Spectral theory of weighted Fourier algebras of (locally) compact quantum groups

Hun Hee Lee (이훈희)

Seoul National University

Based on joint works with U. Franz (Besancon), M. Ghandehari (Delaware)/J. Ludwig (Lorraine)/N. Spronk (Waterloo)/L. Turowska (Gothenberg)

August 9, 2019

Overview

Motivations

The case of compact quantum groups and $SU_q(2)$

The case of non-compact (quantum) groups

Fourier algebras on locally compact groups

- The Fourier algebra $A(G)$ of a locally compact group G is
  1. $A(G) = VN(G)_*$, where $VN(G) \subseteq B(L^2(G))$ is the group von Neumann algebra OR
  2. $A(G) = L^1(\widehat{G})$, where $\widehat{G}$ is the dual quantum group OR
  3. $A(G) = \{f * \check{g} : f, g \in L^2(G)\} \subseteq C_0(G)$, where $\check{g}(x) = g(x^{-1})$.
- $A(G)$ is a (non-closed) subalgebra of $C_0(G)$, which is still a commutative Banach algebra under its own norm.
- (**Prop, Eymard '64**) We have a homeomorphism

$$\mathrm{Spec}A(G) \cong G, \ \ \varphi_x \mapsto x,$$

where φ_x is the evaluation at the point x. Here, $\mathrm{Spec}A(G)$ is the Gelfand spectrum, i.e. all (bounded) non-zero multiplicative linear maps from $A(G)$ into $\mathbb{T}$.

Fourier algebras of compact groups and their weighted versions

- G: a compact group.

$$A(G) = \{f \in C(G) : \|f\|_{A(G)} = \sum_{\pi \in \mathrm{Irr}(G)} d_\pi \|\hat{f}(\pi)\|_1 < \infty\},$$

where $\hat{f}(\pi) = \int_G f(x)\pi(x)^* dx \in M_{d_\pi}$ is the Fourier coefficient of f at π.

- For a weight function $w : \mathrm{Irr}(G) \to [1, \infty)$ we can define the weighted space $A(G, w)$ with the norm

$$\|f\|_{A(G,w)} = \sum_{\pi \in \mathrm{Irr}(G)} w(\pi) d_\pi \|\hat{f}(\pi)\|_1.$$

- When w satisfies a "sub-multiplicativity" we have

$$A(G, w) \subseteq A(G) \subseteq C(G),$$

which are commutative Banach algebras under their own norms.

The spectrum of weighted Fourier algebras on compact groups

- (**Q**) G: a compact group, $w : \mathrm{Irr}(G) \to (0, \infty)$ a weight function

$$\mathrm{Spec}A(G, w) = ?$$

- (**A**) For a compact (Lie) group G we have

$$G \subseteq \mathrm{Spec}A(G, w) \subseteq G_{\mathbb{C}},$$

where $G_{\mathbb{C}}$ is the complexification of G.

- (**Why?**) We have $\mathrm{Pol}(G) \subseteq A(G, w)$ densely and (by Chevalley)

$$\mathrm{Spec}\,\mathrm{Pol}(G) \cong G_{\mathbb{C}}.$$

Examples

- (**Ex**) $G = \mathbb{T}$ and $w_\beta : \mathbb{Z} \to (0,\infty),\ n \mapsto \beta^{|n|},\ \beta \geq 1$:

$$\operatorname{Spec} A(\mathbb{T}, w_\beta) \cong \{c \in \mathbb{C} : \frac{1}{\beta} \leq |c| \leq \beta\} \subseteq \mathbb{C}^* = \mathbb{T}_{\mathbb{C}}.$$

  Moreover, we have

$$\bigcup_{\beta \geq 1} \operatorname{Spec} A(\mathbb{T}, w_\beta) \cong \mathbb{C}^* = \mathbb{T}_{\mathbb{C}}.$$

- (**Ex, Ludwig/Spronk/Turowska, '12**) $G = SU(2)$ with $w_\beta : \mathrm{Irr}(SU(2)) = \frac{1}{2}\mathbb{Z}_+ \to (0,\infty),\ s \mapsto \beta^{2s},\ \beta \geq 1.$

$$\operatorname{Spec} A(SU(2), w_\beta) \cong \{U \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} V : U, V \in SU(2),\ \frac{1}{\beta} \leq |c| \leq \beta\}$$

  and

$$\bigcup_{\beta \geq 1} \operatorname{Spec} A(SU(2), w_\beta) \cong SL_2(\mathbb{C}) = SU(2)_{\mathbb{C}}.$$

- (**Rem**) Bounded weights are not interesting!

