in front:

- hence the above diagrams for classical versions.

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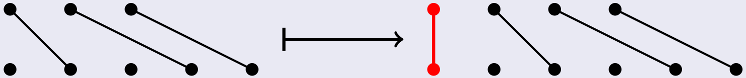
- q_n and $\tilde{\eta}_{k,n}$ are induced by the inclusion $[n-1] \cong \{1, \ldots, n-1\} \subset [n]$
- $\bar{q}_n$ and $\dot{\eta}_{k,n}$ is induced by the inclusion $[n-1] \cong \{2, \ldots, n\} \subset [n]$ plus juxtaposition of 1 in front:



- hence the above diagrams for classical versions.

Lemma

Let $S_n \subsetneq \langle \mathbb{X}_n \rangle \subset S_n^+$. Then $\langle I_{k,n}^+ \rangle = S_n^+$ if there is a commuting diagram:

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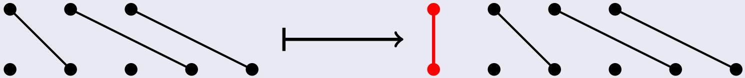

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- the above diagrams holds for classical versions.
- all maps are *-homomorphisms, check commutativity on generators.
- the abelianization maps $C^u(S_n^+) \to C(S_n)$ and $C^u(I_{k,n}^+) \to C(I_{k,n})$ is injective on span of generators and $\mathbb{1}$
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Concluding remarks

We know that

- $\langle I_{k,n}^{+} \rangle = S_n^{+}$ for all $2 \leq k \leq n-2$,
- if some state is invariant wrt generating quantum subset, it is invariant wrt the quantum group it generates
- distributional symmetry wrt S_n^{+} holds iff distributional symmetry wrt $I_{k,n}^{+}$ holds

BUT

(Quantum) spreadability is not just invariance under (quantum) permutations from a generating set $I_{k,n}^{+} \subset S_n^{+}$ via the "completing to permutation" map.

Details: P. Józiak, *Quantum Increasing Sequences generate Quantum Permutation Groups*, arXiv:1904.07721. Accepted in Glasg. Math. J.

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