Quantum extensions

- (**Q**) How about the case of a compact quantum group $\mathbb{G}$?
- (**Preparations**)
    - Discrete dual quantum group $\widehat{\mathbb{G}}$:

$$c_0(\widehat{\mathbb{G}}) = c_0\text{-}\bigoplus_{s\in \mathrm{Irr}(\mathbb{G})} M_{n_s}, \quad \ell^\infty(\widehat{\mathbb{G}}) = \ell^\infty\text{-}\bigoplus_{s\in \mathrm{Irr}(\mathbb{G})} M_{n_s}.$$

    - $c_{00}(\widehat{\mathbb{G}})$: the subalgebra of $c_0(\widehat{\mathbb{G}})$ consisting of finitely supported elements.
    - The right Haar weight $\widehat{h}_R$ on $\ell^\infty(\widehat{\mathbb{G}})$ is given by

$$\widehat{h}_R(X) = \sum_{s\in \mathrm{Irr}(\mathbb{G})} d_s \mathrm{Tr}(X_s Q_s^{-1})$$

      for $X = (X_s)_{s\in \mathrm{Irr}(\mathbb{G})} \in c_{00}(\widehat{\mathbb{G}})$, where Q_s is the deformation matrix for Schur orthogonality and d_s is the quantum dimension.

Preparations: continued

- (**Fourier transform on** $\widehat{\mathbb{G}}$):

$$\mathcal{F} = \mathcal{F}^{\widehat{\mathbb{G}}} : c_{00}(\widehat{\mathbb{G}}) \to \mathrm{Pol}(\mathbb{G}), \;\; X \mapsto (X \cdot \widehat{h}_R \otimes id)\mathbb{U},$$

  where $\mathbb{U} = \oplus_{s \in \mathrm{Irr}(\mathbb{G})} u^{(s)}$ is the multiplicative unitary for a choice of mutually inequivalent irreducible unitary representations of $\mathbb{G}$, $(u^{(s)})_{s \in \mathrm{Irr}(\mathbb{G})}$.

- (**The Fourier algebra** $A(\mathbb{G})$) We define

$$A(\mathbb{G}) = \ell^{\infty}(\widehat{\mathbb{G}})_* = VN(\mathbb{G})_*$$

  equipped with the multiplication $\widehat{\Delta}_*$, which is the preadjoint of $\widehat{\Delta}$, the canonical co-multiplication on $\ell^{\infty}(\widehat{\mathbb{G}})$.

Preparations: continued 2

- We have a natural embedding $c_{00}(\widehat{\mathbb{G}}) \hookrightarrow A(\mathbb{G}),\ X \mapsto X \cdot \widehat{h}_R$, which allows to extend the Fourier transform $\mathcal{F}$ to $A(\mathbb{G})$ as follows.

$$\mathcal{F} : A(\mathbb{G}) \to C_r(\mathbb{G}),\ \ \psi \mapsto (\psi \otimes id)\mathbb{U}.$$

- For the element $X \cdot \widehat{h}_R \in A(\mathbb{G})$ with $X = (X_s) \in c_{00}(\widehat{\mathbb{G}})$ we get the concrete norm formula as follows.

$$||X \cdot \widehat{h}_R||_{A(\mathbb{G})} = \sum_{s \in \mathrm{Irr}(\mathbb{G})} d_s \cdot ||X_s Q_s^{-1}||_1.$$

Weighted Fourier algebras on compact quantum groups

- (**Def/Prop**) For a weight function $w : \mathrm{Irr}(\mathbb{G}) \to [1, \infty)$ satisfying a "sub-multiplicativity" we define

$$\|X \cdot \widehat{h}_R\|_{A(\mathbb{G},w)} = \sum_{s \in \mathrm{Irr}(\mathbb{G})} w(s) d_s \cdot \|X_s Q_s^{-1}\|_1$$

and we have contractive inclusions of Banach algebras

$$A(\mathbb{G}, w) \subseteq A(\mathbb{G}) \subseteq C_r(\mathbb{G}).$$

Spectral theory for $A(\mathbb{G}, w)$: Scenario 1

- (**Q**) Spec$A(\mathbb{G}, w) =$? Any connection to "complexification"?
- (**A**) We can see the complexification of the maximal classical closed subgroup of $\mathbb{G}$.
- (**Why?**) $\mathrm{Pol}(\mathbb{G}) \subseteq A(\mathbb{G}, w)$ densely and Spec Pol($\mathbb{G}$) is actually the (abstract) complexification of $\widetilde{\mathbb{G}} = \mathrm{Spec}A(\mathbb{G})$, which is the maximal classical closed subgroup of $\mathbb{G}$.
- (**Thm**) Let $w_\beta(s) = \beta^{2s}$, $s \in \frac{1}{2}\mathbb{Z}_+$, then we have

$$\mathrm{Spec}A(SU_q(2), w_\beta) \cong \{\rho \in \mathbb{C}\backslash\{0\} : \frac{1}{\beta} \leq |\rho| \leq \beta\}$$

$$\cong \{V = \begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix} \in M_2(\mathbb{C}) : ||V||_\infty \leq \beta\}.$$

Spectral theory for $A(\mathbb{G}, w)$: Scenario 2

- An immediate limitation of Spec$A(\mathbb{G}, w)$ comes from the fact that the algebra $A(\mathbb{G}, w)$ is non-commutative.
- G: a compact Lie group

$$\text{Spec Pol}(G) \cong G_{\mathbb{C}} \cong \text{Spec } C_0(G_{\mathbb{C}}) \cong \mathrm{sp}\, C_0(G_{\mathbb{C}}),$$

  where $\mathrm{sp}\, C_0(G_{\mathbb{C}})$ is the C^*-algebra spectrum, which is the set of equivalence classes of all irreducible $*$-representation $\pi : C_0(G_{\mathbb{C}}) \to B(H)$ for some Hilbert space H.
- $\pi \in \mathrm{sp}\, C_0(G_{\mathbb{C}})$, $\pi : C_0(G_{\mathbb{C}}) \to B(H)$
  $\Rightarrow \exists x \in G_{\mathbb{C}}$ such that $\pi = \varphi_x : C_0(G_{\mathbb{C}}) \to \mathbb{C}$
  $\Rightarrow \varphi_x : H(G_{\mathbb{C}}) \to \mathbb{C}$, where $H(G_{\mathbb{C}})$ is the algebra of holomorphic functions on $G_{\mathbb{C}}$.
  $\Rightarrow \varphi_x : \text{Pol}(G) \to \mathbb{C}$, a homomorphism since $\text{Pol}(G) \subseteq H(G_{\mathbb{C}})$.

Quantum double

- For a compact quantum group $\mathbb{G}$ we have the quantum double $\mathbb{G} \bowtie \widehat{\mathbb{G}}$ by Podles/Woronowicz.
- The associated C^*-algebra is given by $C_0(\mathbb{G} \bowtie \widehat{\mathbb{G}}) := C(\mathbb{G}) \otimes c_0(\widehat{\mathbb{G}})$ with the co-multiplication

$$\Delta_{\mathbb{C}} = (id \otimes \Sigma_{\mathbb{U}} \otimes id)(\Delta \otimes \widehat{\Delta}),$$

  where $\Sigma_{\mathbb{U}}$ is the $*$-isomorphism given by

$$\Sigma_{\mathbb{U}} : C(\mathbb{G}) \otimes c_0(\widehat{\mathbb{G}}) \to c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}), \quad a \otimes x \mapsto \mathbb{U}(x \otimes a)\mathbb{U}^*.$$

- The left (and right) Haar weight on $C_0(\mathbb{G} \bowtie \widehat{\mathbb{G}})$ is given by $h \otimes \widehat{h}_R$, where h is the Haar state on $C(\mathbb{G})$.

The case of $\mathbb{G} = SU_q(2)$, $0 < q < 1$

- Our choice of complexification $\mathbb{G}_{\mathbb{C}}$ is $\mathbb{G} \bowtie \widehat{\mathbb{G}}$, which we write $SL_q(2, \mathbb{C})$.
- $\mathrm{sp}\, C_0(SL_q(2,\mathbb{C})) \cong \mathrm{sp}\, C(SU_q(2)) \times \mathrm{sp}\, c_0(\widehat{SU_q(2)})$.
- $\mathcal{A}$: the $*$-algebra of all elements affilliated to $C_0(SL_q(2,\mathbb{C}))$
  $\mathcal{A}_{\mathrm{hol}}$: a subalgebra of $\mathcal{A}$ generated by the coefficient "function"s $\alpha, \beta, \gamma, \delta$ of $SL_q(2,\mathbb{C})$
  $Q : \mathcal{A}_{\mathrm{hol}} \to \mathrm{Pol}(SU_q(2))$ a bijective homomorphism given by $Q(\alpha) = a_q$, $Q(\beta) = -qc_q^*$, $Q(\gamma) = c_q$ and $Q(\delta) = a_q^*$, where a_q and c_q are canonical $SU_q(2)$ generators.
- From $\pi : C_0(SL_q(2,\mathbb{C})) \to B(H)$
  $\Rightarrow \pi : \mathcal{A} \to B(H)$, the canonical extension
  $\Rightarrow \varphi = \pi \circ Q^{-1} : \mathrm{Pol}(SU_q(2)) \to \mathcal{A}_{\mathrm{hol}} \subseteq \mathcal{A} \to B(H)$, homomorphism
  $\Rightarrow v \in \prod_{s \in \mathrm{Irr}(\mathbb{G})} (M_{n_s} \otimes B(H))$ associated element.

The case of $\mathbb{G} = SU_q(2)$: continued

- We begin with

$$\pi \in \mathrm{sp}\, C_0(SL_q(2,\mathbb{C})) \mapsto (\pi_c, \pi_d) \in \mathrm{sp}\, C(SU_q(2)) \times \mathrm{sp}\, c_0(\widehat{SU_q(2)})$$

with the associated elements
$v, v_c, v_d \in \prod_{s \in \mathrm{Irr}(\mathbb{G})} (M_{n_s} \otimes B(H))$ respectively.

- (**Prop**) We have $v = v_c v_d$ and v_c is a unitary (no contribution to norm).
- For the above reason we may focus on the case

$$\pi = \pi_d \in \mathrm{sp}\, c_0(\widehat{SU_q(2)}) = \{A_s : s \in \frac{1}{2}\mathbb{Z}_+\},$$

where A_s, $s \in \frac{1}{2}\mathbb{Z}_+$ are irreducible AN_q-matrices by Podles/Woronowicz.

The case of $\mathbb{G} = SU_q(2)$: continued 2

- (**Thm**) Let φ_s be the unital homomorphism associated to A_s. Then, φ_s extends to a bounded map on $A(SU_q(2), w_\beta)$ if and only if $|q|^{-s} \leq \beta$. Moreover, we have

$$\begin{aligned} &\mathrm{sp}\, C_0(SL_q(2,\mathbb{C})) \\ &\cong \mathrm{sp}\, C(SU_q(2)) \times \bigcup_{\beta \geq 1} \{A_s : s \in \frac{1}{2}\mathbb{Z}_+, \varphi_s \text{ is bounded on } A(SU_q(2), w_\beta)\}. \end{aligned}$$

Remarks before the journey to non-compact world

- We only focused on a weight function w defined on $\mathrm{Irr}(\mathbb{G})$, which immediately has some problem for a group like $ax + b$-group, whose unitary dual is essentially (support of the Plancherel measure) is a two-points set, so that the weight functions are automatically bounded, which is not interesting.
- However, there is a canonical way of extending "weight"s from (abelian) subgroups, which can be applied to all Lie groups.
- There is another way of producing weights using Laplacian.

The case of $\mathbb{R}$: a prelude for non-compact cases

- $w : \widehat{\mathbb{R}} \to (0, \infty)$ a weight function. $\mathrm{Spec} A(\mathbb{R}, w) = \mathrm{Spec} L^1(\widehat{\mathbb{R}}, w) = ?$
- For $\varphi \in \mathrm{Spec} A(\mathbb{R}, w)$ we have $\varphi : A(\mathbb{R}, w) \longrightarrow \mathbb{C}$.

$$\begin{array}{ccc} A(\mathbb{R}, w) & \xrightarrow{\varphi} & \mathbb{C} \\ \uparrow \mathcal{F}^{\widehat{\mathbb{R}}} & \nearrow & \\ C_c^\infty(\widehat{\mathbb{R}}) & & \end{array}$$

- Note that φ is determined by its restriction $\varphi|_{\mathcal{A}}$ and its transferred version $\psi := \varphi|_{\mathcal{A}} \circ \mathcal{F}^{\widehat{\mathbb{R}}} : C_c^\infty(\widehat{\mathbb{R}}) \to \mathbb{C}$ is a multiplicative linear functional w.r.t. convolution product.
- We can check ψ satisfies the *Cauchy functional equation*

$$\psi(x + y) = \psi(x)\psi(y) \text{ for a.e. } x, y \in \widehat{\mathbb{R}},$$

so that $\psi(x) = e^{icx}$, $x \in \widehat{\mathbb{R}}$ for some $c \in \mathbb{C}$. This observation establishes the correspondence

$$\varphi \in \mathrm{Spec}\, C_c^\infty(\widehat{\mathbb{R}}) \Leftrightarrow c \in \mathbb{C} = \mathbb{R}_{\mathbb{C}}.$$

The case of $\mathbb{R}$: continued

- The Paley-Wiener theorem implies for any $f \in C_c^\infty(\widehat{\mathbb{R}})$ the Fourier transform $\mathcal{F}^{\widehat{\mathbb{R}}}(f)$ extends to an entire function on $\mathbb{C}$ and we have

$$\varphi(\mathcal{F}^{\widehat{\mathbb{R}}}(f)) = \int_{\widehat{\mathbb{R}}} e^{icx} f(x)dx = \mathcal{F}^{\widehat{\mathbb{R}}}(f)(-c).$$

  In other words, the functional φ is nothing but the evaluation at the point $-c \in \mathbb{C}$.

- In summary, we have a dense subalgebra $\mathcal{A}$ in $A(\mathbb{R}, w)$ which leads us to the "abstract Lie" description of the complexification $\mathbb{C} = \mathbb{R}_\mathbb{C}$ via the Cauchy functional equation. Moreover, any elements in $\mathrm{Spec}A(\mathbb{R}, w)$ can be understood as point evaluations on points of $\mathbb{C} = \mathbb{R}_\mathbb{C}$ for the functions in $\mathcal{A}$.
- The final step would be checking the norm condition on φ.

The case of the Heisenberg group $\mathbb{H}$

- $\mathbb{H} = \left\{ (y,z,x) = \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x,y,z \in \mathbb{R} \right\} = (\mathbb{R} \times \mathbb{R}) \rtimes \mathbb{R}.$
- For any $a \in \mathbb{R}^*$ we have an irreducible unitary representation

$$\pi^a(y,z,x)\xi(t) = e^{-ia(ty-z)}\xi(-x+t),\ \xi \in L^2(\mathbb{R}).$$

- The left regular representation λ allows a quasi-equivalence $\lambda \cong \int_{\mathbb{R}^*}^{\oplus} \pi^a |a| da$, which tells us that

$$VN(\mathbb{H}) \cong L^\infty(\mathbb{R}^*, |a|da; B(L^2(\mathbb{R}))),\ A(\mathbb{H}) \cong L^1(\mathbb{R}^*, |a|da; S^1(L^2(\mathbb{R}))).$$

- For $f \in L^1(\mathbb{H})$ we define the group Fourier transfom on $\mathbb{H}$ by

$$\mathcal{F}^{\mathbb{H}}(f) = (\mathcal{F}^{\mathbb{H}}(f)(a))_{a\in\mathbb{R}^*} = (\widehat{f}^{\mathbb{H}}(a))_{a\in\mathbb{R}^*} \in L^\infty(\mathbb{R}^*; B(L^2(\mathbb{R})))$$

and

$$\widehat{f}^{\mathbb{H}}(a) = \int_{\mathbb{H}} f(g)\pi^a(g)dg.$$

The case of the Heisenberg group $\mathbb{H}$: continued

- We have the universal complexification
$$\mathbb{H}_{\mathbb{C}} = \left\{ (y,z,x) = \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x,y,z \in \mathbb{C} \right\}.$$
- We clearly have the following Cartan type decomposition

$$\mathbb{H}_{\mathbb{C}} \cong \mathbb{H} \cdot \exp(i\,\mathfrak{heis}),$$

where $\mathfrak{heis}$ is the Lie algebra of $\mathbb{H}$.

Finding a dense subalgebra of $A(\mathbb{H}, W)$

- The Heisenberg group $\mathbb{H}$ actually have a "background" Euclidean structure $\widehat{\mathbb{R}}^3$, which shares the Haar measures, namely the Lebesgue measure with $\mathbb{H}$.
- This motivates us to begin with the space of test functions $C_c^\infty(\mathbb{R}^3)$ and its $\mathbb{R}^3$-Fourier transform image as a function algebra $\mathcal{A}$ on $\mathbb{H}$.
- The algebra $\mathcal{A}$ can be shown to be inside of $A(\mathbb{H}, W)$ densely regardless of the choice of W, which is highly non-trivial.
- For any $\varphi \in \mathrm{Spec}A(\mathbb{H}, W)$ we have

$$\begin{array}{ccc} \varphi : A(\mathbb{H}, W) & \longrightarrow & \mathbb{C} \\ \uparrow \mathcal{F}^{\mathbb{R}^3} & \nearrow \tilde{\varphi} & \\ \mathcal{A} = C_c^\infty(\mathbb{R}^3) & & \end{array}.$$

  Thus, we get $\tilde{\varphi} = \varphi \circ \mathcal{F}^{\mathbb{R}^3} : C_c^\infty(\mathbb{R}^3) \to \mathbb{C}$ which is multiplicative with respect to $\mathbb{R}^3$-convolution. This leads us to solving a Cauchy type functional equation on $\mathbb{R}^3$ in distribution sense.

Some technicalities on $\mathcal{A}$

- For the density of $\mathcal{A}$ in $A(G, W)$ we need companion spaces

Def

We define $\mathcal{B} := \mathcal{F}^{\mathbb{R}^3}(\mathcal{B}_0) \subseteq C^\infty(\mathbb{H})$, where

$$\mathcal{B}_0 := \{f \in L^1_{\text{loc}}(\mathbb{R}^3) : e^{t(|x|+|y|+|z|)}(\partial^\alpha f)(x, y, z) \in L^2(\mathbb{R}^3), \forall t > 0, \forall \alpha\},$$

where ∂^α refers the partial derivative in the weak sense for the multi-index α. We endow a natural locally convex topology on $\mathcal{B}_0$ given by the family of canonical semi-norms.
We also define the space $\mathcal{D}$ by

$$\mathcal{D} := \text{span}\{P_{mn} \otimes h : m, n \in \mathbb{Z},\ h \in C_c^\infty(\mathbb{R}^*)\} \subseteq C_c^\infty(\mathbb{R}^*; S^1(L^2(\mathbb{R}))),$$

where P_{mn} is the rank 1 operator on $B(L^2(\mathbb{R}))$ given by $P_{mn}\xi = \langle \xi, \varphi_m \rangle \varphi_n$ with respect to the basis $\{\varphi_n\}_{n \geq 0}$ consisting of Hermite functions.

Some technicalities on $\mathcal{A}$: continued

- The space $\mathcal{B}_0$ can be called as the space of functions whose partial derivatives have a "super-exponential" decay. Note that the space $\mathcal{B}_0$ has already been introduced by Jorgensen under the name of "hyper-Schwartz space".
- (Why $\mathcal{B}_0$?) The super-exponential decay property allows us to "absorb" the effect of the weight W which is possibly "exponentially growing".
- (Why $\mathcal{B}_0$??) It contains the space $\mathcal{D}$ whose elements are entire vectors for λ. This allows us to use complex Fourier inversion!

Entire vectors

- $\pi : G \to B(\mathcal{H}_\pi)$: a unitary representation of G.
  A vector $v \in \mathcal{H}_\pi$ is called an entire vector for π if $E_s(v) < \infty$ for all $s > 0$, where

$$E_s(v) := \sum_{m=1}^{\infty} \frac{s^m}{m!} \rho_m(v).$$

  We denote the space of all entire vectors for π by $\mathcal{D}^\infty_{\mathbb{C}}(\pi)$.
- Roughly speaking the mapping $g \in G \mapsto \pi(g)v$ extends to an analytic mapping to the whole $G_{\mathbb{C}}$.

Entire vectors: continued

Thm by Goodman

Let G be a connected solvable Lie group which is separable, type I and unimodular. Let $f \in L^2(G)$ be an entire vector for λ, then we have

$$\int_{\widehat{G}} \sup_{\gamma \in \Omega_t} ||\pi_{\mathbb{C}}^{\xi}(\gamma^{-1})\widehat{f}^G(\xi)||_1 d\mu(\xi) < \infty$$

for any $t > 0$, where $|| \cdot ||_1$ is the trace class norm. Moreover, f is analytically entended to $G_{\mathbb{C}}$ with the analytic continuation $f_{\mathbb{C}}$ given by the absolutely convergent integral

$$f_{\mathbb{C}}(\gamma) = \int_{\widehat{G}} \mathrm{Tr}(\pi_{\mathbb{C}}^{\xi}(\gamma^{-1})\widehat{f}^G(\xi))d\mu(\xi), \ \gamma \in G_{\mathbb{C}}.$$

Entire vectors: continued 2

Thm by Goodman

Let G be a connected solvable Lie group which is separable, type I and unimodular. A function $f \in L^2(G)$ is an entire vector for λ if and only if

$$\begin{cases} \mathrm{ran} \widehat{f}^G(\xi) \subseteq \mathcal{D}^\infty_{\mathbb{C}}(\pi^\xi) \ \mu\text{-almost every } \xi \text{ and} \\ \int_{\widehat{G}} \sup_{\gamma \in \Omega_t} ||\pi^\xi_{\mathbb{C}}(\gamma^{-1})\widehat{f}^G(\xi)||_2^2 d\mu(\xi) < \infty \text{ for any } t > 0, \end{cases}$$

where the set Ω_t is given by $\Omega_t = \{\mathrm{exp} X : X \in \mathfrak{g}_{\mathbb{C}}, \ ||X|| < t\}$

- Using the above we can show that all the elements in $\mathcal{D}$ are entire vectors for λ.

The case of the Heisenberg groups: conclusion

Thm

Let $\mathfrak{h}$ be the Lie subalgebra corresponding to the subgroup $H = H_{Y,Z}$ of $\mathbb{H}$. Then we have

$$\mathrm{Spec} A(\mathbb{H}, W) \cong \{g \cdot \exp(iX') : g \in \mathbb{H}, X' \in \mathfrak{h}, \exp(iX') \in \mathrm{Spec} A(H, W_H)\}$$

Future directions

- Compact quantum groups other than $SU_q(2)$.
- Non-compact quantum groups such as quantum $E(2)$-group. How about their complexification?

Thank you for your attention